F A S C I C U L I M A T H E M A T I C I

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A COUNTERPART OF THE TAYLOR THEOREM AND MEANS

Abstract. For an n-times differentiable real function f defined in an a real interval I, some properties of the Taylor remainder means $T_n^{[f]}$ are considered. It is proved that $T_n^{[f]}$ is symmetric iff $n = 1$, and a conjecture concerning the equality $T_n^{[g]} = T_n^{[f]}$ is formulated. The main result says that if $f^{(n)}$ is one-to-one, there exists a unique mean $M_n^{[f]} : f^{(n)}(I) \times f^{(n)}(I) \to f^{(n)}(I)$ such that, for all $x, y \in I$,

$$f(y) = \sum_{k=0}^{n-1} \frac{f^{(k)}(x)}{k!}(y-x)^k + \frac{M_n^{[f]}\left(f^{(n)}(x), f^{(n)}(y)\right)}{n!}(y-x)^n .$$

The connection between $T_n^{[f]}$ and $M_n^{[f]}$ is given. A functional equation related to $M_2^{[f]}$ is derived and an open problem is posed.

Key words: Taylor theorem, mean, Taylor remainder mean, functional equation.

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1. Introduction

By the Taylor theorem, if a real function f defined on an interval $I \subset \mathbb{R}$ is n-times differentiable and $f^{(n)}$, the n-th derivative of f, is one-to-one, then there exists a unique mean $T_n^{[f]} : I^2 \to I$ such that, for all $x, y \in I$,

$$f(y) = \sum_{k=0}^{n-1} \frac{f^{(k)}(x)}{k!}(y-x)^k + \frac{f^{(n)}\left(T_n^{[f]}(x,y)\right)}{n!}(y-x)^n .$$

The mean $T_n^{[f]}$, called the *Taylor reminder mean*, is continuous and strict. In the first Section we show (Theorem 1) that $T_n^{[f]}$ is symmetric if, and only if, $n = 1$ that if $T_n^{[f]}$ is the Lagrange mean. (Let us mention that A. Horwitz [3], [4], (cf. also P.S. Bullen [2], p. 409) on the basis the Taylor theorem, introduced some symmetric means.) It is known that $T_1^{[g]} = T_1^{[f]}$ iff there are

$a, b, c \in \mathbb{R}$, $a \neq 0$, such that $g(x) = af(x) + bx + c$ for all $x \in I$ (L.R. Berrone and J. Moro [1], also [5]). In Section 2 we conjecture that $T_n^{[g]} = T_n^{[f]}$ for an $n > 1$, iff there are $a \in \mathbb{R}$, $a \neq 0$, and a polynomial p of the degree n, such that

$$g(x) = af(x) + p(x), \quad x \in I.$$

Theorem 3, the main result in Section 3, a counterpart of the Taylor Theorem, reads as follows. *If a real function f is n-times differentiable in an interval I and $f^{(n)}$ is one-to-one, then $f^{(n)}(I)$ is an interval and there exists a unique strict mean $M_n^{[f]} : f^{(n)}(I) \times f^{(n)}(I) \to f^{(n)}(I)$ such that, for all $x, y \in I$,*

$$f(y) = \sum_{k=0}^{n-1} \frac{f^{(k)}(x)}{k!} (y-x)^k + \frac{M_n^{[f]}\left(f^{(n)}(x), f^{(n)}(y)\right)}{n!} (y-x)^n .$$

A formula for the mean $M_n^{[f]}$ and its relation with $T_n^{[f]}$ are also given. Taking $n = 1$ in Theorem 3 we obtain the main result of [7] (cf. also [8]). An application of Theorem 3 for $n = 2$ leads to the equality

$$\frac{f'(x) - f'(y)}{x-y} = \frac{1}{2}\left[M_2^{[f]}\left(f''(x), f''(y)\right) + M_2^{[f]}\left(f''(y), f''(x)\right)\right]$$

for all $x, y \in I$, $x \neq y$. For $f(x) = x^3$, setting $g := f'$, $h := g''$ we obtain

$$\frac{g(x) - g(y)}{x-y} = \frac{1}{2}\left[h\left(\frac{2x+y}{3}\right) + h\left(\frac{2y+x}{3}\right)\right], x, y \in I, x \neq y.$$

It is an open problem to determine all functions $g, h : I \to \mathbb{R}$ satisfying this functional equation.

2. The Taylor remainder means

Recall that a function $M : I^2 \to I$ is called a *mean in* a nontrivial interval $I \subset \mathbb{R}$ if it is *internal*, that is if

$$\min(x, y) \leq M(x, y) \leq \max(x, y) \text{ for all } x, y \in I.$$

The mean M is called *strict* if these inequalities are sharp for all $x, y \in I$, $x \neq y$, and *symmetric* if $M(x, y) = M(y, x)$ for all $x, y \in I$.

The Lagrange mean-value theorem can be formulated in the following way. *If a function $f : I \to \mathbb{R}$ is a differentiable, then there exists a strict symmetric mean $L : I^2 \to I$ such that, for all $x, y \in I, x \neq y$,*

$$\frac{f(x) - f(y)}{x-y} = f'(L(x, y)).$$

If f' is one-to-one then, obviously, $L^{[f]} := L$ is uniquely determined and is called the *Lagrange mean* generated by f.

The classical Taylor theorem can be formulated in the following way.

Theorem 1. *Let $I \subset \mathbb{R}$ be an interval and $n \in \mathbb{N}$ be fixed. If $f : I \to \mathbb{R}$ is n-times differentiable function in I, then there exists a strict mean $T_n^{[f]} : I^2 \to I$ such that, for all $x, y \in I$,*

$$f(y) = \sum_{k=0}^{n-1} \frac{f^{(k)}(x)}{k!} (y-x)^k + \frac{f^{(n)}\left(T_n^{[f]}(x,y)\right)}{n!} (y-x)^n .$$

If moreover the n-th derivative of f is one-to-one, then $T_n^{[f]}$ is uniquely determined and

$$(1)\quad T_n^{[f]}(x,y) = \left(f^{(n)}\right)^{-1} \left(n! \frac{f(y) - \sum\limits_{k=0}^{n-1} \frac{f^{(k)}(x)}{k!} (y-x)^k}{(y-x)^n} \right), \quad x, y \in I,\ x \neq y.$$

The mean $T_n^{[f]}$ is called the *Taylor remainder mean* of n-th order and the function f is called a generator of $T_n^{[f]}$. Clearly, $T_1^{[f]} = L^{[f]}$.

Remark 1. If $f : I \to \mathbb{R}$ is n-times differentiable function in the interval $I \subset \mathbb{R}$ and $f^{(n)}$, the n-th derivative of f, is one-to-one, then $f^{(n)}$is strictly monotonic and continuous (cf. [6], Remark 1) and, consequently, f is of the class C^n in I.

Theorem 2. *Let $I \subset \mathbb{R}$ be an interval and $n \in \mathbb{N}$ be fixed. Suppose that $f : I \to \mathbb{R}$ is n-times differentiable and $f^{(n)}$ is one-to-one. The mean $T_n^{[f]}$ is symmetric if, and only if, $n = 1$.*

Proof. Assume that $T_n^{[f]}(x,y) = T_n^{[f]}(y,x)$ for some $n \in \mathbb{N}$, $n \geq 2$, and for all $x, y \in I$. Then, from 1, we get

$$f(y) - \sum_{k=0}^{n-1} \frac{f^{(k)}(x)}{k!} (y-x)^k = (-1)^n \left[f(x) - \sum_{k=0}^{n-1} \frac{f^{(k)}(y)}{k!} (x-y)^k \right]$$

for all $x, y \in I$, $x \neq y$. Differentiating both sides with respect to x we obtain

$$-\sum_{k=0}^{n-1} \frac{f^{(k+1)}(x)}{k!} (y-x)^k + \sum_{k=1}^{n-1} \frac{f^{(k)}(x)}{(k-1)!} (y-x)^{k-1}$$
$$= (-1)^n \left[f'(x) - \sum_{k=1}^{n-1} \frac{f^{(k)}(y)}{(k-1)!} (x-y)^{k-1} \right]$$

for all $x, y \in I$, $x \neq y$. Differentiating both sides of this equality with respect to y we obtain

$$-\sum_{k=1}^{n-1} \frac{f^{(k+1)}(x)}{(k-1)!}(y-x)^k + \sum_{k=2}^{n-1} \frac{f^{(k)}(x)}{(k-2)!}(y-x)^{k-2}$$
$$= (-1)^n \left[-\sum_{k=1}^{n-1} \frac{f^{(k+1)}(y)}{(k-1)!}(x-y)^{k-1} + \sum_{k=2}^{n-1} \frac{f^{(k)}(y)}{(k-2)!}(x-y)^{k-2} \right],$$

and, after obvious simplification,

$$\frac{f^{(n)}(x)}{(n-2)!}(y-x)^{n-2} = (-1)^n \frac{f^{(n)}(y)}{(n-2)!}(x-y)^{n-2},$$

whence,

$$f^{(n)}(x) = f^{(n)}(y), \quad x, y \in I, \quad x \neq y,$$

which contradicts the injectivity of $f^{(n)}$. This proves that $n = 1$.

If $n = 1$, then clearly, $T_1^{[f]} = L^{[f]}$ is symmetric. ■

Example 1. For $f(x) = x^{n+1}$, $x \in (0, \infty)$, $n \geq 2$, by easy calculations, get

$$T_n^{[f]}(x, y) = \frac{nx + y}{n+1}, \quad x, y > 0.$$

Hence, taking $g(x) = \sum_{k=0}^{n+1} a_k x^k$ where $a_k \in \mathbb{R}$ for $k = 0, 1, \ldots, n+1$ and $a_{n+1} \neq 0$, in view of formula Theorem 3, we get

$$T_n^{[g]}(x, y) = \frac{nx + y}{n+1}, \quad x, y > 0.$$

Example 2. For $f = \exp$ and $n = 2$ we get

$$M_2^{[\exp]}(x, y) = \log \frac{2\left[e^y - e^x - e^x(y-x)\right]}{(y-x)^2}, \quad x, y \in \mathbb{R}.$$

3. A conjecture on the equality of Taylor remainder means and some remarks

It is natural to ask when two Taylor remainder means are equal. We pose the following

Conjecture 1. *Let $I \subset \mathbb{R}$ be an interval and $n \in \mathbb{N}$ be fixed. Suppose that $f : I \to \mathbb{R}$, $g : I \to \mathbb{R}$ are n-times differentiable, $f^{(n)}$ and $g^{(n)}$ are one-to-one. Then*

$$T_n^{[g]} = T_n^{[f]}$$

if, and only if, there are $a \in \mathbb{R}$, $a \neq 0$, and a polynomial p of the degree n, such that

$$g(x) = af(x) + p(x), \quad x \in I.$$

Remark 2. This conjecture holds true for $n = 1$ (cf. Berrone & Moro [1], also [5]).

Remark 3. Note that the "if" part of this conjecture is true.

To show it assume that $g = p + af$ where $a \in \mathbb{R}$, $a \neq 0$, and

$$p(x) = \sum_{k=0}^{n} a_k x^k, \quad x \in \mathbb{R},$$

is a polynomial of the degree n. Then

$$p(y) = \sum_{k=0}^{n} \frac{p^{(k)}(x)}{k!} (y-x)^k, \quad x, y \in \mathbb{R},$$

and, taking into account that $a_n = \frac{p^{(n)}}{n!}$ is constant, for all $x, y \in I$, we have

$$\begin{aligned}
&g(y) - \sum_{k=0}^{n-1} \frac{g^{(k)}(x)}{k!} (y-x)^k \\
&= p(y) + af(y) - \sum_{k=0}^{n-1} \frac{p^{(k)}(x) + af^{(k)}(x)}{k!} (y-x)^k \\
&= \left(p(y) - \sum_{k=0}^{n-1} \frac{p^{(k)}(x)}{k!} (y-x)^k \right) + a \left(f(y) - \sum_{k=0}^{n-1} \frac{f^{(k)}(x)}{k!} (y-x)^k \right) \\
&= a_n (y-x)^n + a \left(f(y) - \sum_{k=0}^{n-1} \frac{f^{(k)}(x)}{k!} (y-x)^k \right),
\end{aligned}$$

whence, for all $x, y \in I$, $x \neq y$,

$$\frac{g(y) - \sum_{k=0}^{n-1} \frac{g^{(k)}(x)}{k!} (y-x)^k}{(y-x)^n} = a_n + a \frac{f(y) - \sum_{k=0}^{n-1} \frac{f^{(k)}(x)}{k!} (y-x)^k}{(y-x)^n}$$

Since

$$\left(g^{(n)}\right)^{-1}(u) = \left(f^{(n)}\right)^{-1} \left(\frac{u - a_n}{a} \right),$$

applying (1), for all $x, y \in I$, $x \neq y$, we hence get

$$\begin{aligned} T_n^{[g]}(x,y) &= \left(g^{(n)}\right)^{-1}\left(n!\frac{g(y)-\sum_{k=0}^{n-1}\frac{g^{(k)}(x)}{k!}(y-x)^k}{(y-x)^n}\right) \\ &= \left(f^{(n)}\right)^{-1}\left(\frac{1}{a}\left(a_n + an!\frac{f(y)-\sum_{k=0}^{n-1}\frac{f^{(k)}(x)}{k!}(y-x)^k}{(y-x)^n} - a_n\right)\right) \\ &= \left(f^{(n)}\right)^{-1}\left(n!\frac{f(y)-\sum_{k=0}^{n-1}\frac{f^{(k)}(x)}{k!}(y-x)^k}{(y-x)^n}\right) = T_n^{[f]}(x,y). \end{aligned}$$

Since $T_n^{[g]}(x,x) = x = T_n^{[f]}(x,x)$ for all $x \in I$, the proof is completed.

Remark 4. Under the assumptions of Conjecture, assume that $T_n^{[g]} = T_n^{[f]}$ and put

$$\psi := f^{(n)} \circ \left(g^{(n)}\right)^{-1}.$$

Hence, taking into account 1 and setting

$$\begin{aligned} F(x,y) &:= n!\frac{f(y)-\sum_{k=0}^{n-1}\frac{f^{(k)}(x)}{k!}(y-x)^k}{(y-x)^n}, \\ G(x,y) &:= n!\frac{g(y)-\sum_{k=0}^{n-1}\frac{g^{(k)}(x)}{k!}(y-x)^k}{(y-x)^n} \end{aligned}$$

we get the equality

$$\psi(G(x,y)) = F(x,y), \quad x,y \in I, \quad x \neq y.$$

Here ψ is continuous and strictly monotonic and the functions F and G are n-times continuously differentiable with respect to y (cf. Remark 1). Thus to prove the conjecture it is enough to show that ψ is an affine function. In this connection let us note the following

Remark 5. Let $I \subset \mathbb{R}$ be an interval and suppose that $f : I \to \mathbb{R}$ is n-times continuously differentiable. Then for every $x \in I$, the function $\varphi : I \to \mathbb{R}$ defined by

$$\varphi(y) := \begin{cases} \frac{f(y)-f(x)}{y-x}, & y \neq x, \\ f'(x), & y = x. \end{cases}$$

is n-times differentiable in $I \setminus \{x\}$ and

$$\varphi^{(n-1)}(x) = \frac{f^{(n)}(x)}{n}.$$

Proof. Let us fix $x \in I$. By the definition of φ it is n-times differentiable in $I \backslash \{x\}$ and

$$f(y) = (y-x)\varphi(y) + f(x), \quad y \in I, \quad y \neq x,$$

Hence, applying the Leibniz formula, we get

$$f^{(n)}(y) = \sum_{k=0}^{n} \binom{n}{k} \left(\frac{d^k}{dy^k}(y-x) \right) \left(\frac{d^{n-k}}{dy^{n-k}} \varphi(y) \right) = (y-x)\varphi^{(n)}(y) + n\varphi^{(n-1)}(y)$$

for all $y \in I$, $y \neq x$. Letting $y \to x$ we obtain

$$f^{(n)}(x) = \lim_{y \to x} f^{(n)}(y) = n \lim_{y \to x} \varphi^{(n-1)}(y)$$

which implies the result. ■

4. A counterpart of Taylor's mean-value theorem

The main result reads as follow.

Theorem 3. *Let $I \subset \mathbb{R}$ be an interval and $n \in \mathbb{N}$ be fixed. If $f : I \to \mathbb{R}$ is n-times differentiable function in I, and $f^{(n)}$ is one-to-one, then $f^{(n)}(I)$ is an interval and there exists a unique strict mean $M_n^{[f]} : f^{(n)}(I) \times f^{(n)}(I) \to f^{(n)}(I)$ such that, for all $x, y \in I$,*

$$f(y) = \sum_{k=0}^{n-1} \frac{f^{(k)}(x)}{k!}(y-x)^k + \frac{M_n^{[f]}\left(f^{(n)}(x), f^{(n)}(y)\right)}{n!}(y-x)^n. \tag{2}$$

Moreover, for all $u, v \in f^{(n)}(I)$, $u \neq v$,

$$M_n^{[f]}(u,v) = n! \frac{f \circ \left(f^{(n)}\right)^{-1}(v) - \sum\limits_{k=0}^{n-1} \frac{f^{(k)} \circ \left(f^{(n)}\right)^{-1}(u)}{k!} \left(\left(f^{(n)}\right)^{-1}(v) - \left(f^{(n)}\right)^{-1}(u) \right)^k}{\left(\left(f^{(n)}\right)^{-1}(v) - \left(f^{(n)}\right)^{-1}(u) \right)^n},$$

and

$$f^{(n)}\left(T_n^{[f]}(x,y)\right) = M_n^{[f]}\left(f^{(n)}(x), f^{(n)}(y)\right), \quad x, y \in I. \tag{3}$$

Proof. The injectivity of $f^{(n)}$ and the Darboux property of the derivative imply that $f^{(n)}$ is continuous, strictly monotonic (cf. [7]) and, consequently, $f^{(n)}(I)$ is an interval. Define $M_n^{[f]} : f^{(n)}(I) \times f^{(n)}(I) \to f^{(n)}(I)$ by

$$M_n^{[f]}(u,v) := f^{(n)}\left(T_n^{[f]}\left(\left(f^{(n)}\right)^{-1}(u), \left(f^{(n)}\right)^{-1}(v) \right) \right), \quad u, v \in f^{(n)}(I).$$

Now formula (2) follows from the Taylor theorem. The uniqueness and strictness of $M_n^{[f]}$ follow form the same properties of $T_n^{[f]}$. ■

Remark 6. For $n = 1$ this result coincides with the main result of [7]

Theorem 2 and formula (3) imply

Remark 7. The mean $M_n^{[f]}$ is symmetric iff $n = 1$.

Example 3. Let $f(x) = \sum_{k=0}^{n+1} a_k x^k$ where $a_k \in \mathbb{R}$ for $k = 0, 1, \ldots, n+1$ and $a_{n+1} \neq 0$. From Example 1, applying 3, we get

$$M_n^{[f]}(u, v) = \frac{nu + v}{n + 1}, \quad u, v \in \mathbb{R}.$$

Example 4. Let $f = \exp$ and $n = 2$. From Example 2 and 3 we get

$$M_2^{[\exp]}(u, v) = 2\frac{v - u - u(\log v - \log u)}{(\log v - \log u)^2}, \quad u, v > 0.$$

Remark 8. Assume that f is twice differentiable in an interval I and that f'' is one-to-one. From Theorem 3 we have, for all $x, y \in I$,

$$f(y) = f(x) + \frac{f'(x)}{1!}(y - x) + \frac{M_2^{[f]}(f''(x), f''(y))}{2!}(y - x)^2,$$

and

$$f(x) = f(y) + \frac{f'(y)}{1!}(x - y) + \frac{M_2^{[f]}(f''(y), f''(x))}{2!}(x - y)^2.$$

Adding the respective sides of these equalities we get

$$\frac{f'(x) - f'(y)}{x - y} = \frac{1}{2}\left[M_2^{[f]}\left(f''(x), f''(y)\right) + M_2^{[f]}\left(f''(y), f''(x)\right)\right].$$

$x, y \in I$, $x \neq y$. Taking $f(x) = x^3$ and setting $g := f'$, $h := g''$ we hence get

$$\frac{g(x) - g(y)}{x - y} = \frac{1}{2}\left[h\left(\frac{2x + y}{3}\right) + h\left(\frac{2y + x}{3}\right)\right], \quad x, y \in I, \quad x \neq y. \tag{4}$$

In this connection we pose the following

Problem. *Find all functions $g, h : I \to \mathbb{R}$ satisfying equation 4.*

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