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OPERATOR VALUED MEASURES AS MULTIPLIERS OF $L_1(I, X)$ WITH ORDER CONVOLUTION *

Abstract. Let $I = (0, \infty)$ with the usual topology and product as max multiplication. Then I becomes a locally compact topological semigroup. Let X be a Banach Space. Let $L_1(I, X)$ be the Banach space of X-valued measurable functions f such that $\int_0^\infty \|f(t)\|dt < \infty$. If $f \in L_1(I)$ and $g \in L_1(I, X)$, we define

$$f * g(s) = f(s) \int_0^s g(t)dt + g(s) \int_0^s f(t)dt.$$

It turns out that $f * g \in L_1(I, X)$ and $L_1(I, X)$ becomes an $L_1(I)$-Banach module. A bounded linear operator T on $L_1(I, X)$ is called a multiplier of $L_1(I, X)$ if $T(f * g) = f * Tg$ for all $f \in L_1(I)$ and $g \in L_1(I, X)$. We characterize the multipliers of $L_1(I, X)$ in terms of operator valued measures with point-wise finite variation and give an easy proof of some results of Tewari[12].

Key words: vector valued multiplier, operator valued Measure, order convolution.

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1. Notations and preliminaries

Throughout the paper, X denotes a separable Banach space and I denotes the interval $(0, \infty)$ and we represent the vector valued functions with capital alphabet letters, any set A as $\mathbb{A}$ and a family of sets or set of functions A by the symbol $\mathfrak{A}$. Let $M(I)$ denote the Banach space with total variation norm of all finite regular complex-valued Borel measures on I. The linear order on the interval $I = (0, \infty)$ determines a convolution on $M(I)$ and it becomes a commutative semi-simple Banach algebra with multiplication as

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order convolution defined by Lardy [7]. More specifically, if $\mu, \nu \in M(I)$, then $\mu * \nu \in M(I)$ is defined by the equations

$$\int_I f(z) d(\mu * \nu)(z) = \int_I \left[\int_I f(x.y) d\mu(x) \right] d\nu(y), \quad (f \in C_0(I)),$$

where $C_0(I)$ denotes the Banach space of continuous complex - valued functions on I with usual supremum norm($\|.\|_\infty$). The Banach subspace $L_1(I)$ of $M(I)$ consisting of the equivalence class of all Lebesgue integrable functions on I is a subalgebra of $M(I)$ with respect to order convolution and hence it is itself a commutative Banach algebra. If $f, g \in L_1(I)$, we have

$$f * g(s) = f(s) \int_0^s g(t) dt + g(s) \int_0^s f(t) dt.$$

The maximal ideal space $\hat{I}$ of $L_1(I)$ can be identified with the interval $(0, \infty]$ and the Gelfand transform f of $L_1(I)$ is defined by

$$\hat{f}(s) = \int_0^s f(t) dt \quad (0 < s \leq \infty).$$

For these and other results that may be used in the sequel, the reader is referred to [7, 11]. The algebra $L_1(I)$ is without identity, but it does have approximate identities. One such approximate identity is the sequence u_n defined by

$$u_n(s) = \begin{cases} n, & \text{if} \quad 0 < s \leq \frac{1}{n}, \\ 0, & \text{if} \quad \frac{1}{n} < s < \infty. \end{cases} \qquad n = 1, 2, \ldots$$

A bounded linear operator T on $L_1(I)$ is called a multiplier of $L_1(I)$ if $T(f * g) = f * Tg$ for all $f, g \in L_1(I)$. Johnson and Lahr [1] characterized the multipliers of $L_1(I)$. In fact, they considered the interval (a, b) in place of I, where a and b may be infinite and I may or may not include one or either of the end points. In their paper, $L_1(a, b)$ was considered as a semisimple convolution measure algebra(CMA) in the sense of Taylor [6]. Johnson and Lahr [1] had proved that the multiplier algebra $M(L_1(a, b))$ is the Banach algebra obtained by adjoining the identity multiplier to the canonical image of $L_1(a, b)$ in $M(L_1(a, b))$. Slightly earlier, Larsen [11] had characterized the multipliers of $L_1[0, 1]$ with order convolution using methods quite different. In [11], Larsen mentions that his idea can be extended to any interval. Using his techniques, in Section 2 we characterize the multiplier algebra of $L_1(I)$. Similarly, in Section 3, we extend Larsen's [11] approach to define the positive multipliers of $L_1(I)$ and in Section 4, we characterize the isometric multipliers of $L_1(I)$.

Let X be a separable Banach Space. Let $L_1(I,X)$ be the Banach space of X-valued measurable functions F such that $\int_0^\infty \|F(t)\|dt < \infty$. If $f \in L_1(I)$ and $F \in L_1(I,X)$, we define

$$f * F(s) = f(s)\int_0^s F(t)dt + F(s)\int_0^s f(t)dt.$$

It turns out that $f * F \in L_1(I,X)$ and $L_1(I,X)$ becomes an $L_1(I)$-Banach module. Let X and Y be Banach spaces. A bounded linear operator T from $L_1(I,X)$ to $L_1(I,Y)$ is called a multiplier of $L_1(I,X)$ to $L_1(I,Y)$ if $T(f * F) = f * TF$ for all $f \in L_1(I)$ and $F \in L_1(I,X)$.

The past thirty to forty years have seen major research efforts in the general direction of "vector valued multiplier operators". The memoir [3] has laid the foundation for the development of a general theory of convolution operators and vector-valued Fourier multipliers.

Tewari [12] had characterized these multipliers in terms of operator valued functions. In Section 5, using Larsen's [11] ideas and the technique of Tewari, Dutta and Vaidya [13], we characterize the multipliers of $L_1(I)$ to $L_1(I,X)$ and then multipliers of $L_1(I,X)$ to $L_1(I,Y)$. We characterize these multipliers in terms of operator valued measures with point-wise finite variation and give an easy proof of some results of Tewari [12].

In [1], Johnson and Lahr had described the multipliers of $L_1(a,b)$, where $I = (a,b)$ is an interval contained in R, a or b may be infinite and the interval I may or may not contain one or either of the end points. In the following Section 2 we extend Larsen's approach to any interval.

2. Multiplier of $L_1(I)$

Johnson and Lahr [1] had proved the following theorem. The proof of the theorem based on the ideas of Larsen [11], is quite different from [1] and discussed in detail in [10].

Theorem 1. *f $T : L^1(I) \to L^1(I)$, then the following are equivalent:*
(i) The mapping T is a multiplier of $L_1(I)$.
(ii) There exists a unique $\mu \in M(I)$ of the form $\mu = \alpha\delta + h$, $\alpha \in \mathbb{C}$, δ the identity of $M(I)$ and $h \in L_1(I)$, such that $Tf = \mu * f$ $\forall f \in L_1(I)$.

Proof. Suppose (ii) holds, then it is easy to verify that $T(f * g) = f * Tg = Tf * g$, $\forall$ f, $\forall$ $gL_1(I)$. Hence T is a multiplier and (i) holds.

Suppose T is a multiplier of $L_1(I)$. Assume that ϕ is such that $(Tf)^\wedge = \phi\hat{f}$, $f \in L_1(I)$. We have $\|Tu_n\| \leq \|T\|$, $n = 1, 2, \ldots$. Thus (Tu_n) is a norm bounded sequence in $M(I)$. By the Banach - Alaglou's Theorem and the separability of $C_0(I)$, there exists a subsequence (Tu_{n_k}) of (Tu_n) and a μ in $M(I)$ such that

$$\lim_k \langle g, Tu_{n_k}\rangle = \int_I g(y)d\mu(y), \quad (g \in C_0(I)).$$

Since T is a multiplier and (u_n) is an approximate identity in $L_1(I)$, we have

$$\lim_k T(u_{n_k} * f) = Tf.$$

Taking $g \in C_0(I)$ and $f \in L_1(I) \subseteq M(I)$, we have

$$\begin{aligned}\langle g, Tf\rangle &= \lim_k \langle g, (Tu_{n_k} * f)\rangle \\ &= \lim_k \left\{ \left\langle g, \hat{f}Tu_{n_k} \right\rangle + \langle g, f.\phi \hat{u_{n_k}}\rangle \right\}.\end{aligned}$$

The sequence $(\hat{u_n})$ converges to 1 point-wise on I and $\|\hat{u_n}\|_\infty = 1$ for each n. We have

$$\langle g, Tf\rangle = \int_I g(y)\hat{f}(y)d\mu(y) + \langle g, \phi f\rangle.$$

For $0 < s < \infty$, we observe that

$$\begin{aligned}\hat{\mu}(s) &= \int_I \chi_{[0,s]}(t)d\mu(t) = \lim_k \int_I \chi_{[0,s]}(t)Tu_{n_k}(t)dt \\ &= \lim_k (T\hat{u}_{n_k})(s),\end{aligned}$$

$$= \lim_k \phi(s)\hat{u_{n_k}}(s) = \phi(s).$$

Thus, we have

$$\langle g, Tf\rangle = \int_I g(t)\hat{f}(t)d\mu(t) + \langle g, f\hat{\mu}\rangle.$$

Since $\mu * f \in M(I)$,we have

$$\begin{aligned}\int_I g(u)d(\mu * f)(u) &= \int_I \left(\int_I g(st)f(s)ds\right) d\mu(t) \\ &= \int_I g(t)\hat{f}(t)d\mu(t) + \int_I g(t)f(t)\hat{\mu}(t)dt.\end{aligned} \tag{1}$$

Hence,

$$\int_I g(u)d(\mu * f)(u) = \langle g, Tf\rangle \quad \forall\ g \in C_0(I). \tag{2}$$

Therefore, $\mu * f \in L_1(I)$. It follows from (1) and (2) that for each $f \in L_1(I)$, the measure $\hat{f}d\mu$ on I is absolutely continuous with respect to the Lebesgue measure on I. Thus for each k there exists some $h_k \in L_1(I)$ such that $\hat{u_{n_k}} d\mu = h_k$.

By Lebesgue's Dominated Convergence Theorem we have for each $g \in L_\infty(I)$, the sequence of numbers

$$\int_I g(t)\hat{u_{n_k}}(t)d\mu(t) = \langle g, h_k\rangle$$

is a Cauchy sequence, that is, h_k is a Cauchy sequence in the weak topology on $L_1(I)$. However, $L_1(I)$ is weakly sequentially complete and so there exists some $h \in L_1(I)$ such that

$$\lim_k \langle g, h_k\rangle = \langle g, h\rangle \quad (g \in L_\infty(I)).$$

In particular, if $g \in C_0(I)$, we have

$$\begin{aligned}\int_I g(t)h(t)dt &= \lim_k \int_I g(t)h_k(t)dy \\ &= \lim_k \int_I g(t)\hat{u_n}(t)d\mu(t) = \int_I g(t)d\mu(t).\end{aligned}$$

Hence μ and h are seen to define the same measure on I. Therefore there exists some $\alpha \in I$ such that $\mu = \alpha\delta + h$, where δ is the identity of $M(I)$ and h can be considered as an element of $L_1(I)$. Hence, $\mu * f \in L_1(I)$ and $Tf = \mu * f \ \forall \ f \in L_1(I)$. To see that μ is unique, suppose $\nu \in M(I)$ such that $Tf = \nu * f, f \in L_1(I)$. Then,

$$\int_0^s d\nu(t) = \hat{\nu}(s) = \hat{\mu}(s) = \alpha + \int_0^s h(t)dt, \quad 0 < s \leq \infty$$

and $\nu(0) = \alpha = \mu(0)$. Suppose $\mu_1 = \mu - \alpha\delta$ and $\nu_1 = \nu - \alpha\delta$, we have $\hat{\mu_1}(s) = \hat{\nu_1}(s)$ i.e. $\mu_1([0, s)) = \nu_1([0, s))$ i.e. $(\mu_1 - \nu_1)([0, s)) = 0$. It can be easily seen that $\mu_1([c, d)) = \nu_1([c, d))$ for any arbitrary $[c, d)$. Therefore, μ_1 and ν_1 agree on each element of the Borel $\sigma-$algebra $\mathfrak{B}(I)$. Thus $\mu_1 = \nu_1$ i.e. $\mu = \nu$.

Similar to Larsen's approach [11], we characterize multipliers on $L_1(I)$ in terms of absolutely continuous functions on $\hat{I}$. Tewari [12] had also noted this. If T is a multiplier of $L_1(I)$ then there exists a unique μ in M(I) of the form $\mu = \alpha\delta + h, \alpha \in \mathbb{C}, h \in L_1(I)$ such that $Tf = \mu * f \ \forall f \in L_1(I)$. Then given $0 < s \leq \infty$, we have for each $f \in L(I)$,

$$(Tf)^\wedge(s) = \hat{\mu}(s)\hat{f}(s) = \left(\alpha + \hat{h}(s)\right)\hat{f}(s).$$

Define ϕ by $\phi(s) = \alpha + \hat{h}(s), 0 < s \leq \infty$ and $\phi(0) = \alpha$. Then ϕ is an absolutely continuous function ϕ on $(0, \infty]$ which is of bounded variation.

Conversely, if ϕ is an absolutely continuous function on $(0,\infty]$ which is of bounded variation, then ϕ determines a multiplier of $L_1(I)$ with order convolution. Indeed, since ϕ and $\hat{f}$ are absolutely continuous functions on $(0,\infty]$, so is $\phi\hat{f}$. Thus the derivative of $\phi\hat{f}$, $(\phi\hat{f})'$ exists almost everywhere on $(0,\infty]$. Since $\phi\hat{f}(0) = 0$ for each $f \in L_1(I)$, we conclude that there exists a $g \in L_1(I)$ such that $\hat{g} = \phi\hat{f}$ and g is almost everywhere equal to the derivative of $\phi\hat{f}$, i.e., $g = (\phi\hat{f})'$. Hence every function $\phi \in AC(0,\infty]$ which is of bounded variation defines a multiplier T of $L_1(I)$ such that $(Tf)^\wedge = \phi\hat{f}\ \forall f \in L_1(I)$. Since ϕ is differentiable almost everywhere and $\phi' \in L_1(I)$, $\lim_{t\to 0^+}\phi(t)$ exists. Let $\phi(0) = \lim_{t\to 0^+}\phi(t)$, then $Tf = \phi(0)f + (\phi\hat{f})'$.

We have $\|T\| \leq \|\mu\|$. Since μ is weak-star limit of a sequence in $M(I)$ bounded in norm by $\|T\|$, and so $\|\mu\| \leq \|T\|$ as norm closed balls in $M(I)$ are weak-star closed. By the definition of ϕ, we have $\mu = \phi(0)\delta + \phi'$. Hence

$$\|T\| = \|\mu\| = |\phi(0)| + \int_I |\phi'(t)|dt.$$

■

Remark 1. The inequality $\|\phi\|_\infty \leq \|T\| = \|\mu\|$ may be strict. For example let $\phi(s) = e^{-s^2}$ then $\|\phi\|_\infty = 1$ but $\|\mu\| = 2$ as $\int_I |\phi'(s)|ds = 1$.

Remark 2. Suppose T is a compact multiplier of $L_1(I)$. We show that $Tf = h * f\ \forall f \in L_1(I)$. Suppose $Tf = \alpha f + h * f$, where $\alpha \neq 0$. Let (u_n) is an approximate identity in $L_1(I)$. Since T is a compact operator, there exists a subsequence $Tu_{n_k} = \alpha u_{n_k} + h * u_{n_k}$ which converges. Hence $\alpha u_{n_k} = Tu_{n_k} - h * u_{n_k}$ is convergent. Since $L_1(I)$ has no identity, u_{n_k} can not converge in $L_1(I)$. Thus the assumption $\alpha \neq 0$ is wrong.

It seems that there is no compact multiplier for $L_1(I)$. However we observed the following:

Proposition 1. *Let h be any integrable function with support $(0,r]$ which is properly contained in I. If $Tf = h * f\ \forall f \in L_1(I)$, then T is non-compact.*

Proof. Let $\mathfrak{K} = \{f : f \in L_1(I), f = 0 \text{ on } (0,r]\}$. Therefore $\mathfrak{K}$ is an infinite dimensional space. Hence, there exists a sequence f_n such that $\|f_n\| \leq 1\ \forall\ n$ and $\{f_n\}$ has no convergent subsequence. If $s \in (0,r]$, we have, $\hat{f}_n(s) = 0\ \forall\ n$, Let $\int_0^r h(t)dt = c(\neq 0)$, we have,

$$h * f_n(s) = \begin{cases} 0, & \text{if } s \in (0,r], \\ cf_n(s), & \text{if } s \in (0,r]'. \end{cases}$$

Thus $h * f_n = cf_n$ has no convergent subsequence. ■

3. Positive multipliers of $L_1(I)$

In this section, we give a characterization of positive multipliers of $L_1(I)$. Larsen [11] had characterized positive multipliers of $L_1([0,1])$ with order convolution. Here we extend Larsen's approach to any interval. It was discussed in detail in [10].

Definition 1. *A multiplier T of $L_1(I)$ is said to be a positive multiplier if $Tf(x) \geq 0$ almost everywhere on I, whenever $f \in L_1(I)$ and $f(x) \geq 0$ almost everywhere.*

In the next theorem, we extend Larsen's approach [11] for a complete description of the positive multipliers on $L_1(I)$. For details we refer to [10].

Theorem 2. *Let T be a multiplier of $L_1(I)$. Then the following are equivalent:*

(i) The multiplier T is positive.

(ii) If ϕ is an absolutely continuous function on I which is of bounded variation such that $(Tf)^\wedge = \phi\hat{f}\ \forall f \in L_1(I)$, then $\phi(x) \geq 0\ \forall x \in I$ and $\phi'(x) \geq 0$ almost everywhere.

(iii) If $\mu = \alpha\delta + h, \alpha \in \mathbb{C}$ and $h \in L_1(I)$ is such that $Tf = \mu * f\ \forall f \in L_1(I)$, then $\alpha \geq 0$ and $h(x) \geq 0$ a.e.

Proof. For each n, we have

$$(Tu_n)^\wedge(s) = \phi(s)\hat{u_n}(s) = \begin{cases} n\phi(s)s, & \text{if } 0 < s \leq \frac{1}{n} \\ \phi(s), & \text{if } \frac{1}{n} < s \leq \infty. \end{cases}$$

Since T is positive, it follows that $\phi(s) \geq 0\ \forall\ s \in (0,\infty]$. Since ϕ is continuous on $\hat{I}$ and $\phi(0) = \lim_{t\to 0^+}\phi(t)$, thus $\phi(0) \geq 0$. Now for almost every $s \in I$, if n is chosen so that $0 < \frac{1}{n} < s$, then

$$Tu_n(s) = (\phi\hat{u_n})'(s) = \phi'(s)\hat{u_n}(s) + \phi(s)u_n(s) = \phi'(s).$$

Since T is positive, we conclude that $\phi'(s) \geq 0$ almost everywhere. Thus (i) implies (ii). Since $\alpha = \phi(0)$ and $h = \phi'$ we see that (ii) implies (iii). It is easy to see (iii) implies (i). ■

Similar to Larsen's remark [11], we see that in the case of a positive multiplier, equality holds in Remark 1.

Corollary 1. *Let T be a positive multiplier of $L_1(I)$ such that $(Tf)^\wedge = \phi\hat{f}\ \forall f \in L_1(I)$. Then $\|\phi\|_\infty = \|T\|$.*

Proof. As T is a positive multiplier we see that $\phi(s) \geq 0$ on I and $\phi'(s) \geq 0$ almost everywhere on I, hence $\|\phi\|_\infty = \lim_{x\to\infty} \phi(s)$. Moreover by Theorem 1, we have

$$\begin{aligned}\|T\| &= |\phi(0)| + \int_I |\phi'(t)|dt \\ &= \phi(0) + \int_I \phi'(t)dt = \lim_{s\to\infty} \phi(s).\end{aligned}$$

■

In [11] Larsen had shown that the converse of the above corollary fails even in the case of I being the closed unit interval (see Corollary 3, [11].)

4. Isometric multipliers of $L_1(I)$

For each $s \in I$, the translation operator τ_s on $L_1(I)$ is defined by $\tau_s f(t) = f(s.y)$. In [11], Larsen had shown that the translation operator is not a multiplier. It is easy to see that every multiple of the identity operator by a constant α of absolute value one, that is, $Tf = \alpha f, f \in L_1(I), |\alpha| = 1$ is an isometric multiplier of $L_1(I)$. Larsen [11] had shown that these are the only isometric multipliers of $L_1([0,1])$ with order convolution. Here we extend Larsen's result to any interval. The proof of the following theorem is based on the ideas of Larsen [11] and discussed in detail in [10] .

Lemma. *Let T be an isometric multiplier of $L_1(I)$. Let $\mu \in M(I)$ such that $Tf = \mu * f \ \forall\ f \in L_1(I)$. If $f \in L_1(I)$ then $|\mu * f(s)| = |\mu| * |f|(s)$ for almost every $s \in I$.*

Theorem 3. *Let T be an isometric multiplier of $L_1(I)$ such that $Tf = \mu * f \ \forall f \in L_1(I)$ and $(Tf)^\wedge = \phi\hat{f} \ \forall f \in L_1(I)$, then T is an isometric multiplier if and only if there exists some $\alpha \in \mathbb{C}, |\alpha| = 1$ such that $\mu = \alpha\delta$ or $\phi(s) = \alpha \ \forall s \in I$.*

Proof. The Sufficiency is obvious. Suppose T is an isometry. We shall show first that $\phi'(s) = 0$ almost everywhere on I and since ϕ is absolutely continuous, therefore it is constant. For $r \in \mathbb{R}$ such that $0 < r < \infty$, define

$$f_r(s) = \begin{cases} ie^{is}, & 0 \leq s \leq r, \\ 0, & \text{otherwise.} \end{cases}$$

And for $0 \leq s \leq r$, where $r < \infty$, we have $\hat{f}_r(s) = e^{is} - 1$. By Lemma, for almost every $s \in I$, we have $\overline{\phi(s)}\phi'(s) \geq 0$ and therefore, $\forall$ s such that $0 \leq s \leq r$, we have

$$\begin{aligned}|(\phi\hat{f}_r)'(s)|^2 &= |\phi'(s)(e^{is} - 1) + \phi(s)ie^{is}|^2 = |\phi'(s)|^2|e^{is} - 1|^2 \\ &\quad -2\mathrm{Re}\left\{\phi'(s)\overline{\phi(s)}(e^{is} - 1)i(e^{-is} - 1)\right\} + |\phi(s)|^2\end{aligned}$$

$$= 2|\phi'(s)|^2(1-\cos s) + 2\overline{\phi(s)}\phi'(s)\sin s + |\phi(s)|^2$$
$$= 4|\phi'(s)|^2(\sin\frac{s}{2})^2 + 2\overline{\phi(s)}\phi'(s)\sin s + |\phi(s)|^2.$$

Since $|f_r| = 1$, for $0 \le s \le r$, where $r < \infty$, we have

$$\begin{aligned}\left\{(|\phi||f_r|^\wedge)'(s)\right\}^2 &= \left\{|\phi'(s)|s + |\phi(s)|\right\}^2 \\ &= |\phi'(s)|^2 s^2 + 2\overline{\phi(s)}\phi'(s)s + |\phi(s)|^2.\end{aligned}$$

Since this holds $\forall\ r$ such that $r < \infty$, by the lemma, for almost every $s \in I$, we have

$$4|\phi'(s)|^2\left\{(\sin\frac{s}{2})^2 - (\frac{s}{2})^2\right\} + 2\overline{\phi(s)}\phi'(s)[\sin s - s] = 0.$$

And since $(\frac{s}{2})^2 - (\sin\frac{s}{2})^2 \ge 0$ and $s - \sin s \ge 0\ \forall\ s \in I$, it follows that $|\phi'(s)|^2 = \overline{\phi(s)}\phi'(s) = 0$ almost everywhere on I. Thus there exists some $\alpha \in \mathbb{C}$ such that $\phi(s) = \alpha\ \forall\ s \in I$. Therefore, $Tf = \alpha f\ \forall\ f \in L_1(I)$ and since $\|Tf\| = \|f\|\ \forall\ f \in L_1(I)$ we have $|\alpha| = 1$. ∎

5. Multipliers of $L_1(I, X)$

Let X be a separable Banach space and the interval $I = (0, \infty)$ be with the usual topology and max multiplication. Let $L_1(I, X)$ be the Banach space of X-valued measurable functions F such that $\int_I \|F(t)\|dt < \infty$. For integration of vector-valued set functions, we follow [3, 5]. Using such integrals, it is possible to define order convolution between various spaces of vector-valued functions and measures on I. If $f \in L_1(I)$ and $F \in L_1(I, X)$, for $s \in I$,we define

$$f * F(s) = f(s)\int_0^s F(t)dt + F(s)\int_0^s f(t)dt.$$

It turns out that $f * F \in L_1(I, X)$ and $L_1(I, X)$ becomes an $L_1(I) -$ Banach module.

We shall make use of the concept of module tensor product and its relation to multipliers (see [8]. Let $\mathbb{A}$ be a commutative Banach algebra. If $\mathbb{V}$ and $\mathbb{W}$ are $\mathbb{A}$-modules, the $\mathbb{A}$-module tensor product $\mathbb{V} \otimes_{\mathbb{A}} \mathbb{W}$ is defined to be quotient Banach space $\mathbb{V} \otimes_\gamma \mathbb{W}/\mathbb{K}$, where $\mathbb{K}$ is the closed linear subspace of the projective tensor product $\mathbb{V} \otimes_\gamma \mathbb{W}$, spanned by the elements of the form $av \otimes w - v \otimes aw$ with $a \in \mathbb{A}$, $v \in \mathbb{V}$ and $w \in \mathbb{W}$. A continuous linear transformation from $\mathbb{V}$ to $\mathbb{W}$ is called an $\mathbb{A}$ - module homomorphism if $T(a * v) = a * T(v)$ for all $a \in \mathbb{A}$ and $v \in \mathbb{V}$.

The theory of vector measures and integration lets us identify the dual of $C_0(I,X)$ with $M(I,X^*)$ where X^* is the dual of X. The identification is given by $\langle \mu, F\rangle = \int_I F d\mu$, for $F \in C_0(I,X)$ and $\mu \in M(I,X^*)$, (see([3, 9]).

The "integral" $\int_I F d\mu \in \mathbb{C}$ is defined via a continuous extension procedure from $C_c(I)\otimes X$ to $C_0(I,X)$, where for $F=\sum_{j=1}^n f_j x_j$ with $f_j \in C_c(I)$ and $x_j \in X$

$$\int_I F d\mu = \sum_{j=1}^n \int_I f_j d\,\langle x_j, \mu\rangle$$

here $\langle x_j,\mu\rangle : \mathfrak{B}(I) \to \mathbb{C}$ is the complex measure, $\mathbb{E} \to \langle x_j, \mu(\mathbb{E})\rangle$ for $\mathbb{E} \in \mathfrak{B}(I)$, (see [3]).

A bounded linear operator T on $L_1(I,X)$ to $L_1(I,X)$ is called a multiplier of $L_1(I,X)$ to $L_1(I,X)$ if $T(f*F) = f*TF$ for all $f \in L_1(I)$ and $F \in L_1(I,X)$. Tewari [12] had characterized these multipliers in terms of operator valued functions. In this section, using Larsen's [11] ideas and the technique of Tewari,Dutta and Vaidya [13], we characterize the multipliers of $L_1(I)$ to $L_1(I,X)$ and then multipliers of $L_1(I,X)$ to $L_1(I,X)$ in terms of operator valued measures with point-wise finite variation.

We know that $\{u_n\}$ is an approximate identity for $L_1(I)$. The following proposition tells us that $\{u_n\}$ acts as an approximate identity for $L_1(I,X)$ (see Proposition 3.1, [12]).

Proposition 2. *Let $\{u_n\}$ be the approximate identity of $L_1(I)$ defined earlier. Suppose $F \in L_1(I,X)$. Then*

$$\|u_n * F - F\|_1 \to 0 \quad as \quad n \to \infty.$$

Definition 2. *Let $F \in L_1(I,X)$ and for each $s \in (0,\infty]$, define*

$$\hat{F}(s) = \int_0^s F(t)dt.$$

The function $\hat{F}$ is called the Gelfand transform of F. Clearly $\hat{F}$ is absolutely continuous. Also $(\hat{F})'(s) = F(s)$ almost everywhere.

Note that $\hat{F}(s) \to 0$ as $s \to 0$. Further, if $\hat{F}(s) = 0$ for all $s \in (0,\infty]$ then $F(s) = 0$ almost everywhere.

The following proposition follows immediately from Proposition 3.2, [12].

Proposition 3. *Suppose T is a multiplier of $L_1(I)$ into $L_1(I,X)$. Then there exists an X-valued bounded continuous function Φ on $(0,\infty)$ such that $(Tf)^\wedge(s) = \Phi(s)\hat{f}(s)$ for all $s \in (0,\infty)$ and $f \in L_1(I)$.*

Using the technique of Larsen [11], we characterize the multipliers $T : L_1(I) \to L_1(I,X)$ as follows:

Theorem 4. *Let X be a Banach Space which has the Radon Nikodym property. If $T : L_1(I) \to L_1(I, X)$ is a linear map, then following are equivalent*:

(*i*) *T is a multiplier of $L_1(I)$ to $L_1(I, X)$ with the order convolution.*

(*ii*) *There exists a unique measure $\mu \in M(I, X)$ of the form $\mu = x\delta + J$, $x \in X, J \in L_1(I, X)$, δ the identity of $M(I)$ such that $Tf = \mu * f$ $\forall f \in L_1(I)$ and $\|T\| = \|\mu\|$.*

Proof. Let $\{u_n\}$ be the approximate identity for $L_1(I)$ defined earlier. Considering the natural embedding of X into its second dual X^{**}, $L_1(I, X)$ can be embedded isometrically in $M(I, X^{**})$ and since $\|Tu_n\|_1 \le \|T\|$, $\{Tu_n\}$ is a norm bounded sequence in $M(I, X^{**})$. By the Banach Alaglou's Theorem and separability of $C_0(I, X^*)$ (see [9]), there exists a subsequence $\{Tu_{n_k}\}$ and a $\mu \in M(I, X^{**})$ such that

$$\lim_k \int_I \langle L(s), Tu_{n_k}(s)\rangle\, ds = \int_I L(s) d\mu(I) \quad \forall L \in C_0(I, X^*).$$

Since T is a multiplier and $\{u_n\}$ is an approximate identity, hence by Proposition 2, we have

$$\lim_k T(u_{n_k} * f) = \lim_k u_{n_k} * Tf = Tf.$$

Let $L \in C_0(I, X^*)$. Since $Tu_{n_k} * f(s) = Tu_{n_k}(s)\hat{f}(s) + f(s)(Tu_{n_k})^\wedge(s)$, hence by Proposition 3, we have

$$\begin{aligned}\langle L, Tu_{n_k} * f\rangle &= \int_I \langle L(s), Tu_{n_k}(s)\rangle \hat{f}(s) ds \\ &\quad + \int_I \langle L(s), f(s)\Phi(s)\rangle \hat{u_{n_k}}(s) ds.\end{aligned}$$

Since $\{\hat{u_n}\}$ converges point-wise to 1, on taking limits, Lebesgue's Dominated Convergence Theorem implies that

$$\begin{aligned}\lim_k \langle L, Tu_{n_k} * f\rangle &= \lim_k \int_I \langle L(s), Tu_{n_k}(s)\rangle \hat{f}(s) ds \\ &\quad + \int_I \langle L(s), f(s)\Phi(s)\rangle ds.\end{aligned}$$

If $x^* \in X^*$, then

$$\lim_k \int_I \chi_{[0,s]}(t) \langle x^*, Tu_{n_k}(t)\rangle dt = \int_I \chi_{[0,s]}(t) d\langle x^*, \mu\rangle(t) = \langle x^*, \hat{\mu}(s)\rangle.$$

Hence

$$\begin{aligned}\langle x^*, \hat{\mu}(s)\rangle &= \lim_k \int_I \chi_{[0,s]}(t) \langle x^*, Tu_{n_k}(t)\rangle \, dt = \lim_k \left\langle x^*, T\hat{u_{n_k}}(s)\right\rangle \\ &= \lim_k \langle x^*, \Phi(s)\hat{u_{n_k}}(s)\rangle = \langle x^*, \Phi(s)\rangle .\end{aligned}$$

Hence, for each $L \in C_0(I, X^*)$, we have

$$\lim_k \langle L, Tu_{n_k} * f\rangle = \int_I L(s)\hat{f}(s)d\mu(s) + \int_I \langle L(s), f(s)\hat{\mu}(s)\rangle \, ds.$$

On the other hand, we have

$$\begin{aligned}\langle L, \mu * f\rangle &= \int_I L(u)d(\mu * f)(u) = \int_I [L(s.t)f(s)] \, d\mu(t) \\ &= \int_I L(t) \left(\int_0^t f(s)ds\right) + \int_I \left(\int_t^\infty L(s)f(s)ds\right) d\mu(t) \\ &= \int_I L(t)\hat{f}(t)d\mu(t) + \int_I L(s)f(s) \left(\int_0^s d\mu(t)\right) ds \\ &= \int_I L(t)\hat{f}(t)d\mu(t) + \int_I \langle L(t), f(t)\hat{\mu}(t)\rangle \, dt.\end{aligned}$$

Hence, we have

$$\int_I L(t)d(\mu * f)(t) = \int_I L(t)Tf(t)dt.$$

Since this holds for each $L \in C_0(I, X^*)$, we conclude that $\mu * f \in L_1(I, X)$. Thus the above expressions imply that for each $f \in L_1(I)$, the measure $\hat{f}d\mu$ on I is absolutely continuous. Therefore, by the Radon Nikodym property of X, for each k there exists some $J_k \in L_1(I, X)$ such that $\hat{u_{n_k}}d\mu = J_k$. Now suppose $L \in L_\infty(I, X^*)$. Since the sequence $\{\hat{u_{n_k}}\}$ converges to 1 point-wise on I and $\|\hat{u_{n_k}}\|_\infty = 1$, Lebesgue's Dominated Convergence Theorem tells us that the sequence of numbers

$$\int_I L(t)\hat{u_{n_k}}d\mu(t) = \int_I \langle L(t), J_k(t)\rangle \, dt$$

is a Cauchy sequence, i.e. $\{J_k\}$ is a Cauchy sequence in the weak topology on $L_1(I, X)$. Since $L_1(I, X)$ is weakly sequentially complete there exists some $J \in L_1(I, X)$ such that

$$\lim_k \int_I \langle L(t), J_k(t)\rangle \, dt = \int_I \langle L(t), J(t)\rangle \, dt \quad \forall L \in L_\infty(I, X^*).$$

In particular, if $L \in C_0(I, X^*)$ then

$$\begin{aligned}\int_I \langle L(t), J(t)\rangle \, dt &= \lim_k \int_I \langle L(t), J_k(t)\rangle \, dt \\ &= \lim_k \int_I L(t)\hat{u_{n_k}}(t) d\mu(t) = \int_I L(t) d\mu(t).\end{aligned}$$

Hence, μ and J are seen to define the same measure on I. Therefore, there exists some $x \in X$ such that $\mu = x\delta + J$. This also tells us that μ is X-valued. Moreover, since δ is the identity of $M(I)$, it is obvious that $\mu * f \in L_1(I, X)$ for each $f \in L_1(I)$ and so $Tf = \mu * f \ \forall f \in L_1(I)$. An easy argument shows that μ is unique.

Let T be a multiplier from $L_1(I)$ to $L_1(I, X)$ and $Tf = \mu * f, f \in L_1(I), \mu \in M(I, X)$. Since $\|\mu * f\| \leq \|\mu\|\|f\|$, $\|T\| \leq \|\mu\|$. Also since μ is the weak-star limit of a sequence in $M(I, X)$ bounded in norm by $\|T\|$, we have $\|\mu\| \leq \|T\|$. ∎

The following definition is taken from Hille and Phillips [2].

Definition 3. *Let Φ be an X-valued function on $(0, \infty]$. Φ is said to be absolutely continuous if $\forall \epsilon > 0$ there exists a $\delta > 0$ such that whenever $\{(s_i, t_i)\}$ is a finite sequence of disjoint open intervals such that $\sum (t_i - s_i) < \delta$, we have*

$$\sum_{i=1}^{n} \|\Phi(t_i) - \Phi(s_i)\| < \epsilon.$$

The following characterization is a special case of Theorem 3.9, [12]. We are giving an easy proof.

Theorem 5. *If $T : L_1(I) \to L_1(I, X)$ is a multiplier with order convolution then there exists a unique, bounded, continuous X-valued function Φ such that*

(i) The function $s \to \Phi(s)$ is absolutely continuous.
(ii) The function $s \to \Phi(s)$ is differentiable almost everywhere.
(iii) If $M_{\Phi'}(f) = \Phi' \hat{f}$ then $M_{\Phi'} : L_1(I) \to L_1(I, X)$ is a bounded linear map.
(iv) $(Tf)^\wedge = \Phi \hat{f} \ \forall s \in (0, \infty)$ and $f \in L_1(I)$.

Conversely, if Φ is a bounded X-valued function on $(0, \infty)$ satisfying (i) to (iii). Then there exists a multiplier T of $L_1(I)$ to $L_1(I, X)$ satisfying (iv).

Proof. Suppose there exists $\mu \in M(I, X)$ of the form $\mu = x\delta + J$, $x \in X$, $J \in L_1(I, X)$ such that $Tf = \mu * f \ \forall f \in L_1(I)$. Then for $s \in \hat{I}$ and $f \in L_1(I)$, we have

$$(Tf)^\wedge(s) = \hat{\mu}(s)\hat{f}(s) \left(x + \hat{J}(s)\right) \hat{f}(s)$$

where $\hat{\mu}(s) = \int_0^s d\mu(t)$. Define Φ on $\hat{I}$ by $\Phi(s) = x + \hat{J}(s)\ \forall s \in \hat{I}$ and $\Phi(0) = x$. Since $J \in L_1(I, X)$, (i), (ii) and (iv) follow immediately.

For $f \in L_1(I)$, we have $(Tf)^\wedge(s) = \Phi(s)\hat{f}(s)$. Therefore,

$$\Phi(s)\hat{f}(s) = \int_0^s (Tf)(t)dt.$$

Differentiating, we have

$$\Phi(s)f(s) + \Phi'(s)\hat{f}(s) = Tf(s) \quad \text{a.e.}$$

Hence $M_{\Phi'}(f) = Tf - \Phi f$. Since Φ is bounded $\Phi f \in L_1(I, X)$. It follows that $M_{\Phi'}(f) \in L_1(I, X)$. We also have

$$\|M_{\Phi'}(f)\|_1 \leq (\|T\| + \|\Phi\|_\infty)\|f\|_1.$$

Conversely, suppose Φ is a bounded X-valued function on $(0, \infty)$ satisfying (i) to (iii). We define

$$T : L_1(I) \to L_1(I, X)$$

by

$$Tf(s) = \Phi(s)f(s) + \Phi'(s)\hat{f}(s) \quad \text{a.e.} \quad \forall f \in L_1(I).$$

It is easy to see that $\|Tf\|_1 \leq [\|\Phi\|_\infty + \|M_{\Phi'}\|]\, \|f\|_1$. Hence T is a bounded linear map of $L_1(I)$ to $L_1(I, X)$. We also see that the derivative of $(\Phi\hat{f})$ equals $\Phi(s)f(s) + \Phi'(s)\hat{f}(s) = Tf(s)$ almost everywhere. Hence $(Tf)^\wedge(s) = \Phi(s)\hat{f}(s)$ for all $s \in (0, \infty]$. This completes the proof of the theorem. ■

We now characterize the multipliers on $L_1(I, X)$ with respect to order convolution using the technique of Tewari, Dutta and Vaidya [13].

Let $T : L_1(I, X) \to L_1(I, X)$ be a multiplier, i.e. $T(f * F) = f * TF\ \forall f \in L_1(I)$ and $F \in L_1(I, X)$. For $x \in X$, define $T_x : L_1(I) \to L_1(I, X)$ by $T_x(f) = T(fx)$. It is easy to see that T_x is a multiplier from $L_1(I)$ to $L_1(I, X)$ and $\|T_x\| \leq \|T\|\|x\|$.

Therefore, by Theorem (4), there exists a measure $\mu_x \in M(I, X)$ of the form $\mu_x = \alpha_x\delta + J_x$ where $\alpha_x \in X$, and $J_x \in L_1(I, X)$ such that $T_x(f) = \mu_x * f$ and $\|\mu_x\| \leq \|T\|\|x\|$. The map $M : X \to M(I, X)$ defined by $M(x) = \mu_x$ is a bounded linear map with $\|M\| \leq \|T\|$ and $T(fx) = M(x) * f\ \forall x \in X$ and $f \in L_1(I)$.

Conversely, let M be a bounded linear operator from X into $M(I, X)$. $M(x) = \alpha_x\delta + J_x$ where $\alpha_x \in X$ and $J_x \in L_1(I, X)$. Consider the map $L_1(I) \times X \to L_1(I, X)$ defined by $(f, x) \to M(x) * f\ \forall f \in L_1(I)$ and $x \in X$. It is easy to see that this is a bilinear map and $\|M(x) * f\| \leq \|M(x)\|\|f\|_1 \leq \|M\|\|x\|\|f\|_1$. Hence, by the universal property of tensor products, we get a bounded linear map T' from $L_1(I) \otimes_\gamma X$ into $L_1(I, X)$

with $\|T'\| \leq \|M\|$ such that $T'(f \otimes x) = M(x) * f$ for any $f \in L_1(I)$ and $x \in X$. However, $L_1(I) \otimes_\gamma X$ is isometrically isomorphic to $L_1(I, X)$ (see [8]). Hence we get a bounded linear operator T of $L_1(I, X)$ with $\|T\| \leq \|M\|$ and $T(fx) = M(x) * f$ $\forall f \in L_1(I)$ and $x \in X$. Let $g \in L_1(I)$. We have

$$\begin{aligned} T(g * fx) &= T((g * f)x) = T_x(g * f) \\ &= M(x) * (g * f) = g * (M(x) * f) = g * T(fx). \end{aligned}$$

Since functions of the form $\sum_{i=1}^{n} f_i x_i$ with $f_i \in L_1(I)$ and $x_i \in X$ are dense in $L_1(I, X)$, it follows that T is multiplier on $L_1(I, X)$. It is easy to see that the bounded linear transformation from X into $M(I, X)$ associated with T is nothing but M and $\|M\| \leq \|T\|$. Therefore $\|T\| = \|M\|$.

Thus we have proved the following.

Theorem 6. *The set of all ultipliers on $L_1(I, X)$ with respect to order convolution is isometrically isomorphic to $L(X, M(I, X))$, the space of bounded linear operators from X into $M(I, X)$ in the following sense. Let T be any multiplier on $L_1(I, X)$ with order convolution such that $T(fx) = \mu_x * f$. Then there exists a bounded linear map M from X into $M(I, X)$ such that $M(x) = \mu_x = \alpha_x \delta + J_x$ where $\alpha_x \in X$ and $J_x \in L_1(I, X)$ and $\|T\| = \|M\|$.*

Now for an operator $M \in L(X, M(I, X))$ where $M(x) = \mu_x$, define μ from the Borel σ- algebra $\mathfrak{B}(I)$ into the space of bounded linear operators on X by $\mu(\mathbb{E})x = \mu_x(\mathbb{E})$ $\forall \mathbb{E} \in \mathfrak{B}(I)$. It is easy to see that $\mu(\mathbb{E})$ is a linear operator. Since

$$\|\mu(\mathbb{E})x\| = \|\mu_x(\mathbb{E})\| = \|M(x)(\mathbb{E})\| \leq \|M\|\|x\|.$$

So, $\mu(\mathbb{E}) \in L(X)$.

Corollary 2. *The set of all multipliers on $L_1(I, X)$ with respect to order convolution is isometrically isomorphic to the space of operator valued measures on I with point-wise finite variation such that $\mu(\mathbb{E})x = \alpha_x \delta(\mathbb{E}) + \int_{\mathbb{E}} J_x(t)dt$, where $x, \alpha_x \in X$ and $J_x \in L_1(I, X)$.*

The following definition is taken from Gaudry [3].

Definition 4. *For any regular operator-valued measure $\mu : \mathfrak{B}(I) \to L(X)$, the operator-valued function $\hat{\mu} : \hat{I} \to \mathfrak{L}(X)$ defined by $\hat{\mu}(s) = \int_0^s d\mu(t)$ is called the Fourier-Stieltjes transform of μ.*

Note. The definition of regularity of an operator valued measure is equivalent to the regularity of the scalar measure $\langle \mu x_1, x_2^* \rangle$ for each $x_1 \in X$ and $x_2^* \in X^*$.

The following definitions are taken from Hille and Phillips [2].

Definition 5. *Let $\Phi(s)$ be an operator valued function on $(0,\infty]$. We say that Φ is of strong bounded variation on $(0,\infty)$ if for each $x \in X$ the function $s \to \Phi(s)x$ is of strong bounded variation, that is,*

$$\sup \sum_{i=1}^{n} \|\Phi(t_i)x - \Phi(t_{i-1})x\| < \infty,$$

where the supremum is taken over all possible finite sets

$$\{t_0, t_1, \ldots, t_n \subset (0,\infty) : t_0 < t_1 < \ldots < t_n\}.$$

Φ is called strongly absolutely continuous if $\forall \epsilon > 0$ there exists a $\delta > 0$ such that whenever $\{(s_i, t_i)\}$ is a finite sequence of disjoint open intervals for which $\sum(t_i - s_i) < \delta$, we have

$$\sum_{i=1}^{n} \|\Phi(t_i)x - \Phi(s_i)x\| < \epsilon.$$

Tewari [12] had proved the following theorem (see Theorem 3.9, [12]). We are giving an easy proof here.

Theorem 7. *Let T be a multiplier of $L_1(I,X)$ into itself with order convolution. Then there exists an operator valued bounded strongly continuous function Φ on $(0,\infty]$ such that*

(i) The function $s \to \Phi(s)x$ is strongly absolutely continuous and hence of strong bounded variation on $(0,\infty)$,

(ii) The function $s \to \Phi(s)x$ is strongly differentiable almost everywhere,

(iii) If $M_{\Phi'}(F) = \Phi'\hat{F}$ then $M_{\Phi'} : L_1(I,X) \to L_1(I,X)$ is a bounded linear map,

(iv) $(TF)^\wedge(s) = \Phi(s)(\hat{F}(s))$ for all $s \in (0,\infty)$ and $F \in L_1(I,X)$.

Conversely, if Φ is a bounded $\mathfrak{L}(X)$ - valued function on $(0,\infty)$ satisfying (i) to (iii) then there exists a multiplier T of $L_1(I,X)$ to itself satisfying (iv).

Proof. Suppose there exists an operator-valued measure μ of the form $\mu(\mathbb{E})x = \mu_x(\mathbb{E})$, where $\mu_x = \alpha_x\delta + J_x, \alpha_x \in X, J_x \in L_1(I,X)$ such that $TF = \mu * F \ \forall F \in L_1(I,X)$. Then for for each $s \in I$ and $F \in L_1(I,X)$, we have $(TF)^\wedge(s) = \hat{\mu}(s)\hat{F}(s)$ where $\hat{\mu}(s) = \int_0^s d\mu(t)$. Define $\Phi(s) = \hat{\mu}(s)$. Therefore, $\Phi(s)x = (\hat{\mu}(s))(x) = \int_0^s d\mu_x(t) = \hat{\mu_x}(s) = \alpha_x + \int_0^s J_x(t)dt$. Since J_x belongs to $L_1(I,X)$, (i) and (ii) follow immediately. Since $\Phi(s)\hat{F}(s) = \int_0^s (TF)(t)dt$, we have $\Phi(s)F(s) + \Phi'(s)\hat{F}(s) = TF(s)$ almost everywhere. Hence $M_{\Phi'}(F) = TF - \Phi F$ and since Φ is bounded we have $M_{\Phi'}(F) \in$

$L_1(I, X)$ and $\|M_{\Phi'}(F)\|_1 \leq (\|T\| + \|\Phi\|_\infty)\|F\|_1$. Thus (iii) holds. Moreover, (iv) is a consequence of the relationship between T and Φ.

Conversely, suppose Φ is a bounded operator valued function on $(0, \infty)$ which satisfies all conditions from (i) to (iii). We define

$$T : L_1(I, X) \to L_1(I, X).$$

by $TF(s) = \Phi(s)F(s) + \Phi'(s)\hat{F}(s)$ a.e. for $F \in L_1(I, X)$. Since Φ is bounded and continuous in the strong operator topology, the function $\Phi(s)F(s)$ is strongly measurable and

$$\int_0^\infty \|\Phi(s)F(s)\|ds \leq \|\phi\|_\infty \|F\|_1.$$

Therefore, $\|TF\|_1 \leq [\|\Phi\|_\infty + \|M_{\Phi'}\|]\|F\|_1$ and we conclude that T is a bounded linear map.

We can easily see that the derivative of $(\Phi\hat{F})$ equals $\Phi(s)F(s) + \Phi'(s)\hat{F}(s)$ $= TF(s)$ almost everywhere. Hence, it follows that $(TF)^\wedge(s) = \Phi(s)\hat{F}(s)$ $\forall s \in (0, \infty]$.

This completes the proof of the theorem. ■

Remark 3. Let T be a compact multiplier from $L_1(I, X)$ with order convolution into itself. We show that $T(fx) = J_x * f$, $J_x \in L_1(I, X)$. Suppose $T(fx) = \alpha_x f + J_x$, $\alpha_x (\neq 0) \in X$ and $J_x \in L_1(I, X)$. Let (u_n) be an approximate identity in $L_1(I)$ and $x \in X$. Since T is a compact operator, there exists a subsequence $T(u_{n_k} x) = \alpha_x u_{n_k} + J_x * u_{n_k}$ which converges. Hence $\alpha_x u_{n_k} = Tu_{n_k} - J_x * u_{n_k}$ is convergent, which is a contradiction. Therefore, $\alpha_x = 0$.

Remark 4. Finally, we remark that the characterization of isometric multipliers of $L_1(I, X)$ with order convolution will be quite interesting. Let $T : L_1(I, X) \to L_1(I, X)$ such that $T(f.x) = \alpha_x f$, where $\alpha_x \in X, \|\alpha_x\| = 1$ and $f \in L_1(I)$. Then T is an isometric multiplier of $L_1(I, X)$ into itself. We feel that these are the only isometric multipliers of $L_1(I, X)$ with order convolution. The characterization of isometric multipliers of $L_1(I, X)$ is an open problem.

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