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**ON NONLINEAR SECOND ORDER  
VOLTERRA-FREDHOLM FUNCTIONAL  
INTEGRODIFFERENTIAL EQUATION WITH  
NONLOCAL CONDITION IN BANACH SPACES**

ABSTRACT. Our purpose in this paper is to study the existence of solution of nonlinear second order mixed functional integrodifferential equation with nonlocal condition in Banach space by employing two different techniques namely the Darbo-Sadovskii's fixed point theorem with Hausdorff's measure of noncompactness and the Leray Schauder Alternative.

KEY WORDS: nonlocal condition; Hausdorff's measure of noncompactness, Darbo-Sadovskii fixed point theorem, Leray-Schauder Alternative.

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## 1. Introduction

Let  $X$  be a Banach space with the norm  $\|\cdot\|$ . Let  $C = C([-r, 0], X)$ ,  $0 < r < \infty$ , be the Banach space of all continuous functions  $x : [-r, 0] \rightarrow X$  with the supremum norm

$$\|x\|_C = \sup \{ \|x(t)\| : -r \leq t \leq 0 \}.$$

We denote the Banach space of all continuous functions  $y : [-r, T] \rightarrow X$  with the supremum norm

$$\|y\|_B = \sup \{ \|y(t)\| : -r \leq t \leq T \}$$

by  $B = C([-r, T], X)$ . For any  $y \in B$  and  $t \in [0, T]$  we denote by  $y_t$  the element of  $C = C([-r, 0], X)$  given by  $y_t(\theta) = y(t + \theta)$  for  $\theta \in [-r, 0]$ . Con-

sider the nonlinear Volterra-Fredholm functional integrodifferential equations with nonlocal condition of the type

$$(1) \quad \frac{d}{dt}[x'(t) - w(t, x_t)] + Ax(t) = f\left(t, x_t, \int_0^t a(t, s)h(s, x_s)ds, \int_0^T b(t, s)k(s, x_s)ds\right), \quad t \in [0, T],$$

$$(2) \quad x(t) + (g(x_{t_1}, \dots, x_{t_p}))(t) = \phi(t), \quad t \in [-r, 0], \quad x'(0) = \xi \in X,$$

where  $0 < t_1 < \dots < t_p \leq T$ ,  $p \in N$ ,  $f : [0, T] \times C \times X \times X \rightarrow X$ ,  $a, b : [0, T] \times [0, T] \rightarrow \mathbb{R}$ ,  $w, h, k : [0, T] \times C \rightarrow X$  are continuous functions,  $g : C^p \rightarrow C$  is given,  $\phi$  is a given element of  $C$ .  $-A$  is the infinitesimal generator of a strongly continuous cosine family  $C(t)$  of bounded linear operators in  $X$ .

The theory of abstract nonlinear second order functional differential and integrodifferential equations has received considerable attention in recent years. Several papers have also appeared for the existence and controllability of solutions of the nonlinear second-order neutral functional differential equations in Banach spaces [1, 2]. In many cases it is advantageous to treat second order abstract differential equations directly rather than to convert them to first order systems. A useful machinery for the study of second order equations is the theory of strongly continuous cosine family [10]. Recently, Rumping Ye and Guowei Zhang[8] studied the existence problem for the neutral functional differential equation of second order of the form

$$\begin{aligned} \frac{d}{dt}[x'(t) + g(t, x_t)] &= Ax(t) + f(t, x_t), \quad t \in J = [0, b], \\ x_0 = \varphi \in B, \quad x'(0) &= z \in X \end{aligned}$$

in Banach spaces using the Hausdorff's measure of noncompactness and Darbo-Sadovskii's fixed point theorem. In [6], Ntouyas and Tsamatos investigated the initial value problem of the form

$$\begin{aligned} x''(t) &= Ax(t) + f(t, x(t), x'(t)), \quad a.e. t \in [0, b], \\ x(0) + g(x) &= x_0, \quad x'(0) = \eta \end{aligned}$$

using the Leray-Schauder Alternative. In both the works  $A$  is the infinitesimal generator of a strongly continuous cosine family  $\{C(t) : t \in R\}$  in a Banach space  $X$  and  $f$  and  $g$  are appropriate given functions. Motivated by this two works we prove the existence of mild solution for the abstract second order neutral functional integrodifferential equations (1)-(2) using the above two techniques.

## 2. Preliminaries

In this section we introduce some definitions, notation, preliminary facts from [3, 4, 10] and hypotheses which are used throughout this paper.

The functions  $a, b$  being continuous on compact domains, there are constants  $\lambda$  and  $\mu$  such that

$$(3) \quad |a(t, s)| \leq \lambda \quad \text{and} \quad |b(t, s)| \leq \mu, \quad \text{for } s, t \in [0, T].$$

**Definition 1.** A one-parameter family  $C(t)$ ,  $t \in R$ , of bounded linear operators in the Banach space  $X$  is called strongly continuous cosine family if and only if

- (a)  $C(s+t) + C(s-t) = 2C(s)C(t)$  for all  $s, t \in R$ ;
- (b)  $C(0) = I$ ;
- (c)  $C(t)x$  is continuous in  $t \in R$  for each fixed  $x \in X$ .

We denote by  $S(t)$ ,  $t \in R$ , the sine family associated to  $C(t)$ ,  $t \in R$  and it is defined as

$$S(t)x = \int_0^t C(s)x ds, \quad x \in X, \quad t \in R$$

The infinitesimal generator of a strongly continuous cosine family  $C(t)$ ,  $t \in R$ , is the operator  $A$  defined by

$$Ax = \frac{d^2}{dt^2} C(0)x,$$

where  $D(A) = \{x \in X : C(t)x \text{ is a twice continuously differentiable function of } t\}$ . We denote by  $N$  and  $\tilde{N}$  certain constants such that

$$(4) \quad \|C(t)\| \leq N \quad \text{and} \quad \|S(t)\| \leq \tilde{N} \quad \text{for every } t \in [0, T].$$

**Definition 2.** The Hausdorff's measure of noncompactness  $\chi_Y$  is defined by  $\chi_Y(S) = \inf\{r > 0, S \text{ can be covered by finite number of balls with radiir } r\}$  for bounded set  $S$  in any Banach space  $Y$ .

**Lemma 1** ([3]). Let  $Y$  be a real Banach space and  $B, C \subseteq Y$  be bounded, then the following properties are satisfied:

- (a)  $B$  is precompact if and only if  $\chi_Y(B) = 0$ ;
- (b)  $\chi_Y(B) = \chi_Y(\bar{B}) = \chi_Y(\text{conv}B)$  where  $\bar{B}$  and  $\text{conv}B$  mean the closure and convex hull of  $B$  respectively;
- (c)  $\chi_Y(B) \leq \chi_Y(C)$  when  $B \subseteq C$ ;
- (d)  $\chi_Y(B + C) \leq \chi_Y(B) + \chi_Y(C)$  where  $B + C = \{x + y; x \in B, y \in C\}$ ;
- (e)  $\chi_Y(B \cup C) \leq \max\{\chi_Y(B), \chi_Y(C)\}$ ;
- (f)  $\chi_Y(\lambda B) = |\lambda| \chi_Y(B)$  for any  $\lambda \in R$ ;

(g) If the map  $Q : D(Q) \subseteq Y \rightarrow Z$  is Lipschitz continuous with constant  $k$  then  $\chi_Z(Q(B)) \leq k \chi_Y(B)$  for any bounded set  $B \subseteq D(Q)$ , where  $Z$  is a Banach space;

(h)  $\chi_Y(B) = \inf\{d_Y(B, C); C \subseteq Y \text{ be precompact}\} = \inf\{d_Y(B, C); C \subseteq Y \text{ be finite valued}\}$ , where  $d_Y(B, C)$  means the nonsymmetric (or symmetric)

Hausdorff distance between  $B$  and  $C$  in  $Y$ ;

(i) If  $\{W_n\}_{n=1}^\infty$  is a decreasing sequence of bounded, closed nonempty subsets of  $Y$  and  $\lim_{n \rightarrow +\infty} \chi_Y(W_n) = 0$ , then  $\bigcap_{n=1}^{+\infty} W_n$  is nonempty and compact in  $Y$ .

**Definition 3.** The map  $Q : W \subseteq Y \rightarrow Y$  is said to be a  $\chi_Y$ -contraction if there exists a positive constant  $k < 1$  such that  $\chi_Y(Q(S)) \leq k \chi_Y(S)$  for any bounded closed subset  $S \subseteq W$  where  $Y$  is a Banach space.

The following lemma known as Darbo-Sadovskii fixed point theorem given in [3] is used while proving Theorem 3.1.

**Lemma 2** ([3]). If  $W \subseteq Y$  is bounded, closed and convex, the continuous map  $Q : W \rightarrow W$  is a  $\chi_Y$ -contraction, then the map  $Q$  has at least one fixed point in  $W$ .

In this paper we use the notations  $\chi$  and  $\chi_B$  to denote the Hausdorff's measure of noncompactness of the Banach space  $X$  and that of the Banach space  $B = C([-r, T], X)$  respectively.

**Lemma 3** ([3]). If  $W \subseteq C([a, b], X)$  is bounded, then

$$\chi(W(t)) \leq \chi_C(W)$$

for all  $t \in [a, b]$ , where  $W(t) = \{u(t); u \in W\} \subseteq X$ . Furthermore if  $W$  is equicontinuous on  $[a, b]$ , then  $\chi(W(t))$  is continuous on  $[a, b]$  and

$$\chi_C(W) = \sup \{\chi(W(t)), t \in [a, b]\}.$$

**Lemma 4** ([3]). If  $W \subseteq C([a, b]; X)$  is bounded and equicontinuous, then  $\chi(W(s))$  is continuous and

$$\chi\left(\int_a^t W(s)ds\right) \leq \int_a^t \chi(W(s))ds$$

for all  $t \in [a, b]$ , where  $\int_a^t W(s)ds = \{\int_a^t x(s)ds : x \in W\}$ .

**Lemma 5.** If the semigroup  $S(t)$  is equicontinuous and  $\eta \in L(0, b; R^+)$ , then the set  $\{\int_0^t S(t-s)u(s)ds, \|u(s)\| \leq \eta(s) \text{ for a.e. } s \in [0, b]\}$  is equicontinuous for  $t \in [0, b]$ .

We prove Theorem 2 using the following lemma known as Leray-Schauder Alternative given in [4]. The advantage of using this lemma lies in the fact that we do not claim conditions which imply  $FU \subset U$ , where  $U$  is a set and  $F$  is an operator.

**Lemma 6.** *Let  $S$  be a convex subset of a normed linear space  $E$  and assume  $0 \in S$ . Let  $F : S \rightarrow S$  be a completely continuous operator and let*

$$\varepsilon(F) = \{x \in S : x = \nu Fx \text{ for some } 0 < \nu < 1\}.$$

*Then either  $\varepsilon(F)$  is unbounded or  $F$  has a fixed point.*

**Definition 4.** *A function  $x \in C([-r, T], X)$  is said to be a mild solution of the nonlocal problem (1)-(2) if it satisfies the following:*

$$\begin{aligned} (5) \quad x(t) &= C(t)[\phi(0) - (g(x_{t_1}, \dots, x_{t_p}))(0)] + S(t)[\xi - w(0, x_0)] \\ &\quad + \int_0^t C(t-s)w(s, x_s)ds \\ &\quad + \int_0^t S(t-s)f\left(s, x_s, \int_0^s a(s, \tau)h(\tau, x_\tau)d\tau, \right. \\ &\quad \left. \int_0^T b(s, \tau)k(\tau, x_\tau)d\tau\right)ds, \quad t \in [0, T] \\ (6) \quad x(t) &+ (g(x_{t_1}, \dots, x_{t_p}))(t) = \phi(t), \quad t \in [-r, 0]. \end{aligned}$$

We shall make use of the following hypotheses to prove our main results:

( $H_1$ ) There exists a continuous function  $l : [0, T] \rightarrow \mathbb{R}_+ = [0, \infty)$  such that

$$\|f(t, \psi, x, y)\| \leq l(t)(\|\psi\|_C + \|x\| + \|y\|)$$

for every  $t \in [0, T]$ ,  $\psi \in C$  and  $x, y \in X$ .

( $H_2$ ) There exists a continuous function  $p : [0, T] \rightarrow \mathbb{R}_+$  such that

$$\|h(t, \psi)\| \leq p(t)H(\|\psi\|_C)$$

for every  $t \in [0, T]$ ,  $\psi \in C$  where  $H : \mathbb{R}_+ \rightarrow (0, \infty)$  is a continuous nondecreasing function.

( $H_3$ ) There exists a continuous function  $q : [0, T] \rightarrow \mathbb{R}_+$  such that

$$\|k(t, \psi)\| \leq q(t)K(\|\psi\|_C)$$

for every  $t \in [0, T]$ ,  $\psi \in C$  where  $K : \mathbb{R}_+ \rightarrow (0, \infty)$  is a continuous nondecreasing function.

( $H_4$ ) For each  $t \in [0, T]$  the function  $f(t, \dots) : C \times X \times X \rightarrow X$  is continuous and for each  $(\psi, x, y) \in C \times X \times X$  the function  $f(\cdot, \psi, x, y) : [0, T] \rightarrow X$  is strongly measurable.

( $H_5$ ) For each  $t \in [0, T]$  the functions  $h(t, \cdot), k(t, \cdot) : C \rightarrow X$  are continuous and for each  $\psi \in C$  the functions  $h(\cdot, \psi), k(\cdot, \psi) : [0, T] \rightarrow X$  are strongly measurable.

( $H_6$ ) There exists a constant  $\rho > 0$  such that

$$\|(g(u_{t_1}, \dots, u_{t_p}))(s) - (g(v_{t_1}, \dots, v_{t_p}))(s)\| \leq \rho \|u - v\|_B$$

for  $u, v \in B, s \in [-r, 0]$ .

( $H_7$ ) There exists a constant  $G$  such that

$$(7) \quad G = \max_{y \in B} \|g(y_{t_1}, \dots, y_{t_p})\|.$$

( $H_8$ ) There exist positive constants  $c_1, c_2$  and  $V$  such that

$$(8) \quad \|w(t, \psi)\| \leq c_1 \|\psi\| + c_2,$$

$$(9) \quad \|w(t, \psi_1) - w(t, \psi_2)\| \leq V \|\psi_1 - \psi_2\|_C$$

for  $t \in [0, T]$  and  $\psi, \psi_1, \psi_2 \in C$ .

( $H_9$ ) Assume that

$$(\tilde{N} + NT) c_1 + \tilde{N} M^* T \left[ 1 + M^* T \liminf_{m \rightarrow \infty} \left( \frac{H(m)}{m} + \frac{K(m)}{m} \right) \right] < 1$$

where

$$(10) \quad M^* = \sup \{M(t), t \in [0, T]\}$$

$$(11) \quad M(t) = \max \{l(t), \lambda p(t), \mu q(t)\} \text{ for each } t \in [0, T].$$

( $H_{10}$ ) There exists integrable functions  $\eta, \eta_1, \eta_2 : [0, T] \rightarrow [0, \infty)$  such that for any bounded set  $W \subset C([-r, T], X)$  and  $s \in [0, T]$  we have

$$\begin{aligned} & \chi\left(S(t-s) f\left(s, W_s, \int_0^s a(s, \tau) h(\tau, W_\tau) d\tau, \int_0^T b(s, \tau) k(\tau, W_\tau) d\tau\right)\right) \\ & \leq \eta(s) \left( \sup_{-r \leq \theta \leq 0} \chi(W(s+\theta)) \right. \\ & \quad + \int_0^s |a(s, \tau)| \eta_1(\tau) \sup_{-r \leq \theta \leq 0} \chi(W(\tau+\theta)) d\tau \\ & \quad \left. + \int_0^T |b(s, \tau)| \eta_2(\tau) \sup_{-r \leq \theta \leq 0} \chi(W(\tau+\theta)) d\tau \right) \end{aligned}$$

(H<sub>11</sub>) Assume that  $C(t), t > 0$  is compact.

(H<sub>12</sub>) The range of  $g$  consists of Lipschitz continuous functions only and

$$L = \max_{y \in B} L_y \text{ where } L_y = \text{Lipschitz constant of } g(y_{t_1}, \dots, y_{t_p}) \in C.$$

(H<sub>13</sub>) For each  $t \in [-r, 0]$  the set

$$\{\phi(t) - (g(y_{t_1}, \dots, y_{t_p}))(t) : y \in B_m\}$$

is precompact in  $X$ , where  $B_m = \{y \in B : \|y\| \leq m\}$ , for every positive integer  $m$ .

(H<sub>14</sub>)  $\phi$  is Lipschitz continuous on  $[-r, 0]$  with Lipschitz constant  $\sigma$  i.e.

$$\|\phi(s_1) - \phi(s_2)\| \leq \sigma |s_1 - s_2| \text{ for } s_1, s_2 \in [-r, 0].$$

## 2. Existence of mild solution

**Theorem 1.** *Suppose that the hypotheses (H<sub>1</sub>) – (H<sub>11</sub>) hold. Then the nonlocal problem (1)-(2) has a mild solution  $x$  on  $[-r, T]$  if*

$$(12) \quad \left\{ \rho_1 + \int_0^T \eta(s) \left[ 1 + \int_0^s \lambda \eta_1(\tau) d\tau + \int_0^T \mu \eta_2(\tau) d\tau \right] ds \right\} < 1$$

where the constant term

$$(13) \quad \rho_1 = (N + 1) \rho + (\tilde{N} + NT) V.$$

**Proof.** We prove the existence of mild solution of the nonlinear mixed integrodifferential equations (1)-(2), by using the Darbo-Sadovskii fixed point theorem and the Hausdorff's measure of noncompactness. Consider the bounded set  $B_m = \{y \in B : \|y\| \leq m\}$  for each  $m \in N$  (the set of all positive integers). Define an operator  $F : B = C([-r, T], X) \rightarrow B$  by  $F = F_1 + F_2$

$$(14) \quad (F_1 x)(t) = \begin{cases} \phi(t) - (g(x_{t_1}, \dots, x_{t_p}))(t), & -r \leq t \leq 0 \\ C(t) [\phi(0) - (g(x_{t_1}, \dots, x_{t_p}))(0)] \\ \quad + S(t) [\xi - w(0, x_0)] \\ \quad + \int_0^t C(t-s) w(s, x_s) ds & 0 \leq t \leq T \end{cases}$$

$$(15) \quad (F_2 x)(t) = \begin{cases} 0, & -r \leq t \leq 0 \\ \int_0^t S(t-s) f \left( s, x_s, \int_0^s a(s, \tau) h(\tau, x_\tau) d\tau, \right. \\ \quad \left. \int_0^T b(s, \tau) k(\tau, x_\tau) d\tau \right) ds & 0 \leq t \leq T \end{cases}$$

From the definition of  $F$  it follows that the fixed point of  $F$  is the mild solution of the nonlocal problem (1) – (2). We first show that  $F : B \rightarrow B$  is continuous. Let  $\{u_n\}$  be a sequence of elements of  $B$  converging to  $u$  in  $B$ . Consider the case when  $t \in [-r, 0]$ , then using hypothesis  $(H_6)$  we have

$$(16) \quad \|(Fu_n)(t) - (Fu)(t)\| \leq \rho \|u - u_n\| \rightarrow 0 \text{ as } n \rightarrow \infty.$$

Now let  $t \in [0, T]$  then using hypotheses  $(H_4)$  and  $(H_5)$  we have

$$\begin{aligned} & f\left(t, u_{n_t}, \int_0^t a(t, s)h(s, u_{n_s})ds, \int_0^T b(t, s)k(s, u_{n_s})\right) \\ & \rightarrow f\left(t, u_t, \int_0^t a(t, s)h(s, u_s)ds, \int_0^T b(t, s)k(s, u_s)\right). \end{aligned}$$

Using the Dominated Convergence Theorem, condition (4), hypotheses  $(H_6)$ ,  $(H_8)$  and the fact that for every  $t \in (0, T]$ ,  $\|x_t\|_C \leq \|x\|_B$ , we have

$$\begin{aligned} (17) \quad & \|(Fu_n)(t) - (Fu)(t)\| = \|C(t)[\phi(0)] - C(t)[(g(u_{n_{t_1}}, \dots, u_{n_{t_p}}))(0)] \\ & + S(t)[\xi] - S(t)[w(0, u_{n_0})] + \int_0^t C(t-s)w(s, u_{n_s})ds \\ & + \int_0^t S(t-s)f\left(s, u_{n_s}, \int_0^s a(s, \tau)h(\tau, u_{n_\tau})d\tau, \int_0^T b(s, \tau)k(\tau, u_{n_\tau})d\tau\right)ds \\ & - C(t)[\phi(0)] + C(t)[(g(u_{t_1}, \dots, u_{t_p}))(0)] \\ & - S(t)[\xi] + S(t)[w(0, u_0)] - \int_0^t C(t-s)w(s, u_s)ds \\ & - \int_0^t S(t-s)f\left(s, u_s, \int_0^s a(s, \tau)h(\tau, u_\tau)d\tau, \int_0^T b(s, \tau)k(\tau, u_\tau)d\tau\right)ds\| \\ & \leq \|C(t)[(g(u_{t_1}, \dots, u_{t_p}))(0) - (g(u_{n_{t_1}}, \dots, u_{n_{t_p}}))(0)]\| \\ & + \|S(t)[w(0, u_0) - w(0, u_{n_0})]\| \\ & + \left\| \int_0^t C(t-s)[w(s, u_{n_s}) - w(s, u_s)]ds \right\| \\ & + \left\| \int_0^t S(t-s)\left[f\left(s, u_{n_s}, \int_0^s a(s, \tau)h(\tau, u_{n_\tau})d\tau, \int_0^T b(s, \tau)k(\tau, u_{n_\tau})d\tau\right) \right. \right. \\ & \left. \left. - f\left(s, u_s, \int_0^s a(s, \tau)h(\tau, u_\tau)d\tau, \int_0^T b(s, \tau)k(\tau, u_\tau)d\tau\right)\right]ds \right\| \\ & \leq N\rho\|u_n - u\|_B + \tilde{N}V\|u_0 - u_{n_0}\|_C + NTV\|u_n - u\|_B \\ & + \tilde{N} \int_0^t \left\| f\left(s, u_{n_s}, \int_0^s a(s, \tau)h(\tau, u_{n_\tau})d\tau, \int_0^T b(s, \tau)k(\tau, u_{n_\tau})d\tau\right) \right. \\ & \left. - f\left(s, u_s, \int_0^s a(s, \tau)h(\tau, u_\tau)d\tau, \int_0^T b(s, \tau)k(\tau, u_\tau)d\tau\right) \right\| ds \Big\} \rightarrow 0 \\ & \text{as } n \rightarrow \infty. \end{aligned}$$



Since  $\|(Fu_n) - (Fu)\|_B = \sup_{t \in [-r, T]} \|(Fu_n)(t) - (Fu)(t)\|$ , inequalities (16) and (17) imply  $Fu_n \rightarrow Fu$  in  $B$  as  $u_n \rightarrow u$  in  $B$ . Therefore  $F$  is continuous.

We shall show that  $F$  is a  $\chi_B$ -contraction on some bounded closed convex subset  $B_m \subseteq B = (C[-r, T], X)$ . And then by using Darbo-Sadovskii's fixed point theorem we get a fixed point of  $F$ . Firstly by using the method of contradiction we obtain a  $m \in N$  such that  $F_{B_m} \subseteq B_m$ . Suppose that for each  $m \in N$  there is a  $y^m \in B_m$  and  $t^m \in [-r, T]$  such that  $\|(Fy^m)(t^m)\| > m$ . If  $t^m \in [-r, 0]$  then using hypothesis  $(H_7)$  we obtain

$$(18) \quad m < \|(Fy^m)(t^m)\| \leq \|\phi(t^m)\| + \|(g(y_{t_1}^m, \dots, y_{t_p}^m))(t^m)\| \leq c + G.$$

where  $c$  denotes  $\|\phi\|_C$ . We also know that if  $\|y^m\|_B \leq m$  then

$$(19) \quad \|y_t^m\|_C \leq m \text{ for all } t \in [0, T]$$

Using hypotheses  $(H_1) - (H_3)$  and conditions (3), (7), (8), (11), (10) and (19) for the case when  $t^m \in [0, T]$  we obtain

$$(20) \quad m < \|(Fy^m)(t^m)\| \leq N [\|\phi(0)\| + \|(g(y_{t_1}^m, \dots, y_{t_p}^m))(0)\|] \\ + \tilde{N} [\|\xi\| + \|w(0, y_0^m)\|] + \int_0^{t^m} \|C(t^m - s)w(s, y_s^m)\| ds \\ + \tilde{N} \int_0^{t^m} \left\| f\left(s, y_s^m, \int_0^s a(s, \tau)h(\tau, y_\tau^m) d\tau, \int_0^T b(s, \tau)k(\tau, y_\tau^m) d\tau\right) \right\| ds \\ \leq N(c + G) + \tilde{N} [\|\xi\| + c_1m + c_2] + N(c_1m + c_2)T + \tilde{N} \int_0^{t^m} l(s) \\ \left( \|y_s^m\|_C + \int_0^s |a(s, \tau)| \|h(\tau, y_\tau^m)\| d\tau + \int_0^T |b(s, \tau)| \|k(\tau, y_\tau^m)\| d\tau \right) ds \\ \leq N(c + G) + \tilde{N} [\|\xi\| + c_1m + c_2] + N(c_1m + c_2)T + \tilde{N} \int_0^{t^m} l(s) \\ \left( \|y_s^m\|_C + \int_0^s \lambda p(\tau) H(\|y_\tau^m\|_C) d\tau + \int_0^T \mu q(\tau) K(\|y_\tau^m\|_C) d\tau \right) ds \\ \leq N(c + G) + \tilde{N}\|\xi\| + (\tilde{N} + NT)(c_1m + c_2) \\ + \tilde{N} \int_0^T M(s) \left( m + \int_0^s M(\tau)H(m) d\tau + \int_0^T M(\tau)K(m) d\tau \right) ds \\ \leq N(c + G) + \tilde{N}\|\xi\| + (\tilde{N} + NT)(c_1m + c_2) \\ + \tilde{N} \int_0^T M^* \left( m + M^*H(m)T + M^*K(m)T \right) ds$$

Now we combine (18) and (20) so that we obtain

$$(21) \quad m < (N + 1)(c + G) + \tilde{N}\|\xi\| + (\tilde{N} + NT)(c_1m + c_2) \\ + \tilde{N}M^*T \left( m + M^*H(m)T + M^*K(m)T \right).$$

Dividing by  $m$  on both sides of (21) we obtain

$$(22) \quad 1 < \frac{(N+1)(c+G) + \tilde{N}\|\xi\|}{m} + (\tilde{N} + NT) \left( c_1 + \frac{c_2}{m} \right) \\ + \tilde{N}M^*T \left( 1 + M^*T \frac{H(m)}{m} + M^*T \frac{K(m)}{m} \right).$$

Now taking  $\liminf$  as  $m \rightarrow \infty$  on both sides of (22) we get

$$1 < (\tilde{N} + NT)c_1 + \tilde{N}M^*T \left[ 1 + M^*T \liminf_{m \rightarrow \infty} \left( \frac{H(m)}{m} + \frac{K(m)}{m} \right) \right].$$

which contradicts the hypothesis  $(H_9)$ . Thus there is a  $m \in N$  such that  $F_{B_m} \subseteq B_m$ . Hereafter we will consider the restriction of  $F$  on this  $B_m$ .

Now we show that  $F_1$  is Lipschitz continuous. Let  $x, y \in B_m$  then using hypothesis  $(H_6)$  we have for  $t \in [-r, 0]$

$$(23) \quad \|(F_1x)(t) - (F_1y)(t)\| \\ = \|(g(y_{t_1}, \dots, y_{t_p}))(t) - (g(x_{t_1}, \dots, x_{t_p}))(t)\| \leq \rho \|y - x\|_B.$$

Now using hypothesis  $(H_6)$  and conditions (4), (9) for  $t \in [0, T]$  we have

$$(24) \quad \|(F_1x)(t) - (F_1y)(t)\| = \|C(t)[(g(y_{t_1}, \dots, y_{t_p}))(0) \\ - (g(x_{t_1}, \dots, x_{t_p}))(0)] + S(t)[w(0, y_0) - w(0, x_0)] \\ + \int_0^t C(t-s)[w(s, x_s) - w(s, y_s)] ds\| \\ \leq [N\rho + \tilde{N}V + NVT] \|x - y\|_B$$

Thus in view of (23) and (24) we obtain

$$\|(F_1x)(t) - (F_1y)(t)\| \leq [(N+1)\rho + \tilde{N}V + NVT] \|x - y\|_B.$$

for all  $t \in [-r, T]$  and  $x, y \in B_m$ . Consequently using (13) we get

$$\|(F_1x) - (F_1y)\| \leq \rho_1 \|x - y\|_B.$$

Thus  $F_1$  is Lipschitzian with Lipschitz constant  $\rho_1$ . Hence using Lemma 1(g) we now have

$$(25) \quad \chi_B(F_1W) \leq \rho_1 \chi_B(W)$$

for any bounded set  $W \subseteq B_m$ .

Further let  $W$  be any bounded subset of  $B_m$ . We first show that  $F_2W$  is bounded. Let  $y \in W \subseteq B_m$  then  $\|y\|_B \leq m$  and so  $\|y_t\|_C \leq m$ ,  $t \in [0, T]$ . Let  $t \in [-r, 0]$  and  $y \in B_m$  then from the definition of  $F_2$  we have

$$\|(F_2y)(t)\| = 0$$

Now for  $t \in [0, T]$  and  $y \in B_m$  we get

$$\begin{aligned} (26) \quad \|(F_2y)(t)\| &\leq \int_0^t \|S(t-s)\| \left\| f\left(s, x_s, \int_0^s a(s, \tau)h(\tau, x_\tau)d\tau, \int_0^T b(s, \tau)k(\tau, x_\tau)d\tau\right) \right\| ds \\ &\leq \tilde{N} \left[ \int_0^t l(s) \left( \|y_s\|_C + \left\| \int_0^s a(s, \tau)h(\tau, y_\tau)d\tau + \int_0^T b(s, \tau)k(\tau, y_\tau)d\tau \right\| \right) ds \right] \\ &\leq \tilde{N} \int_0^t M(s) \left( m + \int_0^s M(\tau)H(m)d\tau + \int_0^T M(\tau)K(m)d\tau \right) ds \\ &\leq \tilde{N} T M^* \left( m + \frac{TM^*H(m)}{2} + TM^*K(m) \right). \end{aligned}$$

The R.H.S. of the inequality (26) being constant we conclude that the set  $\{(F_2y)(t) : y \in W, -r \leq t \leq T\}$  is bounded in  $X$  and hence  $F_2W$  is bounded in  $B$ . Now we prove that  $F_2W$  is equicontinuous. For this let  $y \in W$ ,  $s_1, s_2 \in [-r, T]$  and consider the following cases:

**Case 1.** Suppose  $0 \leq s_1 \leq s_2 \leq T$  then using hypothesis  $(H_1) - (H_3)$  and conditions(10), (11) and (4), we get

$$\begin{aligned} \|(F_2y)(s_2) - (F_2y)(s_1)\| &\leq \int_0^{s_1} \|S(s_2-s) - S(s_1-s)\| \\ &\times \left\| f\left(s, y_s, \int_0^s a(s, \tau)h(\tau, y_\tau)d\tau, \int_0^T b(s, \tau)k(\tau, y_\tau)d\tau\right) \right\| ds \\ &+ \int_{s_1}^{s_2} \|S(s_2-s)\| \left\| f\left(s, y_s, \int_0^s a(s, \tau)h(\tau, y_\tau)d\tau, \int_0^T b(s, \tau)k(\tau, y_\tau)d\tau\right) \right\| ds \\ &\leq \int_0^{s_1} \| [S(s_2-s) - S(s_1-s)] \| \\ &\times \left[ M(s) \left( m + \int_0^s M(\tau)H(m)d\tau + \int_0^T M(\tau)K(m)d\tau \right) \right] ds \end{aligned}$$

$$\begin{aligned}
& + \tilde{N} \int_{s_1}^{s_2} M(s) \left( m + \int_0^s M(\tau)H(m)d\tau + \int_0^T M(\tau)K(m)d\tau \right) ds \\
& \leq \int_0^{s_1} \| [S(s_2 - s) - S(s_1 - s)] \| \left[ M^* \left( m + M^*H(m)s + M^*K(m)T \right) \right] ds \\
& \quad + \tilde{N} \int_{s_1}^{s_2} M^* \left( m + M^*H(m) \int_0^s d\tau \right. \\
& \quad \left. + M^*K(m) \int_0^T d\tau \right) ds \\
& \leq \gamma \int_0^{s_1} \| S(s_2 - s) - S(s_1 - s) \| ds + \tilde{N}\gamma |s_2 - s_1| \} \rightarrow 0 \text{ as } s_2 \rightarrow s_1,
\end{aligned}$$

where  $\gamma = [M^*(m + M^*H(m)T + M^*K(m)T0)]$ . Now from hypothesis  $(H_{11})$ ,  $C(t)$  for  $t > 0$  is compact and therefore  $S(t), t > 0$  is also compact (see [9]). The compactness of  $S(t), t > 0$  implies the continuity in the uniform operator topology. Therefore the right hand side of above equation tends to zero as  $s_2 \rightarrow s_1$ .

**Case 2.** Suppose  $-r \leq s_1 \leq 0 \leq s_2 \leq T$  then we get

$$\begin{aligned}
& \| (F_2y)(s_2) - (F_2y)(s_1) \| = \left\| \int_0^{s_2} S(s_2 - s) \right. \\
& \quad \left. \times f \left( s, y_s, \int_0^s a(s, \tau)h(\tau, y_\tau)d\tau, \int_0^T b(s, \tau)k(\tau, y_\tau)d\tau \right) ds \right\|
\end{aligned}$$

Now proceeding as in Case 1 for the integral on the right hand side of above equality we further obtain

$$\| (F_2y)(s_2) - (F_2y)(s_1) \| \leq \tilde{N}\gamma |s_2 - s_1| \rightarrow 0 \text{ as } s_2 \rightarrow 0_+ \text{ and } s_1 \rightarrow 0_-.$$

**Case 3.** Suppose  $-r \leq s_1 \leq s_2 \leq 0$ . In this case we have

$$(27) \quad \| (F_2y)(s_2) - (F_2y)(s_1) \| = 0.$$

Thus cases (1) – (3) imply that  $\| (F_2y)(s_2) - (F_2y)(s_1) \| \rightarrow 0$  as  $s_1 \rightarrow s_2$ , for all  $s_1, s_2 \in [-r, T]$ . Thus we conclude that  $F_2W$  is an equicontinuous family of functions.

Further for a bounded subset  $W$  of  $B_m$  we define the notations  $W(t) = \{x(t); x \in W\} \subseteq X$  and  $W_t = \{x_t; x \in W\} \subseteq C([-r, 0], X)$ . Also  $S(t)$  is equicontinuous. Now using Lemma 1, Lemma 3 – 5 and hypothesis  $(H_{10})$  we obtain

$$\begin{aligned}
(28) \quad \chi_B(F_2W) & = \sup_{-r \leq t \leq T} \chi(F_2W(t)) \\
& = \sup_{0 \leq t \leq T} \chi \left( \int_0^t S(t-s) f \left( s, W_s, \int_0^s a(s, \tau)h(\tau, W_\tau)d\tau, \right. \right.
\end{aligned}$$

$$\begin{aligned}
& \int_0^T b(s, \tau) k(\tau, W_\tau) d\tau \Big) ds \\
= & \sup_{0 \leq t \leq T} \int_0^t \chi(S(t-s) f \left( s, W_s, \int_0^s a(s, \tau) h(\tau, W_\tau) d\tau, \right. \\
& \left. \int_0^T b(s, \tau) k(\tau, W_\tau) d\tau \right)) ds \\
\leq & \sup_{0 \leq t \leq T} \int_0^t \eta(s) \left( \sup_{-r \leq \theta \leq 0} \chi(W(s+\theta)) \right. \\
& + \int_0^s |a(s, \tau)| \eta_1(\tau) \sup_{-r \leq \theta \leq 0} \chi(W(\tau+\theta)) d\tau \\
& \left. + \int_0^T |b(s, \tau)| \eta_2(\tau) \sup_{-r \leq \theta \leq 0} \chi(W(\tau+\theta)) d\tau \right) ds \\
\leq & \sup_{0 \leq t \leq T} \int_0^t \eta(s) \left( \sup_{s-r \leq s+\theta \leq s} \chi(W(s+\theta)) \right. \\
& + \int_0^s \lambda \eta_1(\tau) \sup_{\tau-r \leq \tau+\theta \leq \tau} \chi(W(\tau+\theta)) d\tau \\
& \left. + \int_0^T \mu \eta_2(\tau) \sup_{\tau-r \leq \tau+\theta \leq \tau} \chi(W(\tau+\theta)) d\tau \right) ds \\
\leq & \sup_{0 \leq t \leq T} \int_0^t \eta(s) \left( \chi_B(W) + \int_0^s \lambda \eta_1(\tau) \chi_B(W) d\tau \right. \\
& \left. + \int_0^T \mu \eta_2(\tau) \chi_B(W) d\tau \right) ds \\
\leq & \chi_B(W) \int_0^T \eta(s) \left( 1 + \int_0^s \lambda \eta_1(\tau) d\tau + \int_0^T \mu \eta_2(\tau) d\tau \right) ds
\end{aligned}$$

Therefore using (12), (25) and (28) we obtain

$$\begin{aligned}
(29) \quad \chi_B(FW) & \leq \chi_B(F_1W) + \chi_B(F_2W) \\
& \leq \left( \rho_1 + \int_0^T \eta(s) \right. \\
& \quad \left. \times [1 + \int_0^s \lambda \eta_1(\tau) d\tau + \int_0^T \mu \eta_2(\tau) d\tau] ds \right) \chi_B(W) \\
& < \chi_B(W)
\end{aligned}$$

for any bounded subset  $W$  of  $B_m$ .

Hence  $F$  is a  $\chi_B$ -contraction. Now applying lemma 2.5 we get a fixed point  $x$  of  $F$  in  $B_m$ . This  $x$  is a mild solution of (1)-(2). The proof of the theorem is complete.  $\blacksquare$

**Theorem 2.** *Suppose that the hypotheses  $(H_1) - (H_8)$  and  $(H_{11}) - (H_{14})$  holds. Then the nonlocal problem (1)-(2) has a mild solution  $x$  on  $[-r, T]$  if  $T$  satisfies*

$$(30) \quad \int_0^T M(s)ds < \int_\alpha^\infty \frac{1 - c_1(\tilde{N} + NT)ds}{\tilde{N}[s + TM^*H(s) + \beta]}$$

where  $M(t)$  and  $M^*$  are as defined in (11) and (10) and

$$\alpha = \frac{N[c + G] + \tilde{N}[\|\xi\| + c_2] + Nc_2T}{1 - c_1[\tilde{N} + NT]} + (c + G)$$

with  $c = \|\phi\|_C$  and a constant  $\beta$  is such that

$$(31) \quad \int_0^T M(s)K(J(s))ds \leq \beta$$

for any continuous function  $J : [0, T] \rightarrow \mathbb{R}_+$ .

**Proof.** To prove the existence of a mild solution of the nonlocal problem (1) – (2) we apply the Leray-Schauder Alternative to the operator equation

$$x(t) = \nu Fx(t), \quad 0 < \nu < 1$$

where we define the operator  $F : B = C([-r, T], X) \rightarrow B$  by

$$(32) \quad (Fx)(t) = \begin{cases} \phi(t) - (g(x_{t_1}, \dots, x_{t_p}))(t), & -r \leq t \leq 0 \\ C(t)[\phi(0) - (g(x_{t_1}, \dots, x_{t_p}))(0)] \\ \quad + S(t)[\xi - w(0, x_0)] + \int_0^t C(t-s)w(s, x_s)ds \\ \quad + \int_0^t S(t-s)f\left(s, x_s, \int_0^s a(s, \tau)h(\tau, x_\tau)d\tau, \right. \\ \quad \left. \int_0^T b(s, \tau)k(\tau, x_\tau)d\tau\right)ds, & 0 \leq t \leq T \end{cases}$$

so that for  $t \in [0, T]$ , we get

$$(33) \quad x(t) = \nu \left\{ C(t)[\phi(0) - (g(x_{t_1}, \dots, x_{t_p}))(0)] \right. \\ \quad + S(t)[\xi - w(0, x_0)] + \int_0^t C(t-s)w(s, x_s)ds \\ \quad + \int_0^t S(t-s)f\left(s, x_s, \int_0^s a(s, \tau)h(\tau, x_\tau)d\tau, \right. \\ \quad \left. \int_0^T b(s, \tau)k(\tau, x_\tau)d\tau\right)ds \left. \right\}.$$

Using hypotheses  $(H_1) - (H_3)$  and conditions (3), (4), (7) and (8) in (33) we obtain, for  $t \in [0, T]$ ,

$$\begin{aligned}
 (34) \quad \|x(t)\| &\leq |\nu| \left\{ N \left[ \|\phi(0)\| + \|(g(x_{t_1}, \dots, x_{t_p}))(0)\| \right] \right. \\
 &\quad + \tilde{N} \left[ \|\xi\| + \|w(0, x_0)\| \right] + \int_0^t \|C(t-s)w(s, x_s)\| ds \\
 &\quad + \int_0^t \|S(t-s)f\left(s, x_s, \int_0^s a(s, \tau)h(\tau, x_\tau) d\tau, \right. \\
 &\quad \left. \left. \int_0^T b(s, \tau)k(\tau, x_\tau) d\tau\right)\| ds \right\} \\
 &\leq N[c + G] + \tilde{N} \left[ \|\xi\| + c_1 \|x_0\| + c_2 \right] \\
 &\quad + N \left[ c_1 \int_0^t \|x_s\| ds + c_2 t \right] + \tilde{N} \int_0^t l(s) \left( \|x_s\|_C \right. \\
 &\quad \left. + \int_0^s \lambda p(\tau) H(\|x_\tau\|_C) d\tau + \int_0^T \mu q(\tau) K(\|x_\tau\|_C) d\tau \right) ds
 \end{aligned}$$

Now define the function  $Z : [0, T] \rightarrow \mathbb{R}$  by

$$Z(t) = \sup \{ \|x(s)\| : -r \leq s \leq t \}.$$

Clearly  $\|x(t)\| \leq Z(t)$ , for all  $t \in [0, T]$ . Now let  $t^* \in [-r, T]$  be such that  $Z(t) = \|x(t^*)\|$  and consider the two cases:

**Case 1.** If  $t^* \in [-r, 0]$  then using (32) and (7) we obtain

$$(35) \quad Z(t) = \sup \{ \|x(s)\| : -r \leq s \leq 0 \} \leq c + G.$$

**Case 2.** If  $t^* \in [0, t]$  then from (34) we have

$$\begin{aligned}
 \|x(t^*)\| &\leq N[c + G] + \tilde{N} \left[ \|\xi\| + c_1 \|x_0\| + c_2 \right] \\
 &\quad + N \left[ c_1 \int_0^{t^*} \|x_s\| ds + c_2 t^* \right] + \tilde{N} \int_0^{t^*} l(s) \left( \|x_s\|_C \right. \\
 &\quad \left. + \int_0^s \lambda p(\tau) H(\|x_\tau\|_C) d\tau + \int_0^T \mu q(\tau) K(\|x_\tau\|_C) d\tau \right) ds \\
 \text{i.e. } Z(t) &\leq N[c + G] + \tilde{N} \left[ \|\xi\| + c_1 \|x_0\| + c_2 \right] \\
 &\quad + N \left[ c_1 \int_0^t \|x_s\| ds + c_2 t \right] + \tilde{N} \int_0^t l(s) \left( \|x_s\|_C \right. \\
 &\quad \left. + \int_0^s \lambda p(\tau) H(\|x_\tau\|_C) d\tau + \int_0^T \mu q(\tau) K(\|x_\tau\|_C) d\tau \right) ds \}.
 \end{aligned}$$

Using condition (11) and the facts that  $\|x_t\|_C \leq Z(t)$ ,  $t \in [0, T]$  and  $Z(t^*) \leq Z(t)$  for  $t^* \leq t$  we get

$$\begin{aligned}
 Z(t) &\leq N[c + G] + \tilde{N}[\|\xi\| + c_1 Z(0) + c_2] + N[c_1 Z(t)T + c_2 T] \\
 &\quad + \tilde{N} \int_0^t M(s) \left( Z(s) + \int_0^s M(\tau) H(Z(\tau)) d\tau \right. \\
 &\quad \left. + \int_0^T M(\tau) K(Z(\tau)) d\tau \right) ds \\
 (36) \quad Z(t) &\leq \frac{1}{[1 - c_1(\tilde{N} + NT)]} \left[ N(c + G) + \tilde{N}[\|\xi\| + c_2] + Nc_2 T \right. \\
 &\quad \left. + \tilde{N} \int_0^t M(s) \left( Z(s) + \int_0^s M(\tau) H(Z(\tau)) d\tau \right. \right. \\
 &\quad \left. \left. + \int_0^T M(\tau) K(Z(\tau)) d\tau \right) ds \right]
 \end{aligned}$$

Thus, in either case, from (35) and (36) we obtain

$$\begin{aligned}
 (37) \quad Z(t) &\leq \frac{1}{[1 - c_1(\tilde{N} + NT)]} \left[ N(c + G) + \tilde{N}[\|\xi\| + c_2] + Nc_2 T \right. \\
 &\quad \left. + \tilde{N} \int_0^t M(s) \left( Z(s) + \int_0^s M(\tau) H(Z(\tau)) d\tau \right. \right. \\
 &\quad \left. \left. + \int_0^T M(\tau) K(Z(\tau)) d\tau \right) ds \right] + (c + G)
 \end{aligned}$$

for  $t \in [0, T]$ . Denoting the R.H.S of the inequality (37) by  $u(t)$  we have

$$Z(t) \leq u(t), \quad t \in [0, T],$$

$$\begin{aligned}
 u(0) &= \frac{N(c + G) + \tilde{N}[\|\xi\| + c_2] + Nc_2 T}{[1 - c_1(\tilde{N} + NT)]} + (c + G) = \alpha, \\
 u'(t) &= \frac{\tilde{N}}{[1 - c_1(\tilde{N} + NT)]} \\
 &\quad \times \left\{ M(t) \left[ Z(t) + \int_0^t M(s) H(Z(s)) ds + \int_0^T M(s) K(Z(s)) ds \right] \right\} \\
 &\leq \frac{\tilde{N}}{[1 - c_1(\tilde{N} + NT)]} \left\{ M(t) \left[ u(t) + \int_0^t M(s) H(u(s)) ds + \beta \right] \right\} \\
 &\leq \frac{\tilde{N}}{[1 - c_1(\tilde{N} + NT)]} \left\{ M(t) \left[ u(t) + TM^* H(u(t)) + \beta \right] \right\}
 \end{aligned}$$



Therefore

$$\frac{[1 - c_1(\tilde{N} + NT)] u'(t)}{\tilde{N}[u(t) + TM^*H(u(t)) + \beta]} \leq M(t), \quad t \in [0, T].$$

Changing variable  $t \rightarrow s$  and integrating from 0 to  $t$  we get

$$\int_0^t \frac{[1 - c_1(\tilde{N} + NT)] u'(s) ds}{\tilde{N}[u(s) + TM^*H(u(s)) + \beta]} \leq \int_0^t M(s) ds.$$

Now let  $u(s) = p$  then we obtain

$$\int_{u(0)}^{u(t)} \frac{[1 - c_1(\tilde{N} + NT)] dp}{\tilde{N}[p + TM^*H(p) + \beta]} \leq \int_0^t M(s) ds.$$

Using this and condition (30) we obtain

$$(38) \quad \int_{\alpha}^{u(t)} \frac{[1 - c_1(\tilde{N} + NT)] ds}{\tilde{N}[s + TM^*H(s) + \beta]} \leq \int_0^t M(s) ds \leq \int_0^T M(s) ds < \int_{\alpha}^{\infty} \frac{[1 - c_1(\tilde{N} + NT)] ds}{\tilde{N}[s + TM^*H(s) + \beta]},$$

for  $t \in [0, T]$ . From the inequality (38), there exists a constant  $\eta$  independent of  $\nu \in (0, 1)$  such that  $u(t) \leq \eta$  for  $t \in [0, T]$  and hence

$$\|x(t)\| \leq Z(t) \leq u(t) \leq \eta, \quad \text{for all } t \in [0, T].$$

Since for every  $t \in [0, T]$ ,  $\|x_t\|_C \leq Z(t)$ , in particular we get  $\|x_0\|_C \leq Z(0)$  for  $t = 0$

$$\text{i.e. } \|x(t)\| \leq Z(0) \leq \eta, \quad \text{for all } t \in [-r, 0].$$

Thus we have

$$(39) \quad \|x\|_B = \sup \{\|x(t)\| : t \in [-r, T]\} \leq \eta.$$

In order to apply Lemma 6. we must prove that  $F$  is a completely continuous operator. Clearly  $F : B = C([-r, T], X) \rightarrow B$  is continuous as seen in the proof of Theorem 1. Now we prove that  $F$  maps a bounded set of  $B$  into a precompact set of  $B$ . Consider the bounded set  $B_m = \{y \in B : \|y\| \leq m\}$  for a positive integer  $m$ . We show that  $F_{B_m}$  is uniformly bounded. Let  $t \in [-r, 0]$  and  $y \in B_m$  then we have

$$(40) \quad \|(Fy)(t)\| \leq c + G.$$

Using hypotheses  $(H_1) - (H_3)$  and conditions (3), (4), (7), (8), (10), (11) and (19), we obtain for  $t \in [0, T]$ ,

$$\begin{aligned}
(41) \quad \|(Fy)(t)\| &= \|C(t)[\phi(0) - (g(y_{t_1}, \dots, y_{t_p}))(0)] \\
&\quad + S(t)[\xi - w(0, y_0)] + \int_0^t C(t-s)w(s, y_s)ds \\
&\quad + \int_0^t S(t-s)f\left(s, y_s, \int_0^s a(s, \tau)h(\tau, y_\tau)d\tau, \right. \\
&\quad \left. \int_0^T b(s, \tau)k(\tau, y_\tau)d\tau\right)ds\| \\
&\leq N[c + G] \\
&\quad + \tilde{N}[\|\xi\| + c_1\|y_0\| + c_2] + N[c_1\|y\| + c_2]T \\
&\quad + \tilde{N}\left[TM^*\left(m + \frac{TM^*H(m)}{2} + TM^*K(m)\right)\right].
\end{aligned}$$

Now combining (40) and (41) we get

$$\begin{aligned}
\|(Fy)(t)\| &\leq [N + 1][c + G] + N[c_1m + c_2]T + \tilde{N}\left[\|\xi\| + c_1m + c_2\right. \\
&\quad \left.+ TM^*\left(m + \frac{TM^*H(m)}{2} + TM^*K(m)\right)\right]
\end{aligned}$$

where  $t \in [-r, T]$ ,  $y \in B_m$ . This implies that the set  $\{(Fy)(t) : \|y\|_B \leq m, -r \leq t \leq T\}$  is bounded in  $X$  and hence  $F_{B_m}$  is uniformly bounded in  $B$ . Next we show that  $F$  maps  $B_m$  into an equicontinuous family of functions with values in  $X$ . For this let  $y \in B_m$ ,  $s_1, s_2 \in [-r, T]$  and consider the following cases:

**Case 1.** Suppose  $0 \leq s_1 \leq s_2 \leq T$  then using hypothesis  $(H_1) - (H_3)$  and conditions (4), (8), (10), (11) and (19), we get

$$\begin{aligned}
\|(Fy)(s_2) - (Fy)(s_1)\| &= \|[C(s_2) - C(s_1)][\phi(0) - (g(y_{t_1}, \dots, y_{t_p}))(0)] \\
&\quad + [S(s_2) - S(s_1)][\xi - w(0, y_0)] + \int_0^{s_2} C(s_2 - s)w(s, y_s)ds \\
&\quad - \int_0^{s_1} C(s_1 - s)w(s, y_s)ds \\
&\quad + \int_0^{s_2} S(s_2 - s)f\left(s, y_s, \int_0^s a(s, \tau)h(\tau, y_\tau)d\tau, \int_0^T b(s, \tau)k(\tau, y_\tau)d\tau\right)ds \\
&\quad - \int_0^{s_1} S(s_1 - s)f\left(s, y_s, \int_0^s a(s, \tau)h(\tau, y_\tau)d\tau, \int_0^T b(s, \tau)k(\tau, y_\tau)d\tau\right)ds\| \\
&\leq \|C(s_2) - C(s_1)\|[c + G] + \|S(s_2) - S(s_1)\|[\|\xi\| + c_1m + c_2]
\end{aligned}$$

$$\begin{aligned}
& + \int_0^{s_1} \|C(s_2 - s) - C(s_1 - s)\| (c_1 m + c_2) ds \\
& + N (c_1 m + c_2) |s_2 - s_1| + \int_0^{s_1} \| [S(s_2 - s) - S(s_1 - s)] \| \\
& \times \left[ M(s) \left( m + \int_0^s M(\tau) H(m) d\tau + \int_0^T M(\tau) K(m) d\tau \right) \right] ds \\
& + \tilde{N} \int_{s_1}^{s_2} M(s) \left( m + \int_0^s M(\tau) H(m) d\tau + \int_0^T M(\tau) K(m) d\tau \right) ds \\
& \leq \left\{ \|C(s_2) - C(s_1)\| [c + G] + \|S(s_2) - S(s_1)\| [\|\xi\| + c_1 m + c_2] \right. \\
& + \int_0^{s_1} \|C(s_2 - s) - C(s_1 - s)\| (c_1 m + c_2) ds \\
& + N (c_1 m + c_2) |s_2 - s_1| + \gamma \int_0^{s_1} \|S(s_2 - s) - S(s_1 - s)\| ds \\
& \left. + \tilde{N} \gamma |s_2 - s_1| \right\} \rightarrow 0 \text{ as } s_2 \rightarrow s_1,
\end{aligned}$$

where  $\gamma$  is as defined in the proof of previous theorem. The compactness of  $C(t)$  and  $S(t)$  for  $t > 0$  implies the continuity in the uniform operator topology. Therefore the right hand side of above equation tends to zero as  $s_2 \rightarrow s_1$ .

**Case 2.** Suppose  $-r \leq s_1 \leq 0 \leq s_2 \leq T$  then we get

$$\begin{aligned}
\|(Fy)(s_2) - (Fy)(s_1)\| & \leq \|C(s_2)[\phi(0) - (g(y_{t_1}, \dots, y_{t_p}))(0)] \\
& - C(0)[\phi(s_1) - (g(y_{t_1}, \dots, y_{t_p}))(s_1)]\| \\
& + \|S(s_2)[\xi - w(0, y_0)]\| + N (c_1 m + c_2) |s_2| \\
& + \left\| \int_0^{s_2} S(s_2 - s) f \left( s, y_s, \int_0^s a(s, \tau) h(\tau, y_\tau) d\tau, \right. \right. \\
& \left. \left. \int_0^T b(s, \tau) k(\tau, y_\tau) d\tau \right) ds \right\|
\end{aligned}$$

Now proceeding as in Case 1 for the integral on the right hand side of above inequality we further obtain

$$\begin{aligned}
\|(Fy)(s_2) - (Fy)(s_1)\| & \leq \left\{ \|C(s_2)[\phi(0) - (g(y_{t_1}, \dots, y_{t_p}))(0)] \right. \\
& - C(0)[\phi(s_1) - (g(y_{t_1}, \dots, y_{t_p}))(s_1)]\| \\
& + \|S(s_2)[\xi - w(0, y_0)]\| + N (c_1 m + c_2) |s_2 - s_1| \\
& \left. + \tilde{N} \gamma |s_2 - s_1| \right\} \rightarrow 0 \text{ as } s_2 \rightarrow 0_+ \text{ and } s_1 \rightarrow 0_-.
\end{aligned}$$

**Case 3.** Suppose  $-r \leq s_1 \leq s_2 \leq 0$  then using hypotheses  $(H_{12})$  and  $(H_{14})$  we get

$$\|(Fy)(s_2) - (Fy)(s_1)\| \leq [(\sigma + L) |s_1 - s_2|] \rightarrow 0 \text{ as } s_2 \rightarrow s_1.$$

Cases 1 – 3 imply that  $\|(Fy)(s_2) - (Fy)(s_1)\| \rightarrow 0$  as  $s_1 \rightarrow s_2$ , for all  $s_1, s_2 \in [-r, T]$ . Thus we conclude that  $F_{B_m}$  is an equicontinuous family of functions with values in  $X$ .

We have already shown that  $F_{B_m}$  is an equicontinuous and uniformly bounded collection. To prove that the set  $F_{B_m}$  is precompact in  $B$  it is sufficient by Arzela-Ascoli's argument to show that the set  $\{(Fy)(t) : y \in B_m\}$  is precompact in  $X$  for each  $t \in [-r, T]$ .

Let  $t \in [-r, 0]$  be fixed then using hypothesis  $(H_{13})$  we have

$$A_t = \{(Fy)(t) : y \in B_m\} = \{\phi(t) - (g(y_{t_1}, \dots, y_{t_p}))(t) : y \in B_m\}$$

is precompact in  $X$  for every  $t \in [-r, 0]$ .

Now let  $0 < t \leq T$  be fixed and  $\varepsilon$  a real number satisfying  $0 < \varepsilon < T$ . For  $y \in B_m$  define

$$\begin{aligned} (F_\varepsilon y)(t) &= C(t)[\phi(0) - (g(y_{t_1}, \dots, y_{t_p}))(0)] \\ &\quad + S(t)[\xi - w(0, y_0)] + \int_0^{t-\varepsilon} C(t-s)w(s, y_s)ds \\ &\quad + \int_0^{t-\varepsilon} S(t-s)f\left(s, y_s, \int_0^s a(s, \tau)h(\tau, y_\tau)d\tau, \right. \\ &\quad \left. \int_0^T b(s, \tau)k(\tau, y_\tau)d\tau\right)ds. \end{aligned}$$

Since  $C(t)$  and  $S(t)$  are compact operators and the set  $F_{B_m}$  is bounded in  $B$ , the set  $Y_\varepsilon(t) = \{(F_\varepsilon y)(t) : y \in B_m\}$  is precompact in  $X$  for every  $\varepsilon$ ,  $0 < \varepsilon < t$ . Moreover for every  $y \in B_m$  we have

$$\begin{aligned} (Fy)(t) - (F_\varepsilon y)(t) &= \int_{t-\varepsilon}^t C(t-s)w(s, y_s)ds \\ &\quad + \int_{t-\varepsilon}^t S(t-s)f\left(s, y_s, \int_0^s a(s, \tau)h(\tau, y_\tau)d\tau, \right. \\ &\quad \left. \int_0^T b(s, \tau)k(\tau, y_\tau)d\tau\right)ds. \end{aligned}$$

Now using hypothesis  $(H_1) - (H_3)$  and conditions (4), (8), (10), (11) and (19) we obtain

$$\begin{aligned}
\|(Fy)(t) - (F_\varepsilon y)(t)\| &\leq \int_{t-\varepsilon}^t \|C(t-s)\| \|w(s, y_s)\| ds \\
&\quad + \int_{t-\varepsilon}^t \|S(t-s)\| \|f\left(s, y_s, \int_0^s a(s, \tau)h(\tau, y_\tau)d\tau, \right. \\
&\qquad \qquad \qquad \left. \int_0^T b(s, \tau)k(\tau, y_\tau)d\tau\right)\| ds \\
&\leq N(c_1m + c_2)\varepsilon \\
&\quad + \tilde{N} \int_{t-\varepsilon}^t M^* \left(m + M^*H(m)s + M^*K(m)T\right) ds \\
&\leq N(c_1m + c_2)\varepsilon + \tilde{N}\gamma\varepsilon.
\end{aligned}$$

This shows that there exist precompact sets arbitrarily close to the set  $\{(Fy)(t) : y \in B_m\}, t \in [0, T]$ . Hence the set  $\{(Fy)(t) : y \in B_m\}$  is precompact in  $X$  for every  $t \in [-r, T]$ . Thus  $F$  is a completely continuous operator. Moreover from the inequality (39) we conclude that the set

$$\varepsilon(F) = \{x \in B : x = \nu Fx, \nu \in (0, 1)\}$$

is bounded in  $B$ . Thus by virtue of Lemma 6 the operator  $F$  has a fixed point  $\tilde{x}$  in  $B$ . This  $\tilde{x}$  is the solution of the nonlocal problem (1) – (2). This completes the proof of the Theorem 2.  $\blacksquare$

**Remark 1.** While obtaining the mild solution of the nonlocal problem (1) – (2) using the Leray-Schauder Alternative we have to assume the Lipschitz continuity of  $\phi$  and  $(g(y_{t_1}, \dots, y_{t_p}))$ , whereas we could relax these conditions while proving the existence of mild solution of the nonlocal problem (1) – (2) using the Daro-Sadovskii's fixed point theorem and the Hausdorff's measure of noncompactness.

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