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GROWTH ESTIMATES OF COMPOSITE ENTIRE FUNCTIONS IN THE LIGHT OF SLOWLY CHANGING FUNCTIONS BASED RELATIVE ORDER, RELATIVE TYPE AND RELATIVE WEAK TYPE

Abstract. In the paper we prove some growth properties of maximum term and maximum modulus of composition of entire functions on the basis of relative L^*-order, relative L^*-type and relative L^*-weak type.

Key words: entire function, maximum term, maximum modulus, composition, growth, relative L^*-order, relative L^*-type, relative L^*-weak type, slowly changing function.

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1. Introduction, definitions and notations

Let $\mathbb{C}$ be the set of all finite complex numbers. For any entire function $f = \sum\limits_{n=0}^{\infty} a_n z^n$ defined on $\mathbb{C}$, the functions $M(r, f)$ and $\mu(r, f)$ are respectively defined as $M(r, f) = \max\limits_{|z|=r} |f(z)|$ and $\mu(r, f) = \max(|a_n| r^n)$.

Let $L \equiv L(r)$ be a positive continuous function increasing slowly *i.e.*, $L(ar) \sim L(r)$ as $r \to \infty$ for every positive constant a. *Singh and Barker* [8] defined it in the following way:

Definition 1 ([8])**.** *A positive continuous function $L(r)$ is called a slowly changing function if for $\varepsilon (> 0)$,*

$$\frac{1}{k^{\varepsilon}} \leq \frac{L(kr)}{L(r)} \leq k^{\varepsilon} \;\; for \;\; r \geq r(\varepsilon)$$

and uniformly for $k (\geq 1)$.

If further, $L(r)$ is differentiable, the above condition is equivalent to

$$\lim_{r \to \infty} \frac{rL'(r)}{L(r)} = 0.$$

Somasundaram and Thamizharasi [9] introduced the notions of L-order and L-type for entire function where $L \equiv L(r)$ is a positive continuous function increasing slowly i.e.,$L(ar) \sim L(r)$ as $r \to \infty$ for every positive constant 'a'. The more generalised concept for L-order and L-type for entire function are L^*-order and L^*-type. Their definitions are as follows:

Definition 2 ([9])**.** *The L^*-order $\rho_f^{L^*}$ and the L^*-lower order $\lambda_f^{L^*}$ of an entire function f are defined as*

$$\rho_f^{L^*} = \limsup_{r\to\infty} \frac{\log^{[2]} M_f(r)}{\log\left[re^{L(r)}\right]} \quad \text{and} \quad \lambda_f^{L^*} = \liminf_{r\to\infty} \frac{\log^{[2]} M_f(r)}{\log\left[re^{L(r)}\right]},$$

where $\log^{[k]} x = \log\left(\log^{[k-1]} x\right)$ *for* $k = 1, 2, 3, \dots$ *and* $\log^{[0]} x = x$.

Using the inequalities $\mu_f(r) \le M_f(r) \le \frac{R}{R-r}\mu_f(R)$ $\{cf.$ [11]$\}$, *for* $0 \le r < R$ *one may verify that*

$$\rho_f^{L^*} = \limsup_{r\to\infty} \frac{\log^{[2]} \mu_f(r)}{\log\left[re^{L(r)}\right]} \quad \text{and} \quad \lambda_f^{L^*} = \liminf_{r\to\infty} \frac{\log^{[2]} \mu_f(r)}{\log\left[re^{L(r)}\right]} .$$

Definition 3 ([9])**.** *The L^*-type $\sigma_f^{L^*}$ of an entire function f is defined as*

$$\sigma_f^{L^*} = \limsup_{r\to\infty} \frac{\log M_f(r)}{\left[re^{L(r)}\right]^{\rho_f^{L^*}}}, \quad 0 < \rho_f^{L^*} < \infty.$$

In order to determine the relative growth of two entire functions of same non zero finite L^*-lower order one may define the L^*-weak type in the following way:

Definition 4. *The L^*-weak type $\tau_f^{L^*}$ of an entire function f is defined as follows:*

$$\tau_f^{L^*} = \liminf_{r\to\infty} \frac{\log M_f(r)}{\left[re^{L(r)}\right]^{\lambda_f^{L^*}}}, \quad 0 < \lambda_f^{L^*} < \infty.$$

If an entire function g is non-constant then $M_g(r)$ is strictly increasing and continuous and its inverse $M_g^{-1} : (|f(0)|, \infty) \to (0, \infty)$ exists and is such that $\lim_{s\to\infty} M_g^{-1}(s) = \infty$.

Bernal [1] introduced the definition of relative order of an entire function f with respect to an entire function g, denoted by $\rho_g(f)$ as follows:

$$\begin{aligned}\rho_g(f) &= \inf\{\mu > 0 : M_f(r) < M_g(r^\mu) \ \text{ for all } \ r > r_0(\mu) > 0\} \\ &= \limsup_{r\to\infty} \frac{\log M_g^{-1} M_f(r)}{\log r}.\end{aligned}$$

The definition coincides with the classical one {cf. [12]} if $g(z) = \exp z$.

Similarly, one can define the relative lower order of an entire function f with respect to an entire function g denoted by $\lambda_g(f)$ as follows:

$$\lambda_g(f) = \liminf_{r\to\infty} \frac{\log M_g^{-1} M_f(r)}{\log r}.$$

Datta and Maji [5] gave an alternative definition of relative order and relative lower order of an entire with respect to another entire in the following way:

Definition 5 ([5]). *The relative order $\rho_g(f)$ and relative lower order $\lambda_g(f)$ of an entire function f with respect to an entire function g are defined as follows:*

$$\rho_g(f) = \limsup_{r\to\infty} \frac{\log \mu_g^{-1} \mu_f(r)}{\log r} \quad \text{and} \quad \lambda_g(f) = \liminf_{r\to\infty} \frac{\log \mu_g^{-1} \mu_f(r)}{\log r}.$$

In the line of Somasundaram and Thamizharasi [9] and Bernal [1] one may define the relative L^*-order of an entire function in the following manner:

Definition 6 ([3], [4]). *The relative L^*-order $\rho_g^{L^*}(f)$ and relative L^*-lower $\lambda_g^{L^*}(f)$ of an entire function f with respect to another entire function g are defined as*

$$\rho_g^{L^*}(f) = \limsup_{r\to\infty} \frac{\log M_g^{-1} M_f(r)}{\log\left[re^{L(r)}\right]} \quad \text{and} \quad \lambda_g^{L^*}(f) = \liminf_{r\to\infty} \frac{\log M_g^{-1} M_f(r)}{\log\left[re^{L(r)}\right]}.$$

Datta, Biswas and Ali [6] also gave an alternative definition of L^*-order and relative L^*-lower order of an entire function which are as follows:

Definition 7 ([6]). *The relative L^*-order $\rho_g^{L^*}(f)$ and the relative L^*-lower order $\lambda_g^{L^*}(f)$ of an entire function f with respect to g are as follows:*

$$\rho_g^{L^*}(f) = \limsup_{r\to\infty} \frac{\log \mu_g^{-1} \mu_f(r)}{\log\left[re^{L(r)}\right]} \quad \text{and} \quad \lambda_g^{L^*}(f) = \liminf_{r\to\infty} \frac{\log \mu_g^{-1} \mu_f(r)}{\log\left[re^{L(r)}\right]}.$$

In order to determine the relative growth of two entire functions having same non zero finite relative L^*-order with respect to another entire function, one may define the concept of the relative L^*-type in the following manner:

Definition 8. *The relative L^*-type $\sigma_g^{L^*}(f)$ of an entire function f with respect to g is defined as follows:*

$$\sigma_g^{L^*}(f) = \limsup_{r\to\infty} \frac{M_g^{-1} M_f(r)}{\left[re^{L(r)}\right]^{\rho_g^{L^*}(f)}}, \quad 0 < \rho_g^{L^*}(f) < \infty.$$

Analogusly, in order to determine the relative growth of two entire functions having same non zero finite relative L^*-lower order with respect to another entire function, one can define the relative L^*-weak type in the following way:

Definition 9. *The relative L^*-weak type $\tau_g^{L^*}(f)$ of an entire function f with respect to g of finite positive relative L^*-lower order $\lambda_g^{L^*}(f)$ is defined as:*

$$\tau_g^{L^*}(f) = \liminf_{r\to 1} \frac{M_g^{-1}M_f(r)}{\left[re^{L(r)}\right]^{\lambda_g^{L^*}(f)}}.$$

Concidering $g = \exp z$ one may easily verify that the Definition 8 and Definition 9 coincide with the classical Definition 3 and Definition 4 respectively.

In the paper we study some relative growth properties of maximum term and maximum modulus of composition of entire functions with respect to another entire function and compare the relative growth of their corresponding left and right factors on the basis of relative L^*-order, relative L^*-type and relative L^*-weak type. We do not explain the standard definitions and notations in the theory of entire functions as those are available in [13].

2. Lemmas

In this section we present some lemmas which will be needed in the sequel.

Lemma 1 ([1])**.** *If f and g are any two entire functions then for all sufficiently large values of r,*

$$M_{f\circ g}(r) \le M_f(M_g(r)).$$

Lemma 2 ([10])**.** *Let f and g be any two entire functions. Then for every $\alpha > 1$ and $0 < r < R$,*

$$\mu_{f\circ g}(r) \le \frac{\alpha}{\alpha-1}\mu_f\left(\frac{\alpha R}{R-r}\mu_g(R)\right).$$

Lemma 3 ([5])**.** *If f be entire and $\alpha > 1$, $0 < \beta < \alpha$, then for all sufficiently large values of r,*

$$\mu_f(\alpha r) \ge \beta\mu_f(r).$$

Lemma 4 ([7])**.** *Let f and h be any two entire functions. Then for any $\alpha > 1$,*

$$(i) \qquad M_h^{-1}M_f(r) \le \mu_h^{-1}\left[\frac{\alpha}{(\alpha-1)}\mu_f(\alpha r)\right] \quad \text{and}$$

(ii) $$\mu_h^{-1}\mu_f(r) \leq \alpha M_h^{-1}\left[\frac{\alpha}{(\alpha-1)}M_f(r)\right].$$

Lemma 5 ([1])**.** *Suppose f is an entire function and $\alpha > 1$, $0 < \beta < \alpha$, then for all sufficiently large values of r,*

$$M_f(\alpha r) \geq \beta M_f(r).$$

2. Theorems

In this section we present the main results of the paper.

Theorem 1. *Let f, g and h be any three entire functions such that $0 < \lambda_h^{L^*}(f) \leq \rho_h^{L^*}(f) < \infty$ and $\sigma_g^{L^*} < \infty$. If $L(\mu_g(\beta r)) = o\left\{r^\alpha e^{\alpha L(r)}\right\}$ as $r \to \infty$ and for some positive $\alpha < \rho_g^{L^*}$ and $\beta > 1$. Then*

$$\limsup_{r\to\infty}\frac{\log \mu_h^{-1}\mu_{f\circ g}(r)}{\log \mu_h^{-1}\mu_f\left(\exp\left(\beta r e^{L(r)}\right)^{\rho_g^{L^*}}\right)} \leq \frac{\rho_h^{L^*}(f)\cdot \sigma_g^{L^*}}{\lambda_h^{L^*}(f)}.$$

Proof. Taking $R = \beta r$ $(\beta > 1)$ in Lemma 2 and in view of Lemma 3 we have for all sufficiently large values of r that

$$\mu_{f\circ g}(r) \leq \left(\frac{\alpha}{\alpha-1}\right)\mu_f\left(\frac{\alpha\beta}{(\beta-1)}\mu_g(\beta r)\right)$$
$$i.e.,\ \mu_{f\circ g}(r) \leq \mu_f\left(\frac{2\alpha^2\beta}{(\alpha-1)(\beta-1)}\mu_g(\beta r)\right).$$

Since $\mu_h^{-1}(r)$ is an increasing function, it follows from above for all sufficiently large values of r that

(1) $$\mu_h^{-1}\mu_{f\circ g}(r) \leq \mu_h^{-1}\mu_f\left(\frac{2\alpha^2\beta}{(\alpha-1)(\beta-1)}\mu_g(\beta r)\right)$$
$$i.e.,\ \log\mu_h^{-1}\mu_{f\circ g}(r) \leq \log\mu_h^{-1}\mu_f\left(\frac{2\alpha^2\beta}{(\alpha-1)(\beta-1)}\mu_g(\beta r)\right)$$

$$i.e.,\ \frac{\log \mu_h^{-1}\mu_{f\circ g}(r)}{\log \mu_h^{-1}\mu_f\left(\exp\left(\beta r e^{L(r)}\right)^{\rho_g^{L^*}}\right)} \leq \frac{\log\mu_h^{-1}\mu_f\left(\frac{2\alpha^2\beta}{(\alpha-1)(\beta-1)}\mu_g(\beta r)\right)}{\log \mu_h^{-1}\mu_f\left(\exp\left(\beta r e^{L(r)}\right)^{\rho_g^{L^*}}\right)}$$
$$= \frac{\log\mu_h^{-1}\mu_f\left(\frac{2\alpha^2\beta}{(\alpha-1)(\beta-1)}\mu_g(\beta r)\right)}{\log\left\{\frac{2\alpha^2\beta}{(\alpha-1)(\beta-1)}\mu_g(\beta r)\,e^{L(\mu_g(\beta r))}\right\}}\frac{\log\mu_g(\beta r) + L(\mu_g(\beta r)) + O(1)}{\left[\beta r e^{L(r)}\right]^{\rho_g^{L^*}}}$$
$$\times\frac{\log\left\{\exp\left(\beta r e^{L(r)}\right)^{\rho_g^{L^*}}\right\}}{\log \mu_h^{-1}\mu_f\left(\exp\left(\beta r e^{L(r)}\right)^{\rho_g^{L^*}}\right)}$$

$$i.e., \frac{\log \mu_h^{-1}\mu_{f\circ g}(r)}{\log \mu_h^{-1}\mu_f\left(\exp\left(\beta r e^{L(r)}\right)^{\rho_g^{L^*}}\right)} \leq \frac{\log \mu_h^{-1}\mu_f\left(\frac{2\alpha^2\beta}{(\alpha-1)(\beta-1)}\mu_g(\beta r)\right)}{\log\left\{\frac{2\alpha^2\beta}{(\alpha-1)(\beta-1)}\mu_g(\beta r)\, e^{L(\mu_g(\beta r))}\right\}}$$
$$\times \frac{\log \mu_g(\beta r) + L(\mu_g(\beta r)) + O(1)}{\left[\beta r e^{L(r)}\right]^{\rho_g^{L^*}}}$$
$$\times \frac{\log\left\{\exp\left(\beta r e^{L(r)}\right)^{\rho_g^{L^*}}\right\} + L\left(\exp\left(\beta r e^{L(r)}\right)^{\rho_g^{L^*}}\right)}{\log \mu_h^{-1}\mu_f\left(\exp\left(\beta r e^{L(r)}\right)^{\rho_g^{L^*}}\right)}$$

$$(2) \qquad i.e., \limsup_{r\to\infty} \frac{\log \mu_h^{-1}\mu_{f\circ g}(r)}{\log \mu_h^{-1}\mu_f\left(\exp\left(\beta r e^{L(r)}\right)^{\rho_g^{L^*}}\right)}$$
$$\leq \limsup_{r\to\infty} \frac{\log \mu_h^{-1}\mu_f\left(\frac{2\alpha^2\beta}{(\alpha-1)(\beta-1)}\mu_g(\beta r)\right)}{\log\left\{\frac{2\alpha^2\beta}{(\alpha-1)(\beta-1)}\mu_g(\beta r)\, e^{L(\mu_g(\beta r))}\right\}}$$
$$\times \limsup_{r\to\infty} \frac{\log \mu_g(\beta r) + L(\mu_g(\beta r)) + O(1)}{\left[\beta r e^{L(r)}\right]^{\rho_g^{L^*}}}$$
$$\times \limsup_{r\to\infty} \frac{\log\left\{\exp\left(\beta r e^{L(r)}\right)^{\rho_g^{L^*}}\right\} + L\left(\exp\left(\beta r e^{L(r)}\right)^{\rho_g^{L^*}}\right)}{\log \mu_h^{-1}\mu_f\left(\exp\left(\beta r e^{L(r)}\right)^{\rho_g^{L^*}}\right)}.$$

As $\alpha < \rho_g^{L^*}$ and since $L(\mu_g(\beta r)) = o\left(r^\alpha e^{\alpha L(r)}\right)$ as $r \to \infty$, we obtain that

$$(3) \qquad \lim_{r\to\infty} \frac{L(\mu_g(\beta r))}{\left[r e^{L(r)}\right]^{\rho_g^{L^*}}} = 0.$$

Now from (2) and (3) it follows that

$$\limsup_{r\to\infty} \frac{\log \mu_h^{-1}\mu_{f\circ g}(r)}{\log \mu_h^{-1}\mu_f\left(\exp\left(\beta r e^{L(r)}\right)^{\rho_g^{L^*}}\right)} \leq \rho_h^{L^*}(f)\, \sigma_g^{L^*} \frac{1}{\lambda_h^{L^*}(f)}.$$

Thus the theorem is established. ∎

In the line of Theorem 1 the following theorem can be proved:

Theorem 2. *Let f, g and h be any three entire functions with $\lambda_h^{L^*}(g) > 0$, $\rho_h^{L^*}(f) < \infty$ and $\sigma_g^{L^*} < \infty$. If $L(\mu_g(\beta r)) = o\left\{r^\alpha e^{\alpha L(r)}\right\}$ as $r \to \infty$ and for some positive $\alpha < \rho_g^{L^*}$ and $\beta > 1$. Then*

$$\limsup_{r\to\infty} \frac{\log \mu_h^{-1}\mu_{f\circ g}(r)}{\log \mu_h^{-1}\mu_g\left(\exp\left(\beta r e^{L(r)}\right)^{\rho_g^{L^*}}\right)} \leq \frac{\rho_h^{L^*}(f) \cdot \sigma_g^{L^*}}{\lambda_h^{L^*}(g)}.$$

The proof is omitted.

In the line of Theorem 1 and Theorem 2 and with the help of Lemma 1 the following two theorems can be proved:

Theorem 3. *Let f, g and h be any three entire functions such that $0 < \lambda_h^{L^*}(f) \leq \rho_h^{L^*}(f) < \infty$ and $\sigma_g^{L^*} < \infty$. If $L(M_g(r)) = o\left\{r^\alpha e^{\alpha L(r)}\right\}$ as $r \to \infty$ and for some positive $\alpha < \rho_g^{L^*}$, then*

$$\limsup_{r\to\infty} \frac{\log M_h^{-1} M_{f\circ g}(r)}{\log M_h^{-1} M_f\left(\exp\left(re^{L(r)}\right)^{\rho_g^{L^*}}\right)} \leq \frac{\rho_h^{L^*}(f)\,\sigma_g^{L^*}}{\lambda_h^{L^*}(f)}.$$

Theorem 4. *Let f, g and h be any three entire functions with $\lambda_h^{L^*}(g) > 0$, $\rho_h^{L^*}(f) < \infty$ and $\sigma_g^{L^*} < \infty$. If $L(M_g(r)) = o\left\{r^\alpha e^{\alpha L(r)}\right\}$ as $r \to \infty$ and for some positive $\alpha < \rho_g^{L^*}$. Then*

$$\limsup_{r\to\infty} \frac{\log M_h^{-1} M_{f\circ g}(r)}{\log M_h^{-1} M_g\left(\exp\left(re^{L(r)}\right)^{\rho_g^{L^*}}\right)} \leq \frac{\rho_h^{L^*}(f)\,\sigma_g^{L^*}}{\lambda_h^{L^*}(g)}.$$

Using the notion of L^*-weak type, we may state the following two theorems without there proof because those can be carried out in the line of Theorem 1 and Theorem 3 respectively.

Theorem 5. *Let f, g and h be any three entire functions such that $0 < \lambda_h^{L^*}(f) \leq \rho_h^{L^*}(f) < \infty$ and $\tau_g^{L^*} < \infty$. If $L(\mu_g(\beta r)) = o\left\{r^\alpha e^{\alpha L(r)}\right\}$ as $r \to \infty$ and for some positive $\alpha < \lambda_g^{L^*}$ and $\beta > 1$. Then*

$$\liminf_{r\to\infty} \frac{\log \mu_h^{-1} \mu_{f\circ g}(r)}{\log \mu_h^{-1} \mu_f\left(\exp\left(\beta re^{L(r)}\right)^{\lambda_g^{L^*}}\right)} \leq \frac{\rho_h^{L^*}(f)\,\tau_g^{L^*}}{\lambda_h^{L^*}(f)}.$$

Theorem 6. *Let f, g and h be any three entire functions with $0 < \lambda_h^{L^*}(f) \leq \rho_h^{L^*}(f) < \infty$ and $\tau_g^{L^*} < \infty$. If $L(M_g(r)) = o\left\{r^\alpha e^{\alpha L(r)}\right\}$ as $r \to \infty$ and for some positive $\alpha < \lambda_g^{L^*}$. Then*

$$\liminf_{r\to\infty} \frac{\log M_h^{-1} M_{f\circ g}(r)}{\log M_h^{-1} M_f\left(\exp\left(re^{L(r)}\right)^{\lambda_g^{L^*}}\right)} \leq \frac{\rho_h^{L^*}(f)\,\tau_g^{L^*}}{\lambda_h^{L^*}(f)}.$$

Similary, the following two theorems can also be carried out in the line of Theorem 2 and Theorem 4 and therefore their proofs are omitted:

Theorem 7. *Let f, g and h be any three entire functions with $\lambda_h^{L^*}(g) > 0$, $\rho_h^{L^*}(f) < \infty$ and $\tau_g^{L^*} < \infty$. If $L(\mu_g(\beta r)) = o\left\{r^{\alpha} e^{\alpha L(r)}\right\}$ as $r \to \infty$ and for some positive $\alpha < \lambda_g^{L^*}$ and $\beta > 1$. Then*

$$\liminf_{r\to\infty} \frac{\log \mu_h^{-1} \mu_{f\circ g}(r)}{\log \mu_h^{-1} \mu_g\left(\exp\left(\beta r e^{L(r)}\right)^{\lambda_g^{L^*}}\right)} \leq \frac{\rho_h^{L^*}(f)\, \tau_g^{L^*}}{\lambda_h^{L^*}(g)}.$$

Theorem 8. *Let f, g and h be any three entire functions with $\lambda_h^{L^*}(g) > 0$, $\rho_h^{L^*}(f) < \infty$ and $\tau_g^{L^*} < \infty$. If $L(M_g(r)) = o\left\{r^{\alpha} e^{\alpha L(r)}\right\}$ as $r \to \infty$ and for some positive $\alpha < \lambda_g^{L^*}$. Then*

$$\liminf_{r\to\infty} \frac{\log M_h^{-1} M_{f\circ g}(r)}{\log M_h^{-1} M_g\left(\exp\left(r e^{L(r)}\right)^{\lambda_g^{L^*}}\right)} \leq \frac{\rho_h^{L^*}(f)\, \tau_g^{L^*}}{\lambda_h^{L^*}(g)}.$$

Theorem 9. *Let f, g and h be any three entire functions such that $\rho_h^{L^*}(g) < \infty$ and $\lambda_h^{L^*}(f \circ g) = \infty$. Then*

$$\lim_{r\to\infty} \frac{\log \mu_h^{-1} \mu_{f\circ g}(r)}{\log \mu_h^{-1} \mu_g(r)} = \infty.$$

Proof. Let us suppose that the conclusion of the theorem does not hold. Then we can find a constant $\beta > 0$ such that for a sequence of values of r tending to infinity

$$\log \mu_h^{-1} \mu_{f\circ g}(r) \leq \beta \log \mu_h^{-1} \mu_g(r). \tag{4}$$

Again from the definition of $\rho_h^{L^*}(g)$ it follows that for all sufficiently large values of r that

$$\log \mu_h^{-1} \mu_g(r) \leq \left(\rho_h^{L^*}(g) + \varepsilon\right) \log\left[r e^{L(r)}\right]. \tag{5}$$

Thus from (4) and (5) we have for a sequence of values of r tending to infinity that

$$\begin{aligned} \log \mu_h^{-1} \mu_{f\circ g}(r) &\leq \beta \left(\rho_h^{L^*}(g) + \varepsilon\right) \log\left[r e^{L(r)}\right] \\ \text{i.e., } \frac{\log \mu_h^{-1} \mu_{f\circ g}(r)}{\log\left[r e^{L(r)}\right]} &\leq \frac{\beta \left(\rho_h^{L^*}(g) + \varepsilon\right) \log\left[r e^{L(r)}\right]}{\log\left[r e^{L(r)}\right]} \\ \text{i.e., } \liminf_{r\to\infty} \frac{\log \mu_h^{-1} \mu_{f\circ g}(r)}{\log\left[r e^{L(r)}\right]} &= \lambda_h^{L^*}(f\circ g) < \infty, \end{aligned}$$

which contradicts the condition $\lambda_h^{L^*}(f \circ g) = \infty$.

So for all Sufficiently large values of r we get that

$$\log \mu_h^{-1} \mu_{f \circ g}(r) > \beta \log \mu_h^{-1} \mu_g(r) . \tag{6}$$

from which the theorem follows. ■

Remark 1. Theorem 9 is also valid with "limit superior" instead of "limit" if $\lambda_h^{L^*}(f \circ g) = \infty$ is replaced by $\rho_h^{L^*}(f \circ g) = \infty$ and the other conditions remaining the same.

In the line of Theorem 9 and Remark 1 the following theorem can also be proved:

Theorem 10. *Let f, g and h be any three entire functions with $\rho_h^{L^*}(g) < \infty$ and $\lambda_h^{L^*}(f \circ g) = \infty$. Then*

$$\lim_{r \to \infty} \frac{\log M_h^{-1} M_{f \circ g}(r)}{\log M_h^{-1} M_g(r)} = \infty.$$

Further if $\rho_h^{L^*}(f \circ g) = \infty$ instead of $\lambda_h^{L^*}(f \circ g) = \infty$ then

$$\limsup_{r \to \infty} \frac{\log M_h^{-1} M_{f \circ g}(r)}{\log M_h^{-1} M_g(r)} = \infty.$$

Corollary 1. *Under the assumptions of Theorem 9 and the first part of Theorem 10,*

$$\lim_{r \to \infty} \frac{\mu_h^{-1} \mu_{f \circ g}(r)}{\mu_h^{-1} \mu_g(r)} = \infty \quad \textit{and} \quad \lim_{r \to \infty} \frac{M_h^{-1} M_{f \circ g}(r)}{M_h^{-1} M_g(r)} = \infty$$

are respectively holds.

Proof. By Theorem 9 we obtain for all sufficiently large values of r and for $K > 1$,

$$\log \mu_h^{-1} \mu_{f \circ g}(r) > K \log \mu_h^{-1} \mu_g(r)$$
$$\textit{i.e.,}\ \mu_h^{-1} \mu_{f \circ g}(r) > \left\{ \mu_h^{-1} \mu_g(r) \right\}^K ,$$

from which the first part of the corollary follows.

Similary, from Theorem 10 the second part of the corollary is established. ■

Corollary 2. *Under the assumptions of Remark 1 and the second part of Theorem 10,*

$$\limsup_{r \to \infty} \frac{\mu_h^{-1} \mu_{f \circ g}(r)}{\mu_h^{-1} \mu_g(r)} = \infty \quad \textit{and} \quad \limsup_{r \to \infty} \frac{M_h^{-1} M_{f \circ g}(r)}{M_h^{-1} M_g(r)} = \infty$$

are respectively holds.

We omit the proof of Corollary 2 as it can be carried out in the line of Corollary 1.

Theorem 11. *Let f, g and h be any three entire functions such that (i) $\rho_h^{L^*}(f) = \rho_g^{L^*}$, (ii) $0 < \sigma_g^{L^*} < \infty$ and (iii) $\sigma_h^{L^*}(f) > 0$.*

Then for any $\alpha, \beta > 1$,

(a) If $L(\mu_g(\beta r)) = o\{\mu_h^{-1}\mu_f(r)\}$ then

$$\liminf_{r\to\infty} \frac{\log \mu_h^{-1}\mu_{f\circ g}(r)}{\mu_h^{-1}\mu_f(r) + L(\mu_g(\beta r))} \le \left[\frac{\alpha\beta(2\alpha-1)}{\alpha-1}\right]^{\rho_h^{L^*}(f)} \frac{\rho_h^{L^*}(f)\,\sigma_g^{L^*}}{\sigma_h^{L^*}(f)},$$

and (b) if $\mu_h^{-1}\mu_f(r) = o\{L(\mu_g(\beta r))\}$ then

$$\liminf_{r\to\infty} \frac{\log \mu_h^{-1}\mu_{f\circ g}(r)}{\mu_h^{-1}\mu_f(r) + L(\mu_g(\beta r))} \le \rho_h^{L^*}(f).$$

Proof. From (1) and the inequality $\mu(r,f) \le M(r,f)$ {cf. [11]} we get for all sufficiently large values of r that

$$\log \mu_h^{-1}\mu_{f\circ g}(r) \le \left(\rho_h^{L^*}(f) + \varepsilon\right)\{\log \mu_g(\beta r) + L(\mu_g(\beta r)) + O(1)\}$$

$$\text{(7)} \quad i.e.,\ \log \mu_h^{-1}\mu_{f\circ g}(r) \le \left(\rho_h^{L^*}(f) + \varepsilon\right) \times \{\log M_g(\beta r) + L(\mu_g(\beta r)) + O(1)\}.$$

Using the definition of L^*-type we obtain from (7) for all sufficiently large values of r that

$$\text{(8)} \quad \log \mu_h^{-1}\mu_{f\circ g}(r) \le \left(\rho_h^{L^*}(f) + \varepsilon\right)\left(\sigma_g^{L^*} + \varepsilon\right)\left[\beta r e^{L(r)}\right]^{\rho_g^{L^*}} + \left(\rho_h^{L^*}(f) + \varepsilon\right) L(\mu(\beta r, g)) + O(1).$$

Now in view of condition (*ii*) we obtain from (8) for all sufficiently large values of r that

$$\text{(9)} \quad \log \mu_h^{-1}\mu_{f\circ g}(r) \le \left(\rho_h^{L^*}(f) + \varepsilon\right)\left(\sigma_g^{L^*} + \varepsilon\right)\left[\beta r e^{L(r)}\right]^{\rho_h^{L^*}(f)} + \left(\rho_h^{L^*}(f) + \varepsilon\right) L(\mu_g(\beta r)) + O(1).$$

Again in view of Lemma 3, Lemma 4 and the definition of relative L^*-type we get for a sequence of values of r tending to infinity that

$$\mu_h^{-1}\left[\frac{\alpha}{(\alpha-1)}\mu_f(\alpha r)\right] \ge M_h^{-1}M_f(r)$$

$$i.e.,\ \mu_h^{-1}\left[\mu_f\left(\frac{(2\alpha-1)\alpha r}{(\alpha-1)}\right)\right] \ge M_h^{-1}M_f(r)$$

$$i.e.,\ \mu_h^{-1}\mu_f(r) \ge M_h^{-1}M_f\left(\frac{(\alpha-1)}{(2\alpha-1)\alpha}r\right)$$

$$i.e.,\ \mu_h^{-1}\mu_f(r) \geq \left(\sigma_h^{L^*}(f)-\varepsilon\right)\left\{\left(\frac{(\alpha-1)}{(2\alpha-1)\alpha}r\right)e^{L(r)}\right\}^{\rho_h^{L^*}(f)}$$

$$(10)\qquad i.e.,\ \left[re^{L(r)}\right]^{\rho_h^{L^*}(f)} \leq \left[\frac{(2\alpha-1)\alpha}{\alpha-1}\right]^{\rho_h^{L^*}(f)}\frac{\mu_h^{-1}\mu_f(r)}{\left(\sigma_h^{L^*}(f)-\varepsilon\right)}.$$

Now from (9) and (10) it follows for a sequence of values of r tending to infinity that

$$\begin{aligned}\log\mu_h^{-1}\mu_{f\circ g}(r) \quad \leq \quad & \left[\frac{(2\alpha-1)\alpha\beta}{\alpha-1}\right]^{\rho_h^{L^*}(f)}\left(\rho_h^{L^*}(f)+\varepsilon\right)\left(\sigma_g^{L^*}+\varepsilon\right)\frac{\mu_h^{-1}\mu_f(r)}{\left(\sigma_h^{L^*}(f)-\varepsilon\right)}\\ & +\left(\rho_h^{L^*}(f)+\varepsilon\right)L\left(\mu_g(\beta r)\right)+O(1)\end{aligned}$$

$$\begin{aligned}(11)\qquad ie.,\ \frac{\log\mu_h^{-1}\mu_{f\circ g}(r)}{\mu_h^{-1}\mu_f(r)+L\left(\mu_g(\beta r)\right)} \leq & \frac{\frac{\left[\frac{(2\alpha-1)\alpha\beta}{\alpha-1}\right]^{\rho_h^{L^*}(f)}\left(\rho_h^{L^*}(f)+\varepsilon\right)\left(\sigma_g^{L^*}+\varepsilon\right)}{\left(\sigma_h^{L^*}(f)-\varepsilon\right)}}{1+\frac{L(\mu_g(\beta r))}{\mu_h^{-1}\mu_f(r)}}\\ & +\frac{\left(\rho_h^{L^*}(f)+\varepsilon\right)}{1+\frac{\mu_h^{-1}\mu_f(r)}{L(\mu_g(\beta r))}}+\frac{O(1)}{\mu_h^{-1}\mu_f(r)+L\left(\mu_g(\beta r)\right)}.\end{aligned}$$

If $L\left(\mu_g(\beta r)\right)=o\left\{\mu_h^{-1}\mu_f(r)\right\}$ then from (11) we get that

$$\liminf_{r\to\infty}\frac{\log\mu_h^{-1}\mu_{f\circ g}(r)}{\mu_h^{-1}\mu_f(r)+L\left(\mu_g(\beta r)\right)} \leq \frac{\left[\frac{(2\alpha-1)\alpha\beta}{\alpha-1}\right]^{\rho_h^{L^*}(f)}\left(\rho_h^{L^*}(f)+\varepsilon\right)\left(\sigma_g^{L^*}+\varepsilon\right)}{\left(\sigma_h^{L^*}(f)-\varepsilon\right)}.$$

Since $\varepsilon\,(>0)$ is arbitrary it follows from above that

$$\liminf_{r\to\infty}\frac{\log\mu_h^{-1}\mu_{f\circ g}(r)}{\mu_h^{-1}\mu_f(r)+L\left(\mu_g(\beta r)\right)} \leq \left[\frac{\alpha\beta(2\alpha-1)}{\alpha-1}\right]^{\rho_h^{L^*}(f)}\frac{\rho_h^{L^*}(f)\,\sigma_g^{L^*}}{\sigma_h^{L^*}(f)}.$$

Thus the first part of Theorem 11 follows. Again if $\mu_h^{-1}\mu_f(r)=o\{L(\mu_g(\beta r))\}$ then from (11) it follows that

$$\liminf_{r\to\infty}\frac{\log\mu_h^{-1}\mu_{f\circ g}(r)}{\mu_h^{-1}\mu_f(r)+L\left(\mu_g(\beta r)\right)} \leq \left(\rho_h^{L^*}(f)+\varepsilon\right).$$

As $\varepsilon\,(>0)$ is arbitrary, we obtain from above that

$$\liminf_{r\to\infty}\frac{\log\mu_h^{-1}\mu_{f\circ g}(r)}{\mu_h^{-1}\mu_f(r)+L\left(\mu_g(\beta r)\right)} \leq \rho_h^{L^*}(f).$$

Thus the second part of Theorem 11 follows. ■

Theorem 12. *Let f, g and h be any three entire functions with (i) $\rho_h^{L^*}(f) = \rho_g^{L^*}$, (ii) $0 < \sigma_g^{L^*} < \infty$ and (iii) $\sigma_h^{L^*}(f) > 0$. Then*

(a) If $L(M_g(r)) = o\left\{M_h^{-1}M_f(r)\right\}$ then

$$\liminf_{r\to\infty} \frac{\log M_h^{-1}M_{f\circ g}(r)}{M_h^{-1}M_f(r) + L(M_g(r))} \le \frac{\rho_h^{L^*}(f)\cdot\sigma_g^{L^*}}{\sigma_h^{L^*}(f)},$$

and (b) if $M_h^{-1}M_f(r) = o\{L(M_g(r))\}$ then

$$\liminf_{r\to\infty} \frac{\log M_h^{-1}M_{f\circ g}(r)}{M_h^{-1}M_f(r) + L(M_g(r))} \le \rho_h^{L^*}(f).$$

Proof of Theorem 12 is omitted as it can be carried out in the line of Theorem 11 and in view of Lemma 1.

Theorem 13. *Let f, g and h be any three entire functions such that (i) $0 < \tau_h^{L^*}(f) < \infty$, (ii) $0 < \tau_h^{L^*}(f\circ g) < \infty$ and (iii) $\lambda_h^{L^*}(f\circ g) = \lambda_h^{L^*}(f)$. Then for any $\alpha > 1$*

$$\liminf_{r\to\infty} \frac{\mu_h^{-1}\mu_{f\circ g}(r)}{\mu_h^{-1}\mu_f(r)} \le \frac{(2\alpha-1)^{2\lambda_h^{L^*}(f)}\cdot\alpha^{\lambda_h^{L^*}(f)+1}}{(\alpha-1)^{2\lambda_h^{L^*}(f)}}\frac{\tau_h^{L^*}(f\circ g)}{\tau_h^{L^*}(f)}$$

and

$$\frac{(\alpha-1)^{2\lambda_h^{L^*}(f)}}{(2\alpha-1)^{2\lambda_h^{L^*}(f)}\,\alpha^{\lambda_h^{L^*}(f)+1}}\frac{\tau_h^{L^*}(f\circ g)}{\tau_h^{L^*}(f)} \le \limsup_{r\to\infty} \frac{\mu_h^{-1}\mu_{f\circ g}(r)}{\mu_h^{-1}\mu_f(r)}.$$

Proof. From the definition of relative L^*-weak type and in view of Lemma 4 and Lemma 5 we obtain for a sequence of values of r tending to infinity that

$$\mu_h^{-1}\mu_{f\circ g}(r) \le \alpha M_h^{-1}\left[\frac{\alpha}{(\alpha-1)}M_{f\circ g}(r)\right] \le \alpha M_h^{-1}\left[M_{f\circ g}\left(\left(\frac{2\alpha-1}{\alpha-1}\right)r\right)\right]$$

$$\text{(12)}\quad i.e.,\ \mu_h^{-1}\mu_{f\circ g}(r) \le \alpha\left(\tau_h^{L^*}(f\circ g)+\varepsilon\right)\left\{\left(\frac{2\alpha-1}{\alpha-1}\right)re^{L(r)}\right\}^{\lambda_h^{L^*}(f\circ g)}$$

and

$$\text{(13)}\quad \mu_h^{-1}\mu_f(r) \le \alpha\left(\tau_h^{L^*}(f)+\varepsilon\right)\left\{\left(\frac{2\alpha-1}{\alpha-1}\right)re^{L(r)}\right\}^{\lambda_h^{L^*}(f)}.$$

Also we obtain for all sufficiently large values of r that

$$\begin{aligned}\mu_h^{-1}\mu_{f\circ g}(r) &\ge M_h^{-1}M_{f\circ g}\left(\frac{(\alpha-1)}{(2\alpha-1)\alpha}r\right)\\ i.e.,\ \mu_h^{-1}\mu_{f\circ g}(r) &\ge \left(\tau_h^{L^*}(f\circ g)-\varepsilon\right)\left\{\left(\frac{\alpha-1}{(2\alpha-1)\alpha}\right)re^{L(r)}\right\}^{\lambda_h^{L^*}(f\circ g)}\end{aligned}$$

$$(14) \qquad i.e.,\ \mu_h^{-1}\mu_{f\circ g}(r) \geq \left(\frac{\alpha-1}{(2\alpha-1)\alpha}\right)^{\lambda_h^{L^*}(f\circ g)} \times \left(\tau_h^{L^*}(f\circ g)-\varepsilon\right)\left[re^{L(r)}\right]^{\lambda_h^{L^*}(f\circ g)}$$

and

$$(15) \qquad \mu_h^{-1}\mu_f(r) \geq \left(\frac{(\alpha-1)}{(2\alpha-1)\alpha}\right)^{\lambda_h^{L^*}(f)}\left(\tau_h^{L^*}(f)-\varepsilon\right)\left[re^{L(r)}\right]^{\lambda_h^{L^*}(f\circ g)}.$$

Now from (12) and (15) it follows for a sequence of values of r tending to infinity that

$$(16) \qquad \frac{\mu_h^{-1}\mu_{f\circ g}(r)}{\mu_h^{-1}\mu_f(r)} \leq \frac{\alpha\left(\tau_h^{L^*}(f\circ g)+\varepsilon\right)\left\{\left(\frac{2\alpha-1}{\alpha-1}\right)re^{L(r)}\right\}^{\lambda_h^{L^*}(f\circ g)}}{\left(\frac{(\alpha-1)}{(2\alpha-1)\alpha}\right)^{\lambda_h^{L^*}(f)}\left(\tau_h^{L^*}(f)-\varepsilon\right)\left[re^{L(r)}\right]^{\lambda_h^{L^*}(f\circ g)}}.$$

In view of the condition (iii) we get from (16) that

$$\liminf_{r\to\infty}\frac{\mu_h^{-1}\mu_{f\circ g}(r)}{\mu_h^{-1}\mu_f(r)} \leq \frac{\alpha\left(\tau_h^{L^*}(f\circ g)+\varepsilon\right)\left(\frac{2\alpha-1}{\alpha-1}\right)^{\lambda_h^{L^*}(f\circ g)}}{\left(\frac{(\alpha-1)}{(2\alpha-1)\alpha}\right)^{\lambda_h^{L^*}(f)}\left(\tau_h^{L^*}(f)-\varepsilon\right)}.$$

As $\varepsilon\,(>0)$ is arbitrary, it follows from above that

$$(17) \qquad \liminf_{r\to\infty}\frac{\mu_h^{-1}\mu_{f\circ g}(r)}{\mu_h^{-1}\mu_f(r)} \leq \frac{(2\alpha-1)^{2\lambda_h^{L^*}(f)}\,\alpha^{\lambda_h^{L^*}(f)+1}}{(\alpha-1)^{2\lambda_h^{L^*}(f)}}\frac{\tau_h^{L^*}(f\circ g)}{\tau_h^{L^*}(f)}.$$

Again from (13) and (14) we get for a sequence of values of r tending to infinity that

$$(18) \qquad \frac{\mu_h^{-1}\mu_{f\circ g}(r)}{\mu_h^{-1}\mu_f(r)} \geq \frac{\left(\frac{(\alpha-1)}{(2\alpha-1)\alpha}\right)^{\lambda_h^{L^*}(f\circ g)}\left(\tau_h^{L^*}(f\circ g)-\varepsilon\right)\left[re^{L(r)}\right]^{\lambda_h^{L^*}(f\circ g)}}{\alpha\left(\tau_h^{L^*}(f)+\varepsilon\right)\left\{\left(\frac{2\alpha-1}{\alpha-1}\right)re^{L(r)}\right\}^{\lambda_h^{L^*}(f)}}.$$

Since $\lambda_h^{L^*}(f\circ g)=\lambda_h^{L^*}(f)$, we obtain from (18) that

$$\limsup_{r\to\infty}\frac{\mu_h^{-1}\mu_{f\circ g}(r)}{\mu_h^{-1}\mu_f(r)} \geq \frac{\left(\frac{(\alpha-1)}{(2\alpha-1)\alpha}\right)^{\lambda_h^{L^*}(f)}\left(\tau_h^{L^*}(f\circ g)-\varepsilon\right)}{\left(\frac{2\alpha-1}{\alpha-1}\right)^{\lambda_h^{L^*}(f)}\alpha\left(\tau_h^{L^*}(f)+\varepsilon\right)}.$$

As $\varepsilon\,(>0)$ is arbitrary, it follows from above that

$$\limsup_{r\to\infty} \frac{\mu_h^{-1}\mu_{f\circ g}(r)}{\mu_h^{-1}\mu_f(r)} \geq \frac{(\alpha-1)^{2\lambda_h^{L^*}(f)}\,\tau_h^{L^*}(f\circ g)}{(2\alpha-1)^{2\lambda_h^{L^*}(f)}\,\alpha^{\lambda_h^{L^*}(f)+1}\tau_h^{L^*}(f)}. \tag{19}$$

Thus the theorem follows from (17) and (19). ■

In the line of Theorem 13, we may state the following theorem without its proof:

Theorem 14. *Let f, g and h be any three entire functions with (i) $0<\tau_h^{L^*}(g)<\infty$, (ii) $0<\tau_h^{L^*}(f\circ g)<\infty$ and (iii) $\lambda_h^{L^*}(f\circ g)=\lambda_h^{L^*}(g)$. Then for any $\alpha>1$*

$$\liminf_{r\to\infty} \frac{\mu_h^{-1}\mu_{f\circ g}(r)}{\mu_h^{-1}\mu_g(r)} \leq \frac{(2\alpha-1)^{2\lambda_h^{L^*}(g)}\cdot\alpha^{\lambda_h^{L^*}(g)+1}}{(\alpha-1)^{2\lambda_h^{L^*}(g)}}\frac{\tau_h^{L^*}(f\circ g)}{\tau_h^{L^*}(g)}$$

and

$$\frac{(\alpha-1)^{2\lambda_h^{L^*}(g)}}{(2\alpha-1)^{2\lambda_h^{L^*}(g)}\,\alpha^{\lambda_h^{L^*}(g)+1}}\frac{\tau_h^{L^*}(f\circ g)}{\tau_h^{L^*}(g)} \leq \limsup_{r\to\infty} \frac{\mu_h^{-1}\mu_{f\circ g}(r)}{\mu_h^{-1}\mu_g(r)}.$$

Theorem 15. *Let f, g and h be any three entire functions such that (i) $0<\tau_h^{L^*}(f)<\infty$, (ii) $0<\tau_h^{L^*}(f\circ g)<\infty$ and (iii) $\lambda_h^{L^*}(f\circ g)=\lambda_h^{L^*}(f)$. Then*

$$\liminf_{r\to\infty} \frac{M_h^{-1}M_{f\circ g}(r)}{M_h^{-1}M_f(r)} \leq \frac{\tau_h^{L^*}(f\circ g)}{\tau_h^{L^*}(f)} \leq \limsup_{r\to\infty} \frac{M_h^{-1}M_{f\circ g}(r)}{M_h^{-1}M_f(r)}.$$

Theorem 16. *Let f, g and h be any three entire functions with (i) $0<\tau_h^{L^*}(g)<\infty$, (ii) $0<\tau_h^{L^*}(f\circ g)<\infty$ and (iii) $\lambda_h^{L^*}(f\circ g)=\lambda_h^{L^*}(g)$. Then*

$$\liminf_{r\to\infty} \frac{M_h^{-1}M_{f\circ g}(r)}{M_h^{-1}M_g(r)} \leq \frac{\tau_h^{L^*}(f\circ g)}{\tau_h^{L^*}(g)} \leq \limsup_{r\to\infty} \frac{M_h^{-1}M_{f\circ g}(r)}{M_h^{-1}M_g(r)}.$$

The proof of Theorem 15 and Theorem 16 are omitted because those can be carried out in the line of Theorem 13 and Theorem 14 respectively.

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