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ON MILD SOLUTIONS OF NONLOCAL SECOND ORDER SEMILINEAR FUNCTIONAL INTEGRO-DIFFERENTIAL EQUATIONS

Abstract. In the present paper, we investigate the existence, uniqueness and continuous dependence on initial data of mild solutions of second order nonlocal semilinear functional integro-differential equations of more general type with delay in Banach spaces. Our analysis is based on the theory of strongly continuous cosine family of operators and modified version of Banach contraction theorem.

Key words: existence, uniqueness, functional, integro- differential equation, fixed point, second order, cosine family.

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1. Introduction

In the present paper we study semilinear delay functional second order integro-differential equations of the type:

$$x''(t) = Ax(t) + f(t, x_t, \int_0^t k(t,s)h(s,x_s)ds), \quad t \in [0,T], \tag{1}$$

$$x(t) + (g(x_{t_1}, \dots, x_{t_p}))(t) = \phi(t), \quad -r \le t \le 0, \tag{2}$$

$$x'(0) = \eta \in X, \tag{3}$$

where $0 < t_1 < t_2 < \dots < t_p \le T$, $p \in N$, A is the infinitesimal generator of a strongly continuous cosine family of bounded linear operators $\{C(t)\}_{t\in\mathbb{R}}$ on X; f, g, h, k and ϕ are given functions satisfying some assumptions and $x_t(\theta) = x(t+\theta)$, for $\theta \in [-r, 0]$ and $t \in [0,T]$.

Equations of the form (1)-(3) or their special forms serve as an abstract formulation of partial integro-differential equations which arise in the problems with memory visco-elasticity and many other physical phenomena, see

[1], [5], [7], [8], [9], [14] and the references given therein. The problems of qualitative properties of solutions of second order functional differential equations have studied by many authors, see[7]-[9], [11], [16]-[19]. It is advantageous to treat second order abstract differential equations directly rather than to convert into first order differential system. For direct applications of second order differential system, one may refer Fitzgibbon[8]. On the other hand, as nonlocal condition is more precise to describe natural phenomena than classical initial condition, the Cauchy problem with nonlocal condition also received much attention in recent years, see [2]-[4], [10], [12]. The nonlocal evolution problems were first studied by L. Byszewski.

The objective of this paper is to obtain existence and uniqueness of a mild solution on initial data of second order integro-differential (1)-(3), using the theory of strongly continuous cosine family of operators and modified version of Banach contraction theorem. We are generalizing and improving some of results reported in [11], [16]-[19].

The paper is organized as follows: Section 2 presents preliminaries and hypotheses. In Section 3, we prove existence and uniqueness of solutions. Section 4, deals with continuous dependence on initial data of mild solutions. Finally in Section 5, we give application based on our result.

2. Preliminaries and hypotheses

Let X be a Banach space with the norm $\|\cdot\|$. Let $C = \mathcal{C}([-r,0],X)$, $0<r<\infty$, be the Banach space of all continuous functions $\psi : [-r,0] \to X$ endowed with supremum norm

$$\|\psi\|_C = \sup\{\|\psi(t)\| : -r \le t \le 0\}.$$

Let $B = \mathcal{C}([-r,T],X)$, $T>0$, be the Banach space of all continuous functions $x : [-r,T] \to X$ with the supremum norm $\|x\|_B = \sup\{\|x(t)\| : -r \le t \le T\}$. For any $x \in B$ and $t \in [0,T]$, we denote by x_t the element of C given by $x_t(\theta) = x(t+\theta)$, for $\theta \in [-r,0]$ and ϕ is a given element of C.

Definition 1. *A one parameter family $\{C(t) : t \in \mathbb{R}\}$ of bounded linear operators in the Banach space X is called strongly continuous cosine family if and only if*

(a) $C(0) = I$ *is the identity operator,*
(b) $C(t+s) + C(t-s) = 2C(t)C(s) \quad \forall t,s \in \mathbb{R}$,
(c) *The map* $t \mapsto C(t)(x)$ *is strongly continuous for each* $x \in X$.

The associated sine function is the family $\{S(t)\}_{t\in\mathbb{R}}$ of operators defined by $S(t)x = \int_0^t C(s)xds$, for $x \in X, t \in \mathbb{R}$. The infinitesimal generator $A : X \to X$ of a cosine family $\{C(t) : t \in \mathbb{R}\}$ is defined by $Ax = \frac{d^2}{dt^2}C(t)x|_{t=0}$,

$x \in D(A)$, where $D(A) = \{x \in X : C(\cdot)x \in C^2(\mathbb{R}, X)\}$. For more information on strongly continuous cosine and sine families, we refer the reader to [[7], [17]-[19]].

In this paper, we assume that, there exist positive constants $M \geq 1$ and N such that $\|C(t)\| \leq M$ and $\|S(t)\| \leq N$, for every $t \in [0, T]$. Also $k : [0, T] \times [0, T] \to \mathbb{R}$ is continuous function and as $[0, T] \times [0, T]$ is compact set, there exists a constant $L_1 > 0$ such that $|k(t, s)| \leq L_1$, for $0 \leq s \leq t \leq T$.

Definition 2. *A function $x : [-r, T] \to X$ is called a mild solution of the system* (1)-(3)*, if it satisfies the following equations*

$$\begin{aligned} x(t) &= C(t)[\phi(0) - (g(x_{t_1}, \ldots, x_{t_p}))(0)] + S(t)\eta \\ &+ \int_0^t S(t-s) f(s, x_s, \int_0^s k(s, \tau) h(\tau, x_\tau) d\tau) ds, \quad t \in [0, T], \end{aligned}$$

$$\begin{aligned} &x(t) + (g(x_{t_1}, \ldots, x_{t_p}))(t) = \phi(t), \quad -r \leq t \leq 0, \\ &x'(0) = \eta \in X. \end{aligned}$$

The following lemma is known as Pachpatte's inequality.

Lemma 1 ([13], p. 33)**.** *Let u, f and g be nonnegative continuous functions defined on $\mathbb{R}^+$, for which the inequality*

$$u(t) \leq u_0 + \int_0^t f(s) u(s) ds + \int_0^t f(s) \Big(\int_0^s g(\sigma) u(\sigma) d\sigma \Big) ds, \quad t \in \mathbb{R}^+,$$

holds, where u_0 is nonnegative constant. Then

$$u(t) \leq u_0 [1 + \int_0^t f(s) exp(\int_0^s [f(\sigma) + g(\sigma)] d\sigma) ds], \quad t \in \mathbb{R}^+.$$

Our results are based on the modified version of the Banach contraction principle.

Lemma 2 ([15], p. 196)**.** *Let X be a Banach space. Let D be an operator which maps the elements of X into itself for which D^r is a contraction, where r is a positive integer. Then D has a unique fixed point.*

We list the following hypotheses for our convenience.

(H_1) Let $f : [0, T] \times C \times X \to X$ be such that for every $w \in B$, $x \in X$ and $t \in [0, T]$, $f(\cdot, w_t, x) \in B$ and there exists a constant $L > 0$ such that

$$\|f(t, \psi, x) - f(t, \phi, y)\| \leq L(\|\psi - \phi\|_C + \|x - y\|), \quad \phi, \psi \in C, \;\; x, y \in X.$$

(H_2) Let $h : [0, T] \times C \to X$ be such that for every $w \in B$ and $t \in [0, T]$, $h(\cdot, w_t) \in B$ and there exists a constant $H > 0$ such that

$$\|h(t, \psi) - h(t, \phi)\| \leq H \|\psi - \phi\|_C, \quad \phi, \psi \in C.$$

(H_3) Let $g : C^p \to C$ be such that exists a constant $G \geq 0$ satisfying

$$\|(g(x_{t_1}, x_{t_2}, \dots, x_{t_p}))(t) - (g(y_{t_1}, y_{t_2}, \dots, y_{t_p}))(t)\| \leq G\|x - y\|_B,$$

$t \in [-r, 0]$.

3. Existence and uniqueness

Theorem 1. *Suppose that the hypotheses* (H_1)-(H_3) *are satisfied. Then the second order integro-differential system* (1)-(3) *has a unique mild solution* x *on* $[-r, T]$.

Proof. Let $x(t)$ be a mild solution of the problem (1)-(3). Then it satisfies the equivalent integral equation

$$(4)\quad \begin{aligned} x(t) \;=\; & C(t)\phi(0) - C(t)(g(x_{t_1}, \dots, x_{t_p}))(0) + S(t)\eta \\ & + \int_0^t S(t-s) f(s, x_s, \int_0^s k(s,\tau) h(\tau, x_\tau) d\tau) ds, \quad t \in [0, T], \end{aligned}$$

$$(5)\qquad x(t) + (g(x_{t_1}, \dots, x_{t_p}))(t) = \phi(t), \quad -r \leq t \leq 0,$$

$$(6)\qquad x'(0) = \eta \in X.$$

Now, we rewrite solution of initial value problem (1)-(3) as follows: For $\phi \in C$, define $\widehat{\phi} \in B$ by

$$(7)\qquad \widehat{\phi}(t) = \begin{cases} \phi(t) - (g(x_{t_1}, \dots, x_{t_p}))(t) & \text{if } \; -r \leq t \leq 0, \\ C(t)[\phi(0) - (g(x_{t_1}, \dots, x_{t_p}))(0)] & \text{if } \; 0 \leq t \leq T. \end{cases}$$

If $y \in B$ and $x(t) = y(t) + \widehat{\phi}(t)$, $t \in [-r, T]$, then it is easy to see that y satisfies

$$(8)\qquad y(t) = 0; \quad -r \leq t \leq 0$$

and

$$(9)\quad \begin{aligned} y(t) \;=\; & S(t)\eta + \int_0^t S(t-s) f\Big(s, y_s + \widehat{\phi}_s, \\ & \qquad \int_0^s k(s,\tau) h(\tau, y_\tau + \widehat{\phi}_\tau) d\tau\Big) ds, \quad t \in [0, T] \end{aligned}$$

if and only if $x(t)$ satisfies the equations (4)-(6).
We define the operator $F : B \to B$, by

$$(10)\quad (Fy)(t) = \begin{cases} 0 & \text{if } \; -r \leq t \leq 0 \\ S(t)\eta + \int_0^t S(t-s) f\Big(s, y_s + \widehat{\phi}_s, & \\ \qquad \int_0^s k(s,\tau) h(\tau, y_\tau + \widehat{\phi}_\tau) d\tau\Big) ds & \text{if } \; t \in [0, T]. \end{cases}$$

From the definition of an operator F defined by the equation (10), it is to be noted that the equations (8)-(9) can be written as

$$y = Fy.$$

Now we show that F^n is a contraction on B for some positive integer n. Let $y, w \in B$ and using hypotheses (H_1) - (H_3), we get,

$$\begin{aligned}
\|(Fy)(t) - (Fw)(t)\| &\leq \|S(t)\eta - S(t)\eta\| \\
&\quad + \int_0^t \|S(t-s)\| \|f(s, y_s + \widehat{\phi}_s, \int_0^s k(s,\tau)h(\tau, y_\tau + \widehat{\phi}_\tau)d\tau) \\
&\quad - f(s, w_s + \widehat{\phi}_s, \int_0^s k(s,\tau)h(\tau, w_\tau + \widehat{\phi}_\tau)d\tau)\|ds \\
&\leq \int_0^t NL[\|(y_s + \widehat{\phi}_s) - (w_s + \widehat{\phi}_s)\|_C \\
&\quad + L_1 \int_0^s \|h(\tau, y_\tau + \widehat{\phi}_\tau) - h(\tau, w_\tau \\
&\quad + \widehat{\phi}_\tau)\|d\tau]ds \\
&\leq NL \int_0^t \|y_s - w_s\|_C ds \\
&\quad + NL \int_0^t L_1 H \int_0^s \|y_\tau - w_\tau\|_C d\tau ds \\
&\leq NL \int_0^t \|y - w\|_B ds + NL \int_0^t L_1 H \int_0^s \|y - w\|_B d\tau ds \\
&\leq NL\|y - w\|_B t + NLL_1H\|y - w\|_B \frac{t^2}{2} \\
&\leq NL\|y - w\|_B t + NLL_1HT\|y - w\|_B \frac{t}{2} \\
&\leq NL\|y - w\|_B t + NLL_1HT\|y - w\|_B t \\
&\leq NL(1 + L_1HT)\|y - w\|_B t,
\end{aligned}$$

$$\begin{aligned}
\|(F^2y)(t) - (F^2w)(t)\| &= \|(F(Fy))(t) - (F(Fw))(t)\| \\
&= \|(F(y_1))(t) - (F(w_1))(t)\| \\
&\leq \int_0^t \|S(t-s)\| \|f(s, y_{1_s} + \widehat{\phi}_s, \int_0^s k(s,\tau)h(\tau, y_{1_\tau} + \widehat{\phi}_\tau)d\tau) \\
&\quad - f(s, w_{1_s} + \widehat{\phi}_s, \int_0^s k(s,\tau)h(\tau, w_{1_\tau} + \widehat{\phi}_\tau)d\tau)\|ds \\
&\leq \int_0^t NL\|y_{1_s} - w_{1_s}\|_C + NL \int_0^t L_1 H \|y_{1_\tau} - w_{1_\tau}\|_C d\tau ds \\
&\leq NL \int_0^t \|y_1 - w_1\|_{\mathcal{C}([-r,s],X)} ds
\end{aligned}$$

$$
\begin{aligned}
&+ NL\int_0^t L_1H\int_0^s \|y_1 - w_1\|_{\mathcal{C}([-r,\tau],X)}d\tau ds \\
&\leq NL\int_0^t \sup_{\tau\in[-r,s]}\|y_1(\tau) - w_1(\tau)\|ds \\
&\quad + NLL_1H\int_0^t\int_0^s \sup_{\eta\in[-r,\tau]}\|y_1(\eta) - w_1(\eta)\|d\tau ds \\
&\leq NL\int_0^t \sup_{\tau\in[-r,s]}\|Fy(\tau) - Fw(\tau)\|ds \\
&\quad + NLL_1H\int_0^t\int_0^s \sup_{\eta\in[-r,\tau]}\|Fy(\eta) - Fw(\eta)\|d\tau ds \\
&\leq NL\int_0^t \sup_{\tau\in[-r,s]}(NL[1 + L_1HT]\|y - w\|_B\tau)ds \\
&\quad + NLL_1H\int_0^t\int_0^s \sup_{\eta\in[-r,\tau]}(NL[1 + L_1HT]\|y - w\|_B\eta)d\tau ds \\
&\leq N^2L^2[1 + L_1HT]\|y - w\|_B[\int_0^t(\sup_{\tau\in[-r,s]}\tau)ds \\
&\quad + \int_0^t L_1H\int_0^s(\sup_{\eta\in[-r,\tau]}\eta)d\tau ds] \\
&\leq N^2L^2[1 + L_1HT]\|y - w\|_B[\int_0^t sds + \int_0^t L_1H\int_0^s \tau d\tau ds] \\
&\leq N^2L^2[1 + L_1HT]\|y - w\|_B[\frac{t^2}{2} + L_1H\frac{t^3}{3!}] \\
&\leq N^2L^2[1 + L_1HT]^2\|y - w\|_B[\frac{t^2}{2} + L_1HT\frac{t^2}{3!}] \\
&\leq N^2L^2[1 + L_1HT]^2\|y - w\|_B[\frac{t^2}{2!} + L_1HT\frac{t^2}{2!}] \\
&\leq \frac{(NL[1 + L_1HT]t)^2}{2!}\|y - w\|_B.
\end{aligned}
$$

Continuing in this way, we get,

$$
\begin{aligned}
\|(F^ny)(t) - (F^nw)(t)\| &\leq \frac{(NL(1 + L_1HT)t)^n}{n!}\|y - w\|_B \\
&\leq \frac{(NL(1 + L_1HT)T)^n}{n!}\|y - w\|_B.
\end{aligned}
$$

For n large enough, $\frac{(NL(1+L_1HT)T)^n}{n!} < 1$. Thus there exists a positive integer n such that F^n is a contraction in B. By virtue of Lemma 2, the operator

F has a unique fixed point $\tilde{y}$ in B. Then $\tilde{x} = \tilde{y} + \widehat{\phi}$ is a solution of second order integro-differential system (1)-(3). This completes the proof. ■

4. Continuous dependence on initial data

Theorem 2. *Suppose that the functions f, h and g satisfy the hypotheses (H_1)-(H_3). Then for each $\phi_1, \phi_2 \in C$ and for the corresponding mild solutions x_1, x_2 of the problems*

$$x''(t) = Ax(t) + f(t, x_t, \int_0^t k(s,t)h(t, x_t)dt), \quad t \in [0, T], \tag{11}$$

$$x(t) + (g(x_{t_1}, \ldots, x_{t_p}))(t) = \phi_i(t), \quad -r \leq t \leq 0, \quad (i = 1, 2), \tag{12}$$

$$x'(0) = \eta_i \in X, \quad i = 1, 2, \tag{13}$$

the inequality

$$\begin{aligned} \|x_1 - x_2\|_B \; \leq \; & [M\|\phi_1 - \phi_2\|_C + MG\|x_1 - x_2\|_B \\ & + \; N\|\eta_1 - \eta_2\|] \left[1 + NLTe^{(NL+L_1H)T}\right] \end{aligned} \tag{14}$$

is true.

Moreover, if $G = 0$, then it reduces to classical inequality

$$\begin{aligned} \|x_1 - x_2\|_B \; \leq \; & [M\|\phi_1 - \phi_2\|_C \\ & + \; N\|\eta_1 - \eta_2\|] \left[1 + NLTe^{(NL+L_1H)T}\right]. \end{aligned} \tag{15}$$

Proof. Let ϕ_i $(i = 1, 2)$ be arbitrary functions in C and let x_i $(i = 1, 2)$ be the mild solutions of the problem (11)-(13). Then for $t \in [-r, 0]$,

$$\begin{aligned} x_1(t) - x_2(t) \; = \; & \phi_1(t) - (g(x_{1t_1}, \ldots, x_{1t_p}))(t) \\ & - \phi_2(t) + (g(x_{2t_1}, \ldots, x_{2t_p}))(t) \end{aligned} \tag{16}$$

and for $t \in [0, T]$,

$$\begin{aligned} x_1(t) - x_2(t) \; = \; & C(t)[\phi_1(0) - \phi_2(0) - (g(x_{1t_1}, \ldots, x_{1t_p}))(0) \\ & - (g(x_{2t_1}, \ldots, x_{2t_p}))(0)] + S(t)(\eta_1 - \eta_2) \\ & + \int_0^t S(t-s)[f(s, x_{1s}, \int_0^s k(s,\tau)h(\tau, x_{1s_\tau})d\tau)) \\ & - f(s, x_{2s}, \int_0^s k(s,\tau)h(\tau, x_{2s_\tau})d\tau)]ds. \end{aligned} \tag{17}$$

From (17) and hypotheses (H_1)-(H_3), we get, for $t \in [0,t]$,

$$
\begin{aligned}
(18) \qquad \|x_1(t) - x_2(t)\| &= \|C(t)\|\|\phi_1 - \phi_2\|_C \\
&\quad + G\|C(t)\|\|x_1 - x_2\|_B + \|S(t)\|\|\eta_1 - \eta_2\| \\
&\quad + \int_0^t \|T(t-s)\|\|f(s, x_{1s}, \int_0^s k(s,\tau)h(\tau, x_{1\tau})d\tau) \\
&\quad - f(s, x_{2s}, \int_0^s k(s,\tau)h(\tau, x_{2\tau})d\tau)\|ds \\
&\le M\|\phi_1 - \phi_2\|_C + MG\|x_1 - x_2\|_B + N\|\eta_1 - \eta_2\| \\
&\quad + \int_0^t NL\left[\|x_{1s} - x_{2s}\|_C + L_1 H \int_0^s \|x_{1\tau} - x_{2\tau}\|_C d\tau\right] ds.
\end{aligned}
$$

Define the function $z : [-r, T] \to \mathbb{R}$ by $z(t) = \sup\{\|x_1(s) - x_2(s)\| : -r \le s \le t\}$, $t \in [0,T]$. Let $t^* \in [-r,t]$ be such that $z(t) = \|x_1(t^*) - x_2(t^*)\|$. If $t^* \in [0,t]$, then from inequality (18),we have

$$
\begin{aligned}
(19) \quad z(t) &= \|x_1(t^*) - x_2(t^*)\| \\
&\le M\|\phi_1 - \phi_2\|_C + MG\|x_1 - x_2\|_B + N\|\eta_1 - \eta_2\| \\
&\quad + \int_0^{t^*} NL\left[\|x_{1s} - x_{2s}\|_C + L_1 H \int_0^s \|x_{1\tau} - x_{2\tau}\|_C d\tau\right] ds \\
&\le M\|\phi_1 - \phi_2\|_C + MG\|x_1 - x_2\|_B + N\|\eta_1 - \eta_2\| \\
&\quad + \int_0^{t} NL\left[\|x_{1s} - x_{2s}\|_C + L_1 H \int_0^s \|x_{1\tau} - x_{2\tau}\|_C d\tau\right] ds \\
&\le K\|\phi_1 - \phi_2\|_C + MG\|x_1 - x_2\|_B + N\|\eta_1 - \eta_2\| \\
&\quad + \int_0^{t} NL\left[z(s) + L_1 H \int_0^s z(\tau) d\tau\right] ds.
\end{aligned}
$$

If $t^* \in [-r, 0]$ then $z(t) \le \|\phi_1 - \phi_2\|_C + G\|x_1 - x_2\|_B$ and since $M \ge 1$ the inequality (19) holds good. Thus for $t^* \in [-r, T]$, the inequality (19) holds good. Thanks to Pachpatte's inequality given in Lemma 1 and applying it to inequality (19) we get,

$$
\begin{aligned}
z(t) &\le [M\|\phi_1 - \phi_2\|_C + MG\|x_1 - x_2\|_B \\
&\quad + N\|\eta_1 - \eta_2\|]\left[1 + \int_0^t NL e^{\int_0^s (NL + L_1 H) d\tau} ds\right] \\
&\le [M\|\phi_1 - \phi_2\|_C + MG\|x_1 - x_2\|_B \\
&\quad + N\|\eta_1 - \eta_2\|]\left[1 + \int_0^t NL e^{(NL + L_1 H)T} ds\right] \\
&\le [M\|\phi_1 - \phi_2\|_C + MG\|x_1 - x_2\|_B \\
&\quad + N\|\eta_1 - \eta_2\|]\left[1 + NLT e^{(NL + L_1 H)T}\right].
\end{aligned}
$$

Consequently,

$$\begin{aligned}(20)\qquad \|x_1 - x_2\|_B \;\leq\; & [M\|\phi_1 - \phi_2\|_C + MG\|x_1 - x_2\|_B \\ & +\; N\|\eta_1 - \eta_2\|] \left[1 + NLTe^{(NL+L_1H)T}\right].\end{aligned}$$

Hence the inequality (14) holds. Finally inequality (15) is a consequence of the inequality (20). Hence the proof is complete. ■

5. Application

To illustrate the application of our result proved in Section 3, consider the following semilinear partial functional integro-differential problem of the form

$$\begin{aligned}(21)\quad \frac{\partial^2}{\partial t^2}w(u,t) \;=\; & \frac{\partial^2}{\partial u^2}w(u,t) + H\Big(t, w(u,t-r), \\ & \int_0^t k(t,s)P(s,w(s-r))ds\Big), 0 \leq u \leq \pi, t \in [0,T]\end{aligned}$$

$$(22)\qquad w(0,t) = w(\pi,t) = 0, \quad 0 \leq t \leq T,$$

$$(23)\qquad w(u,t) + \sum_{i=1}^{p} w(u,t_i + t) = \phi(u,t), \quad 0 \leq u \leq \pi, \quad -r \leq t \leq 0,$$

$$(24)\qquad \frac{\partial}{\partial t}w(u,0) = \eta(u) \quad 0 \leq u \leq \pi,$$

where $0 < t_1 \leq t_2 \leq t_p \leq T$, the function $H : [0,T] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is continuous. We assume that the functions H and P satisfy the following conditions:

For every $t \in [0,T]$ and $u, v, x, y \in \mathbb{R}$, there exists a constant $l, p > 1$ such that

$$|H(t,u,x) - H(t,v,y)| \leq l(|u-v| + |x-y|),$$

$$|P(t,u) - P(t,v)| \leq p(|u-v|).$$

Let us take $X = L^2[0,\pi]$. Define the operator $A : X \to X$ by $Az = z_{uu}$ with domain $D(A) = \{z \in X : z, z_u$ are absolutely continuous, $z_{uu} \in X$ and $z(0) = z(\pi) = 0\}$. Then the operator A is the infinitesimal generator of a strongly continuous cosine family $\{C(t)\}_{t\in\mathbb{R}}$ on X. Moreover A has a discrete spectrum, the eigenvalues are $-n^2$, $n \in \mathbb{N}$, with corresponding eigenvectors

$z_n(u) = (\sqrt{2/\pi})\sin(nu)$. The set $\{z_n : n \in \mathbb{N}\}$ is an orthonormal basis of X and the following properties hold:

(a) If $z \in D(A)$, then $Az = -\sum_{n=1}^{\infty} n^2(z, z_n)z_n$.

(b) For every $z \in X$, $C(t)z = \sum_{n=1}^{\infty} \cos nt(z, z_n)z_n$.

(c) For every $z \in X$, $S(t)z = \sum_{n=1}^{\infty} \frac{\sin nt}{n}(z, z_n)z_n$.

Consequently, $\|C(t)\| = \|S(t)\| \leq 1$ and $S(t)$ is compact for $t \in \mathbb{R}$.

Define the function $f : [0,T] \times C \times X \to X$, as follows

$$f(t, \psi, x)(u) = H(t, \psi(-r)u, x(u)),$$

$$h(t, \phi)(u) = P(t, \phi(-r)u),$$

for $t \in [0,T]$, $\psi, \phi \in C$, $x \in X$ and $0 \leq u \leq \pi$. With these choices of the functions, the equations(21)-(24) can be formulated as an abstract integro-differential equations (1)-(3) in Banach space X. Since all the hypotheses of Theorem 1 are satisfied, Theorem 1, can be applied to guarantee the existence of mild solution $w(u,t) = x(t)u, \quad t \in [0,T], u \in [0,\pi]$, of the semilinear partial differential problem (21)-(24).

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