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APPROXIMATION OF FIXED POINTS OF SOME CLASSES OF NONLINEAR MAPPINGS

ABSTRACT. We introduce a new class of nonlinear mappings, the class of generalized strongly successively Φ -hemicontractive mappings in the intermediate sense and prove the convergence of Mann type iterative scheme with errors to their fixed points. This class of nonlinear mappings is more general than those defined by several authors. In particular, the class of generalized strongly successively Φ -hemicontractive mappings in the intermediate sense introduced in this study is more general than the class defined by Liu *et al.* [Z. Liu, J. K. Kim and K. H. Kim, Convergence theorems and stability problems of the modified Ishikawa iterative sequences for strictly successively hemicontractive mappings, Bull. Korean Math. Soc. 39 (2002), No. 3, pp. 455-469].

KEY WORDS: generalized strongly successively Φ -hemicontractive mappings in the intermediate sense, Banach spaces, Mann type iterative scheme with errors, strong convergence, unique fixed point.

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1. Introduction

Let E be an arbitrary real normed linear space with dual E^* . We denote by J the normalized duality mapping from E into 2^{E^*} defined by

$$(1) \quad J(x) := \{f^* \in E^* : \langle x, f^* \rangle = \|x\|^2 = \|f^*\|^2\},$$

where $\langle \cdot, \cdot \rangle$ denotes the generalized duality pairing.

In the sequel, we give the following definitions which will be useful in this study

Definition 1. Let C be a nonempty subset of real normed linear space E . A mapping $T : C \rightarrow E$ is said to be

(a) *generalized Φ -pseudocontractive [1]* if for all $x, y \in C$, there exist strictly increasing function $\Phi : [0, \infty) \rightarrow [0, \infty)$ with $\Phi(0) = 0$ and $j(x - y) \in J(x - y)$ satisfying

$$\langle Tx - Ty, j(x - y) \rangle \leq \|x - y\|^2 - \Phi(\|x - y\|)$$

(b) *generalized Φ -hemicontractive [7] if $F(T) := \{x \in C : x = Tx\} \neq \emptyset$ and for all $x \in C$ and $x^* \in F(T)$, there exist strictly increasing function $\Phi : [0, \infty) \rightarrow [0, \infty)$ with $\Phi(0) = 0$ and $j(x - x^*) \in J(x - x^*)$ satisfying*

$$\langle Tx - x^*, j(x - x^*) \rangle \leq \|x - x^*\|^2 - \Phi(\|x - x^*\|).$$

Let E be a real normed linear space, C be a nonempty subset of E and $T : C \rightarrow E$ be a uniformly continuous generalized Φ -hemicontractive mapping, Chidume and Chidume [7] proved that T has at most one fixed point in C .

(c) *strongly successively pseudo-contractive [11] if for all $x, y \in E$, there exists $j(x - y) \in J(x - y)$ and a constant $k \in (0, 1)$ such that*

$$(2) \quad \langle T^n x - T^n y, j(x - y) \rangle \leq (1 - k)\|x - y\|^2;$$

(d) *strongly successively ϕ -pseudocontractive [11] if for all $x, y \in E$, there exists $j(x - y) \in J(x - y)$ and a strictly increasing function $\phi : [0, \infty) \rightarrow [0, \infty)$ with $\phi(0) = 0$ such that*

$$(3) \quad \langle T^n x - T^n y, j(x - y) \rangle \leq \|x - y\|^2 - \phi(\|x - y\|)\|x - y\|.$$

The class of strongly successively ϕ -pseudocontractive maps includes the class of strongly successively pseudocontractive mappings by setting $\phi(s) = ks$ for all $s \in [0, \infty)$.

(e) *generalized strongly successively Φ -pseudocontractive [11] if for all $x, y \in E$, there exists $j(x - y) \in J(x - y)$ and a strictly increasing function $\Phi : [0, \infty) \rightarrow [0, \infty)$ with $\Phi(0) = 0$ such that*

$$(4) \quad \langle T^n x - T^n y, j(x - y) \rangle \leq \|x - y\|^2 - \Phi(\|x - y\|).$$

By setting $n = 1$, then the class of generalized strongly successively Φ -pseudocontractive maps reduces to the class of generalized strongly Φ -pseudocontractive mappings. Moreover, the class of generalized strongly (successively) Φ -pseudocontractive mappings includes the class of strongly (successively) ϕ -pseudocontractive mappings by setting $\Phi(s) = s\phi(s)$ for all $s \in [0, \infty)$.

(f) *T is said to be strictly hemicontractive [8] if $F(T) \neq \emptyset$ and if there exists $t > 1$ such that for all $x \in C$ and $p \in F(T)$, there exists $j(x - p) \in J(x - p)$ satisfying*

$$(5) \quad \operatorname{Re}\langle Tx - p, j(x - p) \rangle \leq \frac{1}{t}\|x - p\|^2.$$

(g) *T is said to be strictly successively hemicontractive [13] if $F(T) \neq \emptyset$ and if there exists $t > 1$ and $n_0 \in \mathbb{N}$ such that for any $x \in C$ and $p \in F(T)$, there exists $j(x - p) \in J(x - p)$ satisfying*

$$(6) \quad \operatorname{Re}\langle T^n x - p, j(x - p) \rangle \leq \frac{1}{t}\|x - p\|^2, \quad n \geq n_0.$$

The class of strictly successively hemicontractive mappings was introduced in 2002 by Liu *et al.* [13]. Observe that if $T^n = T$ for all $n \in \mathbb{N}$, then the class of strictly successively hemicontractive maps is reduced to the class of strictly hemicontractive maps. Moreover, Liu *et al.* [13] provided an example to show that the class of strictly hemicontractive maps is a subset of the class of strictly successively hemicontractive maps.

The map $T : C \rightarrow C$ is said to be asymptotically nonexpansive mappings in the intermediate sense [2] if it is continuous and the following inequality holds:

$$(7) \quad \limsup_{n \rightarrow \infty} \sup_{x, y \in C} (\|T^n x - T^n y\| - \|x - y\|) \leq 0.$$

Observe that if we define

$$(8) \quad \xi_n = \max \left\{ 0, \sup_{x, y \in C} (\|T^n x - T^n y\| - \|x - y\|) \right\},$$

then $\xi_n \rightarrow 0$ as $n \rightarrow \infty$. It follows that (7) is reduced to

$$(9) \quad \|T^n x - T^n y\| \leq \|x - y\| + \xi_n, \quad \forall n \geq 1, \quad \forall x, y \in C.$$

In 1993, Bruck *et al.* [2] introduced the class of *asymptotically nonexpansive mappings in the intermediate sense*.

Sahu *et al.* [22] in 2009 introduced the class of *asymptotically strict pseudocontractive mappings in the intermediate sense* as follows

Let C be a nonempty subset of a Hilbert space H . A mapping $T : C \rightarrow C$ will be called an asymptotically k -strict pseudocontractive mappings in the intermediate sense with sequence $\{\gamma_n\}$ if there exists a constant $k \in [0, 1)$ and a sequence $\{\gamma_n\}$ in $[0, \infty)$ with $\lim_{n \rightarrow \infty} \gamma_n = 0$ such that

$$(10) \quad \limsup_{n \rightarrow \infty} \sup_{x, y \in C} (\|T^n x - T^n y\|^2 - (1 + \gamma_n)\|x - y\|^2 - k\|x - T^n x - (y - T^n y)\|^2) \leq 0.$$

Assume that

$$(11) \quad c_n := \max \left\{ 0, \sup_{x, y \in C} (\|T^n x - T^n y\|^2 - (1 + \gamma_n)\|x - y\|^2 - k\|x - T^n x - (y - T^n y)\|^2) \right\}.$$

Then $c_n \geq 0$ for all $n \in \mathbb{N}$, $c_n \rightarrow 0$ as $n \rightarrow \infty$ and (10) reduces to the relation

$$(12) \quad \|T^n x - T^n y\|^2 \leq (1 + \gamma_n)\|x - y\|^2 + k\|x - T^n x - (y - T^n y)\|^2 + c_n$$

for all $x, y \in C$ and $n \in \mathbb{N}$.

Qin *et al.* [19] introduced the class of asymptotically pseudocontractive mappings in the intermediate sense.

Definition 2 ([19]). A mapping $T : C \rightarrow C$ is said to be asymptotically pseudocontractive mapping in the intermediate sense if

$$(13) \quad \limsup_{n \rightarrow \infty} \sup_{x, y \in C} (\langle T^n x - T^n y, x - y \rangle - k_n \|x - y\|^2) \leq 0,$$

where $\{k_n\}$ is a sequence in $[1, \infty)$ such that $k_n \rightarrow 1$ as $n \rightarrow \infty$. Put

$$(14) \quad \nu_n = \max \left\{ 0, \sup_{x, y \in C} (\langle T^n x - T^n y, x - y \rangle - k_n \|x - y\|^2) \right\}.$$

It follows that $\nu_n \rightarrow 0$ as $n \rightarrow \infty$. Then, (14) is reduced to the following:

$$(15) \quad \langle T^n x - T^n y, x - y \rangle \leq k_n \|x - y\|^2 + \nu_n, \quad \forall n \geq 1, \quad x, y \in C.$$

They proved weak convergence theorems for this class of nonlinear mappings. They also established some strong convergence results without any compact assumption by considering the hybrid projection methods. Zegeye *et al.* [23] in 2011 obtained some strong convergence results of the Ishikawa type iterative scheme for the class of asymptotically pseudocontractive mappings in the intermediate sense without resorting to the hybrid method which was the main tool of Qin *et al.* [19]. Olaleru and Okeke [17] in 2012 established a strong convergence of Noor type scheme for a uniformly L -Lipschitzian and asymptotically pseudocontractive mappings in the intermediate sense without assuming any form of compactness. The results established in [17] improves the results of Olaleru and Mogbademu [16], Zegeye *et al.* [23] and several others in literature.

Numerous convergence results have been established for approximating fixed points of Lipschitz type pseudocontractive type nonlinear mappings (see, e.g. Chidume and Chidume [7], Chidume and Chidume [8], Gu [10], Moore and Nnoli [14] and the references therein). The most general result for generalized Φ -hemiccontractive mappings without uniform continuity was established by Chidume and Chidume [7] in a uniformly smooth Banach space. Ofoedu [15] proved strong convergence results for uniformly L -Lipschitzian asymptotically pseudocontractive maps. Chang *et al.* [6] improved the results of Ofoedu [15] for uniformly L -Lipschitzian mappings. Kim *et al.* [12] introduced the class of asymptotically generalized Φ -pseudocontractive mappings and the class of asymptotically generalized Φ -hemiccontractive mappings. They established strong convergence theorems of the iterative sequence generated by these mappings in a general Banach space.

Motivated by the above facts, we now introduce the following class of nonlinear mappings.

Definition 3. Let C be a nonempty subset of real normed linear space E . A mapping $T : C \rightarrow C$ is said to be generalized strongly successively

Φ -hemicontractive mapping in the intermediate sense if $F(T) \neq \emptyset$ and for each $n \in \mathbb{N}$, $x \in C$ and $p \in F(T)$, there exists a strictly increasing function $\Phi : [0, \infty) \rightarrow [0, \infty)$ with $\Phi(0) = 0$ and $j(x - p) \in J(x - p)$ satisfying

$$(16) \quad \limsup_{n \rightarrow \infty} \sup_{x, p \in C \times F(T)} (\langle T^n x - p, j(x - p) \rangle - \|x - p\|^2 + \Phi(\|x - p\|)) \leq 0.$$

Put

$$(17) \quad \tau_n = \max \left\{ 0, \sup_{x, p \in C \times F(T)} (\langle T^n x - p, j(x - p) \rangle - \|x - p\|^2 + \Phi(\|x - p\|)) \right\}.$$

It follows that $\tau_n \rightarrow 0$ as $n \rightarrow \infty$. Hence (17) is reduced to the following

$$(18) \quad \langle T^n x - p, j(x - p) \rangle \leq \|x - p\|^2 + \tau_n - \Phi(\|x - p\|).$$

We remark that if $\tau_n = 0$ for all $n \in \mathbb{N}$, the class of generalized strongly successively Φ -hemicontractive in the intermediate sense is reduced to the class of generalized strongly successively Φ -hemicontractive maps.

Example 1. Let $E = \mathbb{R}^1$ and $C = [c, \infty)$, where $c > 0$ is any given constant. Define the mapping $T : C \rightarrow 2^E$ by

$$Tx = \begin{cases} [0, c], & \text{if } x = c, \\ \frac{k(x-c)^2}{1+(x-c)}, & \text{if } x > c, \end{cases}$$

where $k \in (0, 1)$. Clearly, T has a unique fixed point $p = c \in C$. Define $\Phi : [0, \infty) \rightarrow [0, \infty)$ by $\Phi(t) = \frac{t^2}{(1+t)}$. Clearly, Φ is strictly increasing and $\Phi(0) = 0$. Now, for each $x \in C$, we have

$$\begin{aligned} \langle T^n x - T^n p, j(x - p) \rangle &= \frac{k(x - c)^3}{1 + (x - c)} \\ &= k^n \left(|x - c|^2 - \frac{|x - c|^2}{1 + |x - c|} \right) \\ &\leq k^n |x - p|^2 - \Phi(|x - p|) + k^n \\ &\leq |x - p|^2 - \Phi(|x - p|) + k^n. \end{aligned}$$

This implies that T satisfies (1.19). Hence, T is a generalized strongly successively Φ -hemicontractive mapping in the intermediate sense.

Example 2. Let $E = (-\infty, \infty)$ with the usual norm, $\Phi(t) = \frac{t^2}{4}$ for each $t \in [0, \infty)$ and $a_n = 2^{-n}$ for $n \geq 0$. Take $C = [0, 1] \cup \{2\}$

$$Tx = \begin{cases} 0 & \text{if } x \in \{0, 4\}, \\ 4 & \text{if } x = 1, \\ a_n - x, & \text{if } x \in [\frac{1}{2}(a_{n+1} + a_n), a_n), \\ x - a_{n+1}, & \text{if } x \in [a_{n+1}, \frac{1}{2}(a_{n+1} + a_n)). \end{cases}$$

for each $n \geq 0$, observe that $F(T) := \{x \in C : Tx = x\} = \{0\}$ and T is not continuous at $x = 1$. It is easy to see that T is generalized strongly successively Φ -hemicontractive mapping in the intermediate sense with sequence $\tau_n = \frac{1}{n^2}$, but not generalized strongly successively Φ -hemicontractive mapping.

Let C be a nonempty subset of a normed linear space E . A mapping $T : C \rightarrow E$ is said to be *Lipschitzian* if there exists a constant $L > 0$ such that

$$(19) \quad \|Tx - Ty\| \leq L\|x - y\|$$

for all $x, y \in C$ and *generalized Lipschitzian* [12] if there exists a constant $L > 0$ such that

$$(20) \quad \|Tx - Ty\| \leq L(\|x - y\| + 1)$$

for all $x, y \in C$. A mapping $T : C \rightarrow C$ is called *uniformly L-Lipschitzian* [12] if for each $n \in \mathbb{N}$, there exists a constant $L > 0$ such that

$$(21) \quad \|T^n x - T^n y\| \leq L\|x - y\|$$

for all $x, y \in C$.

Clearly, every Lipschitzian mapping is a generalized Lipschitzian mapping. Every mapping with a bounded range is a generalized Lipschitzian mapping. The following example was given by Chang *et al.* [5] to show that the class of generalized Lipschitzian mappings properly contains the class of Lipschitzian mappings and that of mappings with bounded range.

Example 3 ([5]). Let $E = (-\infty, \infty)$ and $T : E \rightarrow E$ be defined by

$$Tx = \begin{cases} x - 1 & \text{if } x \in (-\infty, -1), \\ x - \sqrt{1 - (x + 1)^2} & \text{if } x \in [-1, 0), \\ x + \sqrt{1 - (x - 1)^2} & \text{if } x \in [0, 1], \\ x + 1 & \text{if } x \in (1, \infty). \end{cases}$$

Then T is a generalized Lipschitzian mapping which is not Lipschitzian and whose range is not bounded.

Sahu [20] introduced a new class of nonlinear mappings which is more general than the class of generalized Lipschitzian mappings and the class of uniformly L -Lipschitzian mappings.

Definition 4 ([20]). *Let C be a nonempty subset of a Banach space E and fix a sequence $\{a_n\}$ in $[0, \infty)$ with $a_n \rightarrow 0$.*

(a) *A mapping $T : C \rightarrow C$ is said to be nearly Lipschitzian with respect to $\{a_n\}$ if for each $n \in \mathbb{N}$, there exists a constant $k_n > 0$ such that*

$$(22) \quad \|T^n x - T^n y\| \leq k_n(\|x - y\| + a_n)$$

for all $x, y \in C$.

The infimum of constants k_n in (23) is called nearly Lipschitz constant and is denoted by $\eta(T^n)$.

(b) *A nearly Lipschitzian mapping T with sequence $\{(a_n, \eta(T^n))\}$ is said to be nearly uniformly L -Lipschitzian if $k_n = L$ for all $n \in \mathbb{N}$, i.e.*

$$(23) \quad \|T^n x - T^n y\| \leq L(\|x - y\| + a_n)$$

and nearly asymptotically nonexpansive if $k_n \geq 1$ for all $n \in \mathbb{N}$ with $\lim_{n \rightarrow \infty} k_n = 1$.

(c) *A mapping $T : C \rightarrow E$ will be called generalized (M, L) -Lipschitzian if there exists two constants $L, M > 0$ such that*

$$(24) \quad \|Tx - Ty\| \leq L(\|x - y\| + M)$$

for all $x, y \in C$.

Observe that the class of generalized (M, L) -Lipschitzian mappings is a generalization of the class of Lipschitzian mappings. Clearly, the class of nearly uniformly L -Lipschitzian mappings properly contains the class of generalized (M, L) -Lipschitzian mappings and the class of uniformly L -Lipschitzian mappings. We remark that every nearly asymptotically nonexpansive mapping is nearly uniformly L -Lipschitzian.

It has been shown by Sahu [20] that the class of nearly uniformly L -Lipschitzian is not necessarily continuous. Sahu [20] extended the results of Goebel and Kirk [9] to demicontinuous mappings and proved that if C is a nonempty closed convex bounded subset of a uniformly convex Banach space, then every demicontinuous nearly asymptotically nonexpansive self-mapping of C has a fixed point.

We define the modified Mann iteration with errors as follows:

$$(25) \quad u_{n+1} = (1 - \alpha_n - \gamma_n)u_n + \alpha_n T^n u_n + \gamma_n \xi_n,$$

and the modified Ishikawa iteration with errors by

$$(26) \quad \begin{cases} x_{n+1} = (1 - \alpha_n - \gamma_n)x_n + \alpha_n T^n y_n + \gamma_n \nu_n, \\ y_n = (1 - \alpha'_n - \gamma'_n)x_n + \alpha'_n T^n x_n + \gamma'_n \omega_n, \end{cases}$$

where the sequences $\{\alpha_n\}$, $\{\alpha'_n\}$, $\{\gamma_n\}$, $\{\gamma'_n\} \subseteq [0, 1]$ satisfy

$$(27) \quad \lim_{n \rightarrow \infty} \alpha_n = \lim_{n \rightarrow \infty} \alpha'_n = 0, \quad \sum_{n=1}^{\infty} \alpha_n = \infty, \quad \gamma_n = o(\alpha_n), \quad \lim_{n \rightarrow \infty} \gamma'_n = 0,$$

and the sequences $\{\xi_n\}$, $\{\nu_n\}$, $\{\omega_n\}$ are bounded.

Huang [11] in 2007 obtained the following convergence results for the class of generalized strongly successively Φ -pseudocontractive mappings.

Theorem 1 ([11]). *Let E be a real uniformly smooth space and let $T : E \rightarrow E$ be a generalized strongly successively Φ -pseudocontractive mapping with bounded range. The sequences $\{u_n\}$ and $\{x_n\}$ are defined by (27) and (28) respectively, with $\{\alpha_n\}$, $\{\alpha'_n\}$, $\{\gamma_n\}$, $\{\gamma'_n\} \subseteq [0, 1]$ satisfying (29) and $\{\xi_n\}$, $\{\nu_n\}$, $\{\omega_n\}$ being bounded. Then for $u_1, x_1 \in E$, the following two assertions are equivalent:*

- (i) *modified Mann iteration with errors (27) converges to the fixed point $x^* \in F(T)$;*
- (ii) *modified Ishikawa iteration with errors (28) converges to the fixed point $x^* \in F(T)$.*

It is our purpose in this study to use the concept of nearly uniformly L -Lipschitzian (not necessarily continuous) mappings to prove a strong convergence result for the class of *generalized strongly successively Φ -hemicontractive mappings in the intermediate sense* in a general Banach space. Our results is an improvement of several other results in literature.

The following Lemmas will be useful in this study

Lemma 1 ([4]). *Let E be a Banach space. Then for each $x, y \in E$, there exists $j(x + y) \in J(x + y)$ such that*

$$\|x + y\|^2 \leq \|x\|^2 + 2\langle y, j(x + y) \rangle.$$

Lemma 2 ([18]). *Let $\{\delta_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ be three sequences of nonnegative numbers such that*

$$\delta_{n+1} \leq (1 + \beta_n)\delta_n + \gamma_n$$

for all $n \in \mathbb{N}$. If $\sum_{n=1}^{\infty} \beta_n < \infty$ and $\sum_{n=1}^{\infty} \gamma_n < \infty$, then $\lim_{n \rightarrow \infty} \delta_n$ exists.

Lemma 3 ([14]). *Let $\{\theta_n\}$ be a sequence of nonnegative real numbers and $\{\lambda_n\}$ a real sequence in $[0, 1]$ such that $\sum_{n=1}^\infty \lambda_n = \infty$. If there exists a strictly increasing function $\phi : [0, \infty) \rightarrow [0, \infty)$ with $\phi(0) = 0$ such that*

$$\theta_{n+1}^2 \leq \theta_n^2 - \lambda_n \phi(\theta_{n+1}) + \sigma_n$$

for all $n \geq n_0$, where n_0 is some nonnegative integer and $\{\sigma_n\}$ is a sequence of nonnegative numbers such that $\sigma_n = o(\lambda_n)$, then $\lim_{n \rightarrow \infty} \theta_n = 0$.

Lemma 4 ([12]). *Let $\{\delta_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$ and $\{\sigma_n\}$ be four sequences of nonnegative numbers such that*

$$\delta_{n+1}^2 \leq (1 + \beta_n)\delta_n^2 + \gamma_n(\delta_n + \sigma_n)^2$$

for all $n \in \mathbb{N}$. If $\sum_{n=1}^\infty \beta_n < \infty$, $\sum_{n=1}^\infty \gamma_n < \infty$ and $\{\sigma_n\}$ is bounded, then $\lim_{n \rightarrow \infty} \delta_n$ exists.

2. Main results

We prove the following Lemma which will be needed in this study.

Lemma 5. *Let $\{\delta_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$, $\{\sigma_n\}$ and $\{\rho_n\}$ be five sequences of nonnegative numbers such that*

$$(28) \quad \delta_{n+1}^2 \leq (1 + \beta_n)\delta_n^2 + \gamma_n(\delta_n + \sigma_n)^2 + \rho_n^2$$

for all $n \in \mathbb{N}$. If $\sum_{n=1}^\infty \beta_n < \infty$, $\sum_{n=1}^\infty \gamma_n < \infty$, $\sum_{n=1}^\infty \rho_n < \infty$ and $\{\sigma_n\}$ is bounded, then $\lim_{n \rightarrow \infty} \delta_n$ exists.

Proof. Using (30), we obtain

$$(29) \quad \begin{aligned} \delta_{n+1}^2 &\leq (1 + \beta_n)\delta_n^2 + \gamma_n(\delta_n + \sigma_n)^2 + \rho_n^2 \\ &\leq (1 + \beta_n)\delta_n^2 + 2\gamma_n(\delta_n^2 + \sigma_n^2) + \rho_n^2 \\ &\leq (1 + \beta_n + 2\gamma_n)\delta_n^2 + 2\gamma_n\sigma_n^2 + \rho_n^2. \end{aligned}$$

Since $\{\sigma_n\}$ is bounded and $\sum_{n=1}^\infty \rho_n < \infty$, then by Lemma 2, it follows that $\lim_{n \rightarrow \infty} \delta_n$ exists. The proof of Lemma 5 is completed. ■

Theorem 2. *Let C be a nonempty convex subset of a real Banach space E and $T : C \rightarrow C$ a nearly uniformly L -Lipschitzian mapping with sequence $\{a_n\}$ and generalized strongly successively Φ -hemicontractive mapping in the intermediate sense with sequence $\{\tau_n\}$ as defined in (19) and $F(T) \neq \emptyset$. Let $\{x_n\}$ be the sequence in E generated from arbitrary $x_1 \in C$ defined by*

$$(30) \quad x_{n+1} = (1 - \alpha_n - \delta_n)x_n + \alpha_n T^n x_n + \delta_n u_n, \quad n \in \mathbb{N},$$

where $\{\alpha_n\}$, $\{\delta_n\}$ are sequences in $[0,1]$ and $\{u_n\}$ is a bounded sequence in E . Assume that the following conditions are satisfied:

- (i) $\alpha_n + \delta_n < 1$, $\sum_{n=1}^{\infty} \alpha_n = \infty$,
- (ii) $\left\{ \frac{\delta_n(1+\alpha_n L)}{\alpha_n L} \right\}$ is bounded,
- (iii) $\sum_{n=1}^{\infty} \alpha_n^2 < \infty$ and $\sum_{n=1}^{\infty} \tau_n < \infty$.

Then the sequence $\{x_n\}$ in C defined by (30) converges strongly to a unique fixed point of T .

Proof. Fix $p \in F(T)$, using (25) and (30) and set

$$A_n := \alpha_n^2 L$$

and

$$B_n := 1 - 2\alpha_n - \alpha_n^2 L.$$

$$\begin{aligned}
 (31) \quad \|x_{n+1} - x_n\| &= \|(1 - \alpha_n - \delta_n)x_n + \alpha_n T^n x_n + \delta_n u_n - x_n\| \\
 &= \|(-\alpha_n - \delta_n)x_n + \alpha_n T^n x_n + \delta_n u_n\| \\
 &\leq \alpha_n \|x_n - p\| + \alpha_n \|T^n x_n - p\| \\
 &\quad + \delta_n \|x_n - p\| + \delta_n \|u_n - p\| \\
 &\leq \alpha_n \|x_n - p\| + \alpha_n \{L(\|x_n - p\| + a_n)\} \\
 &\quad + \delta_n \|x_n - p\| + \delta_n \|u_n - p\| \\
 &\leq (\alpha_n + \alpha_n L + \delta_n) \|x_n - p\| + \delta_n \|u_n - p\| + a_n L.
 \end{aligned}$$

Using (19), (24), (32), (33), Lemma 1 and for each $j(x_{n+1} - p) \in J(x_{n+1} - p)$, we obtain

$$\begin{aligned}
 (32) \quad \|x_{n+1} - p\|^2 &= \|(1 - \alpha_n - \delta_n)(x_n - p) + \alpha_n(T^n x_n - p) + \delta_n(u_n - p)\|^2 \\
 &\leq (1 - \alpha_n - \delta_n)^2 \|x_n - p\|^2 \\
 &\quad + 2\langle \alpha_n(T^n x_n - p) + \delta_n(u_n - p), j(x_{n+1} - p) \rangle \\
 &= (1 - \alpha_n - \delta_n)^2 \|x_n - p\|^2 \\
 &\quad + 2\alpha_n \langle T^n x_n - p, j(x_{n+1} - p) \rangle + 2\delta_n \langle u_n - p, j(x_{n+1} - p) \rangle \\
 &\leq (1 - \alpha_n)^2 \|x_n - p\|^2 + 2\alpha_n \{ \langle T^n x_{n+1} - p, j(x_{n+1} - p) \rangle \\
 &\quad + \langle T^n x_n - T^n x_{n+1}, j(x_{n+1} - p) \rangle \} \\
 &\quad + 2\delta_n \langle u_n - p, j(x_{n+1} - p) \rangle \\
 &\leq (1 - \alpha_n)^2 \|x_n - p\|^2 + 2\alpha_n \{ \|x_{n+1} - p\|^2 + \tau_n - \Phi(\|x_{n+1} - p\|) \\
 &\quad + L(\|x_{n+1} - x_n\| + a_n) \|x_{n+1} - p\| \} + 2\delta_n \|u_n - p\| \|x_{n+1} - p\| \\
 &\leq (1 - \alpha_n)^2 \|x_n - p\|^2 + 2\alpha_n \{ \|x_{n+1} - p\|^2 + \tau_n - \Phi(\|x_{n+1} - p\|) \\
 &\quad + L[(\alpha_n + \alpha_n L + \delta_n) \|x_n - p\| + \delta_n \|u_n - p\| + a_n L + a_n] \\
 &\quad \times \|x_{n+1} - p\| + 2\delta_n \|u_n - p\| \|x_{n+1} - p\| \}
 \end{aligned}$$

$$\begin{aligned}
 &= (1 - \alpha_n)^2 \|x_n - p\|^2 + 2\alpha_n \{ \|x_{n+1} - p\|^2 + \tau_n - \Phi(\|x_{n+1} - p\|) \} \\
 &\quad + 2\alpha_n L [(\alpha_n + \alpha_n L + \delta_n) \|x_n - p\| + \frac{\delta_n(1 + \alpha_n L)}{\alpha_n L} \|u_n - p\| \\
 &\quad + a_n(L + 1)] \|x_{n+1} - p\| \\
 &\leq (1 - \alpha_n)^2 \|x_n - p\|^2 + 2\alpha_n \{ \|x_{n+1} - p\|^2 + \tau_n - \Phi(\|x_{n+1} - p\|) \} \\
 &\quad + \alpha_n^2 L \{ (\alpha_n + \alpha_n L + \delta_n) \|x_n - p\| \\
 &\quad + \frac{\delta_n(1 + \alpha_n L)}{\alpha_n L} \|u_n - p\| + a_n(L + 1) \}^2 + \|x_{n+1} - p\|^2 \}.
 \end{aligned}$$

From (33), we obtain

$$\begin{aligned}
 \|x_{n+1} - p\|^2 &\leq \frac{(1 - \alpha_n)^2}{B_n} \|x_n - p\|^2 + \frac{2\alpha_n \tau_n}{B_n} - \frac{2\alpha_n}{B_n} \Phi(\|x_{n+1} - p\|) \\
 &\quad + \frac{2\alpha_n^2 L}{B_n} \left[(\alpha_n + \alpha_n L + \delta_n) \|x_n - p\| \right. \\
 &\quad \quad \left. + \frac{\delta_n(1 + \alpha_n L)}{\alpha_n L} \|u_n - p\| + a_n(L + 1) \right]^2 \\
 &\leq \left(1 + \frac{A_n}{B_n} \right) \|x_n - p\|^2 + \frac{2\alpha_n \tau_n}{B_n} - \frac{2\alpha_n}{B_n} \Phi(\|x_{n+1} - p\|) \\
 &\quad + \frac{2\alpha_n^2 L}{B_n} \left[(\alpha_n + \alpha_n L + \delta_n) \|x_n - p\| \right. \\
 &\quad \quad \left. + \frac{\delta_n(1 + \alpha_n L)}{\alpha_n L} \|u_n - p\| + a_n(L + 1) \right]^2.
 \end{aligned}$$

Since $B_n := 1 - 2\alpha_n - \alpha_n^2 L \rightarrow 1$, there exists a number $n_0 \in \mathbb{N}$ such that $\frac{1}{2} < B_n \leq 1$ for each $n \geq n_0$. From (34), we have

$$\begin{aligned}
 (33) \quad \|x_{n+1} - p\|^2 &\leq (1 + 2A_n) \|x_n - p\|^2 + 4\alpha_n \tau_n - 2\alpha_n \Phi(\|x_{n+1} - p\|) \\
 &\quad + 4\alpha_n^2 L \left[(\alpha_n + \alpha_n L + \delta_n) \|x_n - p\| \right. \\
 &\quad \quad \left. + \frac{\delta_n(1 + \alpha_n L)}{\alpha_n L} \|u_n - p\| + a_n(L + 1) \right]^2.
 \end{aligned}$$

Hence,

$$\begin{aligned}
 (34) \quad \|x_{n+1} - p\|^2 &\leq (1 + 2A_n) \|x_n - p\|^2 + 4\alpha_n \tau_n \\
 &\quad + 4\alpha_n^2 L \left[(\alpha_n + \alpha_n L + \delta_n) \|x_n - p\| \right.
 \end{aligned}$$

$$+ \frac{\delta_n(1 + \alpha_n L)}{\alpha_n L} \|u_n - p\| + a_n(L + 1) \Big]^2.$$

The conditions $\sum_{n=1}^{\infty} \alpha_n^2 < \infty$ and $\sum_{n=1}^{\infty} \tau_n < \infty$ imply that $\sum_{n=1}^{\infty} A_n < \infty$. Hence, from (34) and Lemma 5, it follows that $\lim_{n \rightarrow \infty} \|x_n - p\|$ exists. So that $\{x_n\}$ is bounded. Next, we set $M_1 := \sup\{\|x_n - p\| : n \in \mathbb{N}\}$ and $M_2 := \sup\{\|u_n - p\| : n \in \mathbb{N}\}$. Using (35), we have

$$(35) \quad \|x_{n+1} - p\|^2 \leq \|x_n - p\|^2 + 4\alpha_n \tau_n - 2\alpha_n \Phi(\|x_{n+1} - p\|) \\ + 4\alpha_n^2 L \left[(\alpha_n + \alpha_n L + \delta_n) M_1 \right. \\ \left. + \frac{\delta_n(1 + \alpha_n L)}{\alpha_n L} M_2 + a_n(L + 1) \right]^2.$$

We now take $\theta_n = \|x_n - p\|$, $\lambda_n = 2\alpha_n$ and $\sigma_n = 4\alpha_n^2 L \left[(\alpha_n + \alpha_n L + \delta_n) M_1 + \frac{\delta_n(1 + \alpha_n L)}{\alpha_n L} M_2 + a_n(L + 1) \right]^2 + 4\alpha_n \tau_n + 2A_n M_1^2$, (37) reduces to

$$(36) \quad \theta_{n+1}^2 \leq \theta_n^2 - \lambda_n \phi(\theta_{n+1}) + \sigma_n.$$

Using Lemma 3, it follows that $\|x_n - p\| \rightarrow 0$. The proof of Theorem 1 is completed. \blacksquare

For the class of generalized strongly successively Φ -pseudocontractive mappings, the following results is immediate.

Corollary 1. *Let C be a nonempty convex subset of a real Banach space E and $T : C \rightarrow C$ a nearly uniformly L -Lipschitzian mapping with sequence $\{a_n\}$ and generalized strongly successively Φ -pseudocontractive mapping as defined in (4) and $F(T) \neq \emptyset$. Let $\{x_n\}$ be the sequence in E generated from arbitrary $x_1 \in C$ defined by*

$$(37) \quad x_{n+1} = (1 - \alpha_n - \delta_n)x_n + \alpha_n T^n x_n + \delta_n u_n, \quad n \in \mathbb{N},$$

where $\{\alpha_n\}$, $\{\delta_n\}$ are sequences in $[0, 1]$ and $\{u_n\}$ is a bounded sequence in E . Assume that the following conditions are satisfied:

- (i) $\alpha_n + \delta_n < 1$, $\sum_{n=1}^{\infty} \alpha_n = \infty$,
- (ii) $\left\{ \frac{\delta_n(1 + \alpha_n L)}{\alpha_n L} \right\}$ is bounded,
- (iii) $\sum_{n=1}^{\infty} \alpha_n^2 < \infty$ and $\sum_{n=1}^{\infty} \tau_n < \infty$.

Then the sequence $\{x_n\}$ in C defined by (39) converges strongly to a unique fixed point of T .

Corollary 2. Let C be a nonempty convex subset of a real Banach space E and $T : C \rightarrow C$ a nearly uniformly L -Lipschitzian mapping with sequence $\{a_n\}$ and strictly successively hemicontractive as defined in (6) and $F(T) \neq \emptyset$. Let $\{x_n\}$ be the sequence in E generated from arbitrary $x_1 \in C$ defined by

$$(38) \quad x_{n+1} = (1 - \alpha_n - \delta_n)x_n + \alpha_n T^n x_n + \delta_n u_n, \quad n \in \mathbb{N},$$

where $\{\alpha_n\}$, $\{\delta_n\}$ are sequences in $[0,1]$ and $\{u_n\}$ is a bounded sequence in E . Assume that the following conditions are satisfied:

- (i) $\alpha_n + \delta_n < 1$, $\sum_{n=1}^{\infty} \alpha_n = \infty$,
- (ii) $\left\{ \frac{\delta_n(1+\alpha_n L)}{\alpha_n L} \right\}$ is bounded,
- (iii) $\sum_{n=1}^{\infty} \alpha_n^2 < \infty$ and $\sum_{n=1}^{\infty} \tau_n < \infty$.

Then the sequence $\{x_n\}$ in C defined by (40) converges strongly to a unique fixed point of T .

Remark 1. Theorem 1 improves and generalizes the convergence results obtained by Huang [11] and several others in literature.

Example 4. Let $E = \mathbb{R}$ and $C = [0,1]$. For all $x \in C$, we define $T : C \rightarrow C$ by

$$Tx = \begin{cases} \frac{x}{2} & \text{if } x \in [0,1). \\ 0 & \text{if } x = 1. \end{cases}$$

Clearly, T is a discontinuous mapping with unique fixed point $x = 0$. Sahu and Beg [21] proved that T is not Lipschitzian, but it is nearly uniformly $\frac{1}{2}$ -Lipschitzian with sequence $\{\frac{1}{2^n}\}$. It is easy to see that T is *generalized strongly successively Φ -hemicontractive mapping in the intermediate sense* with sequences $\{\tau_n\} = \frac{1}{n^2}$ and $\alpha_n = \frac{1}{n}$. Put $\Phi(t) = \frac{t^2}{3}$ for each $t \in [0, \infty)$.

We can see that the conditions (i), (ii) and (iii) of Theorem 1 are satisfied.

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