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DENSITY OF SMOOTH FUNCTIONS IN SOBOLEV SPACES "WITH MIXED FUNCTIONS"

Abstract. The results presented in this paper concern approximation by smooth functions in the Sobolev spaces defined by means of a modular (1). These spaces can be a natural medium to study the partial differential equations with rapidly or slowly increasing coefficients (i.e. the coefficients are of a nonpolynomial type).

Key words: modular space, Sobolev space.

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1. Basic notions

Let A and B denote arbitrary open and bounded intervals in $R = (-\infty, +\infty)$ and $\Omega = A \times B$. $L(\Omega)$ denote the space of Lebesgue integrable real functions on Ω, with equality almost everywhere. Let real functions $\varphi : A \times R \to [0, +\infty)$ and $\psi : B \times R \to [0, +\infty)$ satisfy the following conditions:
1. φ and ψ are measurable functions of the first variable for every fixed value of the second one;
2. $\varphi(t, u)$ and $\psi(t, u)$ are even, convex and continuous at zero with respect to the second variable, $\varphi(t, 0) = \psi(t, 0) = 0$, $\varphi(t, u) > 0$ and $\psi(t, u) > 0$ if $u \neq 0$ for a.e. t.
3. $\int_A \varphi(t, u)\, dt < \infty$, $\int_B \psi(t, u)\, dt < \infty$ for every u

For any function $f \in L(\Omega)$ we define a functional

$$I_{\varphi,\psi}(f) = \int_A \varphi\left(x, \int_B \psi(y, f(x, y))\, dy\right) dx.$$

The functional $I_{\varphi,\psi}$ is a convex modular in $L(\Omega)$, ([4]). We denote by $L_{\varphi,\psi}(\Omega)$ the vector space of all functions f in $L(\Omega)$ such that $I_{\varphi.\psi}(\lambda f) < \infty$ for some $\lambda > 0$, ([3], [4]).

Convergence $f_n \to f$ in $L_{\varphi,\psi}(\Omega)$ we mean as the convergence in the sense of the modular $I_{\varphi,\psi}$:

$$I_{\varphi,\psi}(\lambda(f_n - f)) \to 0, \quad n \to \infty \ \text{ for some } \ \lambda > 0.$$

Let k be an arbitrary nonnegative integer number and let φ and ψ satisfy the conditions 1 - 3. We denote by $W^k_{\varphi,\psi}(\Omega)$ the space of all functions $f \in L_{\varphi,\psi}(\Omega)$ possessing distributional derivatives $D^\alpha f$ up to order k belonging to the space $L_{\varphi,\psi}(\Omega)$. The space $W^k_{\varphi,\psi}(\Omega)$ we call the Sobolev space "with mixed functions", ([2]). We consider a functional $I^{(k)}_{\varphi,\psi}$

$$I^{(k)}_{\varphi,\psi}(f) = \sum_{|\alpha|\le k} \int_A \varphi\left(x, \int_B \psi\left(y, D^\alpha f(x,y)\right) dy\right) dx \tag{1}$$

for $f \in W^k_{\varphi,\psi}(\Omega)$. Obviously $I^{(k)}_{\varphi,\psi}$ is a convex modular; convergence in the space $W^k_{\varphi,\psi}(\Omega)$ is defined as the convergence in sense of the modular $I^{(k)}_{\varphi,\psi}$, i.e. the sequence (f_n) is convergent to f if there holds the following condition

$$I^{(k)}_{\varphi,\psi}(\lambda(f_n - f)) \to 0 \quad \text{as} \quad n \to \infty \tag{2}$$

for some $\lambda > 0$.

2. Selected properties of $L_{\varphi,\psi}(\Omega)$

Let $S(\Omega)$ be the set of all simple functions in $L(\Omega)$. We have $S(\Omega) \subset L_{\varphi,\psi}(\Omega)$.

Lemma 1. *The set $S(\Omega)$ is dense in $L_{\varphi,\psi}(\Omega)$ in the sense of the modular $I_{\varphi,\psi}$.*

Proof. Let $f \in L_{\varphi,\psi}(\Omega)$, $f \ge 0$. Thus there exists a constant $\lambda > 0$ such that $I_{\varphi.\psi}(\lambda f) < \infty$. Let (f_n) be a sequence of nonnegative simple functions increasing to f on Ω. Then $f(x,y) \ge f(x,y) - f_n(x,y)$ for arbitrary n and every $(x,y) \in \Omega$. Hence $\psi(y, \lambda f(x,y)) \ge \psi(y, \lambda(f(x,y) - f_n(x,y))) \to 0$ as $n \to \infty$ for any $\lambda > 0$ and $(x,y) \in \Omega$. Since $f \in L_{\varphi,\psi}(\Omega)$ we conclude that

$$\int_B \psi(y, \lambda f(x,y))\, dy < \infty$$

for some $\lambda > 0$ and a.e. $x \in A$. By the dominated convergence theorem we obtain

$$\int_B \psi(y, \lambda(f(x,y) - f_n(x,y)))\, dy \to 0$$

as $n \to \infty$ for a.e. $x \in A$. Using continuity of φ with respect to the second variable, we have

$$\varphi\left(x, \int_B \psi(y, \lambda(f(x,y) - f_n(x,y)))\, dy\right) \to 0 \quad \text{as} \quad n \to \infty.$$

Moreover

$$\varphi\left(x,\int_B\psi\left(y,\lambda\left(f\left(x,y\right)-f_n\left(x,y\right)\right)\right)dy\right)\leq\varphi\left(x,\int_B\psi\left(y,\lambda f\left(x,y\right)\right)dy\right)$$

and $\int_A\varphi\left(x,\int_B\psi\left(y,\lambda f\left(x,y\right)\right)dy\right)dx<\infty$ for sufficiently small $\lambda>0$. Applying the dominated convergence theorem again, we obtain $I_{\varphi,\psi}\left(\lambda\left(f_n-f\right)\right)\to 0$ as $n\to\infty$ for small $\lambda>0$. Thus (f_n) is convergent to f in the sense of the modular $I_{\varphi,\psi}$. If $f\in L_{\varphi,\psi}(\Omega)$ is arbitrary, then we may split f into positive and negative parts and apply the above result. ■

The real function $\Phi(\cdot,:)$convex with respect to the second variable - defined on a product $I\times R$, where I is a bounded interval, $I\subset R$ - satisfies the condition $(\star)$, if there exist constants $k>1$ and $\sigma>0$ such that

$$\Phi\left(t-v,u\right)\leq\Phi\left(t,ku\right)+g\left(t,v\right)$$

for $t\in I$, $u\in R$ and $\mid v\mid<\sigma$, where

$$h\left(v\right)=\int_I g\left(t,v\right)dt\to 0$$

as $v\to\infty$ and $H=sup_{|v|<\sigma}h\left(v\right)<\infty$, ([5]). The condition $(\star)$ is satisfied, for example, by any function Φ that does not depend on the parameter t; nontrivial examples are given e.g. in [1].

We define the family $\left(\tau_{(s,t)}\right)_{(s,t)\in R^2}$ of translation operators as follows

$$\tau_{(s,t)}f\left(x,y\right)=\begin{cases}f\left(x+s,y+t\right) & \text{for}\quad (x,y)\in[A\cap(A-s)]\times[B\cap(B-t)]\\ 0 & \text{elsewhere in }\Omega,\end{cases}$$

for every $f\in L_{\varphi,\psi}(\Omega)$.

Let $\mathcal{V}$ be a filter of neighborhoods of zero in R^2. The family $\left(\tau_{(s,t)}\right)_{(s,t)\in R^2}$ of translation operators will be called $\mathcal{V}$-bounded, if there exists a number $K>1$ and a function $G:\Omega\to[0,+\infty)$ such that $G(s,t)\to 0$ with respect to $\mathcal{V}$, and for every $f\in L_{\varphi,\psi}(\Omega)$ there is a set $V\in\mathcal{V}$ for which

$$I_{\varphi,\psi}\left(\tau_{(s,t)}f\right)\leq I_{\varphi,\psi}\left(Kf\right)+G\left(s,t\right)$$

for all $(s,t)\in V$.

Lemma 2. *Let φ and ψ satisfy the conditions 1-3 and $(\star)$. Then the family $\left(\tau_{(s,t)}\right)_{(s,t)\in R^2}$ of translation operators is $\mathcal{V}$-bounded.*

Proof. Applying the condition $(\star)$, we obtain

$$\begin{aligned} I_{\varphi,\psi}\left(\tau_{(s,t)}f\right) &= \int_{A\cap(A-s)}\varphi\left(x,\int_{B\cap(B-t)}\psi\left(y,f\left(x+s,y+t\right)\right)dy\right)dx \\ &\le \int_{(A+s)\cap A}\varphi\left(x-s,\int_B\psi\left(y,k_1f\left(x,y\right)\right)dy+\int_B g_1\left(y,t\right)dy\right)dx \\ &\le \int_A\varphi\left(x,k_2\int_B\psi\left(y,k_1f\left(x,y\right)\right)dy\right. \\ &+\left. k_2\int_B g_1\left(y,t\right)dy\right)dx+\int_A g_2\left(x,s\right)dx \\ &\le \int_A\varphi\left(x,2k_2\int_B\psi\left(y,k_1f\left(x,y\right)\right)dy\right)dx \\ &+\int_A\varphi\left(x,2k_2\int_B g_1\left(y,t\right)dy\right)dx+\int_A g_2\left(x,s\right)dx. \end{aligned}$$

Let us denote $\widetilde{h}_1(t)=\int_A\varphi(x,2k_2\int_B g_1(y,t)dy)dx$ and $h_2(s)=\int_A g_2(x,s)dx$. Thus, by $(\star)$, we have $h_2(s)\to 0$ as $s\to 0$. Moreover, the condition $(\star)$ for the function ψ yields $h_1(t)=\int_B g_1(y,t)dy\to 0$ as $t\to 0$ and $H_1=\sup_{|t|<\sigma}h_1(t)<\infty$. Hence

$$\varphi\left(x,k_3h_1\left(t\right)\right)\le\varphi\left(x,k_3H_1\right)\quad\text{and}\quad\int_A\varphi\left(x,k_3H_1\right)dx<\infty.$$

Applying dominated convergence theorem we have $\widetilde{h}_1\left(t\right)\to 0$ as $t\to 0$. Thus writing $G\left(s,t\right)=\widetilde{h}_1\left(t\right)+h_2\left(s\right)$, we have $G\left(s,t\right)\to 0$ with respect to the filter $\mathcal{V}$ and

$$I_{\varphi,\psi}\left(\tau_{(s,t)}f\right)\le I_{\varphi,\psi}\left(Kf\right)+G\left(s,t\right)$$

for $(s,t)\in R^2$ and $K\ge 1$. ■

Lemma 3. *Let φ and ψ satisfy the conditions 1-3 and $(\star)$. Then $\tau_{(s,t)}f\to f$ in the sense of the modular $I_{\varphi,\psi}$ with respect to the filter $\mathcal{V}$ for every characteristic function f of a measurable subset of Ω.*

Proof. Let $C\subset\Omega$ and $f=\chi_C$ be the characteristic function of C. We denote $C_{(s,t)}=C\dot{-}\left(C-(s,t)\right)$ for any $(s,t)\in R^2$. Then

$$I_{\varphi,\psi}\left(\tau_{(s,t)}f-f\right)=\int_A\varphi\left(x,\int_B\psi\left(y,\chi_{C_{(s,t)}}\left(x,y\right)\right)dy\right)dx.$$

By Jensen's inequality, we have

$$\begin{aligned} I_{\varphi,\psi}\left(\tau_{(s,t)}f-f\right) &\le \frac{1}{L}\int_A\int_B\varphi\left(x,L\chi_{C_{(s,t)}}\left(x,y\right)\right)\psi\left(y,1\right)dxdy \\ &= \frac{1}{L}\iint_{C_{(s,t)}}\varphi\left(x,L\right)\psi\left(y,1\right)dxdy<\infty, \end{aligned}$$

where $L = \int_B \psi(y,1)dy$. Since $|\, C_{(s,t)} \,| \to 0$ as $(s,t) \to 0$, then $I_{\varphi,\psi}(\tau_{(s,t)}\chi_C - \chi_C) \to 0$ in the sense of the filter $\mathcal{V}$. ■

Theorem 1. *Let φ and ψ satisfy the conditions 1-3 and $(\star)$. Then $\tau_{(s,t)}f \to f$ in the sense of the modular $I_{\varphi,\psi}$ with respect to the filter $\mathcal{V}$ for every $f \in L_{\varphi,\psi}(\Omega)$.*

Proof. Let $f \in L_{\varphi,\psi}(\Omega)$. Then, by Lemma 1, there exists a sequence (f_n) of functions $f_n \in S(\Omega)$ such that $f_n \to f$ in the sense of $I_{\varphi,\psi}$. Thus there exists a number $\lambda_0 > 0$ such that $I_{\varphi,\psi}(\lambda_0 (f_n - f)) \to 0$ as $n \to \infty$. Applying Lemma 2, we obtain for $0 < \lambda \leq \lambda_0$ and an arbitrary positive integer n

$$\begin{aligned} I_{\varphi,\psi}\left(\frac{\lambda}{3K}\left(\tau_{(s,t)}f - f\right)\right) &\leq \frac{1}{3} I_{\varphi,\psi}\left(\frac{\lambda_0}{K}\tau_{(s,t)}(f - f_n)\right) \\ &+ \frac{1}{3} I_{\varphi,\psi}\left(\frac{\lambda_0}{K}\left(\tau_{(s,t)}f_n - f_n\right)\right) + \frac{1}{3} I_{\varphi,\psi}\left(\frac{\lambda_0}{K}(f_n - f)\right) \\ &\leq I_{\varphi,\psi}(\lambda_0 (f - f_n)) + I_{\varphi,\psi}\left(\lambda_0\left(\tau_{(s,t)}f_n - f_n\right)\right) + G(s,t). \end{aligned}$$

Let $\varepsilon > 0$ be given. Now, we choose n_0 such that $I_{\varphi,\psi}(\lambda_0 (f - f_{n_o})) < \varepsilon$. Applying Lemma 3 we find sets $V_1 \in \mathcal{V}$ and $V_2 \in \mathcal{V}$such that $I_{\varphi,\psi}\left(\lambda_0\left(\tau_{(s,y)}f_{n_0} - f_{n_0}\right)\right) < \varepsilon$ for $(s,t) \in V_1$ and $G(s,t) < \varepsilon$ for $(s,t) \in V_2$. Hence

$$I_{\varphi,\psi}\left(\frac{\lambda}{3K}\left(\tau_{(s,t)}f - f\right)\right) < 3\varepsilon$$

for $(s,t) \in V_1 \cap V_2 \in \mathcal{V}$. ■

From Theorem 1 follows immediately

Corollary. *If φ and ψ satisfy the assumptions of Theorem 1, then for every $f \in L_{\varphi,\psi}(\Omega)$ there exists a number $c > 0$ such that*

$$\sup_{\substack{|\, s \,| \,<\, \sigma \\ |\, t \,| \,<\, \sigma}} \int_A \varphi\left(x, \int_B \psi(y, c(f(x+s, y+t) - f(x,y)))\, dy\right) dx \to 0$$

as $\sigma \to 0$.

3. Density of $C_0^\infty(\Omega)$ in Sobolev space $W^k_{\varphi,\psi}$

Let ρ be a nonnegative, real-valued function belonging to $C_0^\infty(R^2)$ and having the following properties:

1. $\rho(x,y) = 0$ if $|\, (x,y) \,| \geq 1$

2. $\iint_{R^2} \rho(x,y)\,dxdy = 1$.

If $\sigma > 0$, the function $\rho_\sigma(x,y) = \sigma^{-2}\rho\left(\frac{x}{\sigma}, \frac{y}{\sigma}\right)$ belongs to $C_0^\infty(R^2)$ also and $\iint_{R^2} \rho_\sigma(x,y)\,dxdy = 1$. The convolution

$$f_\sigma(x,y) = (\rho_\sigma \star f)(x,y) = \iint_{R^2} \rho_\sigma(x-s, y-t)\, f(s,t)\, dsdt$$

is the regularization of f for which the right side makes sense.

Lemma 4. *Let $f \in W^k_{\varphi,\psi}(\Omega)$ and φ and ψ satisfy the conditions 1-3 and $(\star)$. Then $\rho_\sigma \star f \to f$ in the sense of the modular $I^{(k)}_{\varphi,\psi}$ with respect to the filter $\mathcal{V}$ in $W^k_{\varphi,\psi}(\Omega')$ if $\Omega' = A' \times B'$, where A' and B' are open intervals such that $\overline{A'} \subset A$, $\overline{B'} \subset B$.*

Proof. Let $f \in W^k_{\varphi,\psi}(\Omega)$ and $\delta < dist(\Omega', \partial\Omega)$. For $(x,y) \in \Omega'$ and α such that $|\alpha| \le k$ we have

$$\begin{aligned} D^\alpha f_\sigma(x,y) &= \iint_{R^2} D^\alpha_{(x,y)} \rho_\sigma(x-s, y-t)\, \widetilde{f}(s,t)\, dsdt \\ &= (-1)^{|\alpha|} \iint_{R^2} D^\alpha_{(s,t)} \rho_\sigma(x-s, y-t)\, \widetilde{f}(s,t)\, dsdt \\ &= \iint_{R^2} \rho_\sigma(x-s, y-t)\, D^\alpha_{(s,t)} \widetilde{f}(s,t)\, dsdt \\ &= \iint_{R^2} \rho_\sigma(s,t)\, D^\alpha_{(x,y)} \widetilde{f}(x+s, y+t)\, dsdt, \end{aligned}$$

where $\widetilde{f}$ is the zero extension of f outside Ω. Hence, we have for $(x,y) \in \Omega'$

$$\begin{aligned} &D^\alpha f_\sigma(x,y) - D^\alpha f(x,y) \\ &\qquad = \iint_{|(s,t)|<\sigma} \rho_\sigma(s,t) \left(D^\alpha f(x+s, y+t) - D^\alpha f(x,y)\right) dsdt. \end{aligned}$$

Writing $\triangle_{(s,t)} D^\alpha f(x,y) = D^\alpha f(x+s, y+t) - D^\alpha f(x,y)$ and applying Jensen's inequality, we obtain

$$\begin{aligned} &\int_{A'} \varphi\left(x, \int_{B'} \psi\left(y, D^\alpha f_\sigma(x,y) - D^\alpha f(x,y)\right) dy\right) dx \\ &= \int_{A'} \varphi\left(x, \int_{B'} \psi\left(y, \iint_{|(s,t)|<\sigma} \rho_\sigma(s,t)\, \triangle_{(s,t)}\, D^\alpha f(x,y) dsdt\right) dy\right) dx \\ &\le \iint_{|(s,t)|<\sigma} \rho_\sigma(s,t)\, dsdt \\ &\quad \times \sup_{\substack{|s| < \sigma \\ |t| < \sigma}} \int_{A'} \varphi\left(x, \int_{B'} \left(\psi\left(\triangle_{(s,t)} D^\alpha f(x,y)\right)\right) dy\right) dx \end{aligned}$$

for any $\mid \alpha \mid \leq k$. By Corollary, for any $f \in W^k_{\varphi,\psi}(\Omega)$ there exists a set $V \in \mathcal{V}$ such that $I^{(k)}_{\varphi,\psi}(c(f_\sigma - f)) < \varepsilon$ for $(s,t) \in V$ and $(x,y) \in \Omega'$. ■

From Lemma 4 follows immediately

Theorem 2. *Let φ and ψ satisfy the conditions 1-3 and $(\star)$. If $\overline{\Omega'} \subset \Omega$, then $C_0^\infty(\Omega)$ is dense in $W^k_{\varphi,\psi}(\Omega')$ in the sense of $I^{(k)}_{\varphi,\psi}$ with respect to the filter $\mathcal{V}$.*

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