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FIXED POINT THEOREMS FOR KANNAN TYPE MAPPINGS IN 2-MENGER SPACES

Abstract. In this paper we show that the results of Choudhary and Das [2] hold under more general situation.

Key words: 2-Menger space, Φ-function, Ψ-function, Kannan type mappings, weakly compatible mappings, n-th order t-norms, t-norm of Hadzic-type.

AMS Mathematics Subject Classification: 54H25, 47H10.

1. Introduction

The theory of probabilistic metric spaces is an important part of stochastic Analysis, and so it is of interest to develop the fixed point theory in such spaces. The first result from the fixed point theory in probabilistic metric spaces is obtained by Sehgal and Bharucha- Reid [44]. Since then many fixed points theorem for single valued and multi valued mappings in probabilistic metric spaces have been proved in [11] - [16], [35].

The notion of a 2-metric space was introduced by Gahler [10]. A 2-metric is not a continuous function in all of its variables, whereas an ordinary metric spaces and a 2-metric space is not topologically equivalent to an ordinary metric. In particular, the fixed point theorems on 2-metric spaces and metric spaces may be unrelated easily. In 1942, Menger proposed a generalization of metric fixed point theory by replacing the number $d(x, y)$ by a real function $F(x, y)$, a probability distribution function, whose value $F(x, y; t)$ for any number $t \geq 0$ is interpreted that the distance between x and y is less than t and $0 \leq F(x, y; t) \leq 1$.

Definition 1 ([45]). *A mapping* $F : \Re \to \Re^+$ *is called distribution function if it is non decreasing and left continuous with* $\inf \{F(t) : t \in \Re\} = 0$ *and* $\sup \{F(t) : t \in \Re\} = 1$.

Let L_+ be the set of all distribution functions whereas H be the set of specific distribution function (Also known as Heaviside function) defined by

$$H(t) = \begin{cases} 0, & \text{if} \quad t \leq 0 \\ 1, & \text{if} \quad t > 0 \end{cases}$$

In 1987, Zeng [48] gave the generalization of 2-metric to Probabilistic 2-metric akin to the notion of probabilistic metric spaces as given Menger.

A probabilistic 2-metric space is an ordered pair (X, F), where X is an arbitrary set and F is a mapping from X^3 into the set of distribution functions. The distribution function $F_{x,y,z}(t)$ will denote the value of $F_{x,y,z}$ at the real number t. The function $F_{x,y,z}$ are assumed to satisfy the following conditions:

(*i*) $F_{x,y,z}(0) = 0$ for all $x, y, z \in X$

(*ii*) $F_{x,y,z}(t) = 1$ for all $t > 0$ iff at least two of three points x, y, z are equal

(*iii*) For distinct points $x, y \in X$, there exists a point $z \in X$ such that $F_{x,y,z}(t) \neq 1$ for $t > 0$

(*iv*) $F_{x,y,z}(t) = F_{x,z,y}(t) = F_{z,y,x}(t)$ for all $x, y, z \in X$ and $t > 0$

(*v*) $F_{x,y,w}(t_1) = 1, \quad F_{x,w,z}(t_2) = 1, \quad F_{w,y,z}(t_3) = 1$ then $F_{x,y,z}(t_1 + t_2 + t_3) = 1$ for all $x, y, z, w \in X$ and $t_1, t_2, t_3 > 0$.

In 2003, Ren and Wang [47] gave the notion of n-th order t-norm as follows:

Definition 2. *A mappings* $\Delta : \Pi_{i=1}^{n}[0, 1] \to [0, 1]$ *is called a n-th order t-norm if following conditions are satisfied:*

(*i*) $\Delta(0, 0, \ldots, 0) = 0$, $\Delta(a, 1, 1, \ldots, 1) = a$ *for all* $a \in [0, 1]$

(*ii*) $\Delta(a_1, a_2, a_3, \ldots, a_n) = \Delta(a_2, a_1, a_3, \ldots, a_n)$

$= \Delta(a_2, a_3, a_1, \ldots, a_n) = \ldots = \Delta(a_2, a_3, a_4, \ldots, a_n, a_1)$

(*iii*) $a_i \geq b_i$, $i = 1, 2, 3 \ldots, n$ *implies*
$\Delta(a_1, a_2, a_3, \ldots, a_n) \geq \Delta(b_1, b_2, b_3, \ldots, b_n)$

(*iv*) $\Delta(\Delta(a_1, a_2, a_3, \ldots, a_n), b_2, b_3, \ldots, b_n)$
$= \Delta(a_1, \Delta(a_2, a_3, \ldots, a_n, b_2), b_3, \ldots, b_n)$
$= \Delta(a_1, a_2, \Delta(a_3, a_4, \ldots, a_n, b_2, b_3,), b_4, \ldots, b_n)$
$= \ldots$
$= \Delta(a_1, a_2, a_3, \ldots, a_{n-1}\Delta(a_n, b_2, b_3,), b_4, \ldots, b_n)$.

For $n = 2$, we have a binary t-norm, which is commonly known as t-norm.

Basics examples of t-norm are the Lukasiewicz t-norm $\Delta_L, \Delta_L(a, b) = \max(a + b - 1, 0)$, t-norms $\Delta_P, \Delta_P(a, b) = ab$ and t-norms $\Delta_M, \Delta_M(a, b) = \min(a, b)$

Definition 3 ([15]). *Let Δ be a t-norm and let $\Delta_n : [0,1] \to [0,1] (n \in N)$ be defined in the following way:*

$$\Delta_1(x) = \Delta(x,x), \quad \Delta_{n+1}(x) = \Delta(\Delta_n(x), x) \quad (n \in N, x \in [0,1]).$$

We say that the t-norm Δ is of Hadzic-type if Δ is continuous and the family $\{\Delta_n(x), n \in N\}$ is equicontinuous at $x = 1$. Recall that the family $\{\Delta_n(x), n \in N\}$ is equicontinuous at $x = 1$ if for every $\lambda \in (0,1)$, there exist $\delta(\lambda) \in (0,1)$ such that the following implication holds:

$$x > 1 - \delta(\lambda) \quad \text{implies} \quad \Delta_n(x) > 1 - \lambda \quad \text{for all} \quad n \in N.$$

A trivial example of t-norm of H-type is $\Delta = \Delta_M$.

Remark 1. Every minimum t-norm is Hadzic t-norm, but the converse is not true.

Definition 4 ([15]). *If Δ is a t-norm and $(x_1, x_2, \ldots, x_n) \in [0,1]^n$ $(n \in N)$, then $\Delta_{i=1}^n x_i$ is defined recursively by 1, if $n = 1$ and $\Delta_{i=1}^n x_i = \Delta(\Delta_{i=1}^{n-1} x_i, x_n)$ for all $n \geq 2$. If $\{b_n\}_{n \in N}$ is a sequence of numbers from $[0,1]$, then $\Delta_{i=1}^\infty x_i$ is defined as $\lim_{n\to\infty} \Delta_{i=1}^n x_i$ (this limit always exists) and $\Delta_{i=1}^\infty x_i$ as $\Delta_{i=1}^\infty x_{n+i}$.*

Definition 5. *Let X is any non-empty set and D_+ denotes the set of all distribution functions. A triplet (X, F, Δ) is said to be a 2-Menger space if the probabilistic 2-metric space (X, F) satisfies the following condition:*

(iv) $F_{x,y,z}(t) \geq \Delta(F_{x,y,w}(t_1), F_{x,w,z}(t_2), F_{w,y,z}(t_3))$,

where $t_1, t_2, t_3 > 0$, $t_1 + t_2 + t_3 = t$ and $x, y, z, w \in X$ and Δ is the 3^{rd} order t-norm.

Definition 6. *A sequence $\{x_n\}$ in a 2-Menger space (X, F, Δ) is said to be*

(i) convergent with limit x if $\lim_{n\to\infty} F_{x_n,x,p}(t) = 1$ for all $t > 0$ and for every $p \in X$

(ii) Cauchy sequence in X if given $\epsilon > 0$, $\lambda > 0$, there exists a positive integer $N_{\epsilon,\lambda}$ such that $F_{x_n,x_m,p}(\epsilon) > 1 - \lambda$ for all $m, n > N_{\epsilon,\lambda}$ and for every $p \in X$

(iii) Complete if every Cauchy sequence in X is convergent in X.

In 1984, Khan, Swaleh and Sessa [23] introduced the concept of altering distance function in metric spaces and call it as a control function and it alters the distance between two points in metric space. There are several works in fixed point theory involving altering distance function, some of these are noted in [38], [39].

Recently, Choudhary, Das [2] extended the concept of altering distance function in the context of Menger spaces as follows:

Definition 7. *A function $\phi : R \to R^+$ is said to be a ϕ-function if it satisfies the following conditions:*

(i) $\phi(t) = 0$ *if and if* $t = 0$

(ii) $\phi(t)$ *is strictly monotonic increasing and* $\phi(t) \to \infty$ *as* $t \to \infty$

(iii) ϕ *is left continuous in* $(0, \infty)$

(iv) ϕ *is continuous at* 0.

In 2008, Choudhury and Das [3] introduced a new type of contraction mappings in Menger spaces which is known as ϕ -contraction. The idea of control function has opened possibilities of proving new fixed point results in Menger spaces. Some recent results using ϕ-function are noted in [4], [5] and [48].

Definition 8. *A function $\Psi : [0,1] \times [0,1] \to [0,1]$ is said to be a Ψ-function if*

(i) Ψ *is monotonic increasing and continuous*

(ii) $\Psi(x, x) > x$ *for all* $0 < x < 1$

(iii) $\Psi(1,1) = 1, \Psi(0,0) = 0$.

Definition 9. *Let (X, F, Δ) be a complete 2-Menger space, where Δ is the 3^{rd} order minimum t-norm and the mapping $T : X \to X$ be a self mapping which satisfies the following inequality for all $x, y, p \in X$*

$$F_{Tx,Ty,p}(\phi(t)) \geq \Psi\Big(F_{x,Tx,p}(\phi(\frac{t_1}{a})), F_{y,Ty,p}(\phi(\frac{t_2}{b}))\Big)$$

where $t_1, t_2, t > 0$ with $t = t_1 + t_2$, $a, b > 0$ with $0 < a + b < 1$, Ψ is a Ψ-function and ϕ is a ϕ- function. Then the mapping T is called a generalized Kannan type mappings.

Definition 10 ([25], [26])**.** *Let (X, d) be a metric space and f be a mapping on X. The mapping f is called a Kannan type mapping if there exists $0 \leq \alpha < \frac{1}{2}$ such that*

$$d(fx, fy) \leq \alpha[d(x, fx) + d(y, fy)] \quad \textit{for all} \quad x, y \in X.$$

There are a large number of works dealing with Kannan type mapping. Several examples of these works are noted in [6], [22], [41].

In 1998, Jungck and Rhoades [20] introduced the notion of weakly compatible mappings as follows:

Two mappings are said to be weakly compatible if they commute at their coincidence point.

Note that compatible mappings are weakly compatible, but the converse is not true in general. In 2007, Kohli et. al [21] introduced the notion of variants of R-weak commutative maps as follows:

Definition 11. *A pair of self-mappings* (f, g) *of a Menger space* (X, F, Δ) *is said to be:*

(i) *Weakly commuting if* $F(fgx, gfx, t) \geq F(fx, gx, t)$

(ii) *R-Weakly commuting if there exists some* $R > 0$ *such that* $F(fgx, gfx, t) \geq F(fx, gx, \frac{t}{R})$

(iii) *R-Weakly commuting mappings of the type* (i) *if there exists some* $R > 0$ *such that* $F(gfx, ffx, t) \geq F(gx, fx, \frac{t}{R})$

(iv) *R-Weakly commuting mappings of the type* (ii) *if there exists some* $R > 0$ *such that* $F(fgx, ggx, t) \geq F(fx, gx, \frac{t}{R})$

(v) *R-Weakly commuting mappings of the type* (iii) *if there exists some* $R > 0$ *such that* $F(ffx, ggx, t) \geq F(fx, gx, \frac{t}{R})$ *for all* $x \in X$ $t > 0$.

In a similar mode, we state R-Weakly commuting mappings and R-Weakly commuting mappings of the type (A_g), type (A_f) and type (P) in setting of 2-Menger spaces

Definition 12. *A pair of self-mappings* (f, g) *of a 2-Menger space* (X, F, Δ) *is said to be:*

(i) *Weakly commuting if* $F(fgx, gfx, p, t) \geq F(fx, gx, p, t)$

(ii) *R-Weakly commuting if there exists some* $R > 0$ *such that* $F(fgx, gfx, p, t) \geq F(fx, gx, p, \frac{t}{R})$

(iii) *R-Weakly commuting mappings of the type* (A_g) *if there exists some* $R > 0$ *such that* $F(gfx, ffx, p, t) \geq F(gx, fx, p, \frac{t}{R})$

(iv) *R-Weakly commuting mappings of the type* (A_f) *if there exists some* $R > 0$ *such that* $F(fgx, ggx, p, t) \geq F(fx, gx, p, \frac{t}{R})$

(v) *R-Weakly commuting mappings of the type* (P) *if there exists some* $R > 0$ *such that* $F(ffx, ggx, p, t) \geq F(fx, gx, p, \frac{t}{R})$, *for all* $x \in X$, $p \in X$ *and* $t > 0$.

In our further discussion, we adopt the terminology from the paper of Imdad et.al. [30] and rename R-Weakly commuting mappings of the type (i), R-Weakly commuting mappings of the type (ii) and R-Weakly commuting mappings of the type (iii) by R-Weakly commuting mappings of the type (A_g), R-Weakly commuting mappings of the type (A_f) and R-Weakly commuting mappings of the type (P) respectively.

One can notice that definition 12 inspired by Imdad et.al. [18], [30] and Pathak et. al. [34].

It is obvious that point wise R-weakly commuting maps commute at their coincidence points and point wise R-weak commutativity is equivalent to commutativity at coincidence points.

In 2002, Aamri and Moutawakil [1] generalized the notion of non compatible mapping by introducing the notion of E.A. property.

Definition 13 ([1]). *Let f and g be two self-mappings of a metric (X, d). The maps f and g satisfy E.A. property if there exists a sequence $\{x_n\}$ in X such that*

$$\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = u, \quad \textit{for some} \quad u \in X.$$

Now in a similar mode we state E.A. property in 2-Menger space as follows:

Definition 14. *A pair of self-mapping (f, g) of 2-Menger spaces (X, F, Δ) is said to satisfy E.A. property, if there exists a sequence $\{x_n\}$ in X such that $\lim_{n\to\infty} F_{fx_n,gx_n,p}(t) = 1$ for some $t \in X$.*

Example 1. Let $X = [0, \infty)$ be the usual metric space. Define $f, g : X \to X$ by $fx = \frac{x}{4}$ and $gx = \frac{3x}{4}$ for all $x \in X$. Consider the sequence $\{x_n\} = \frac{1}{n}$. Since

$$\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = 0.$$

Then f and g satisfy the E.A. property.

Although E.A property is generalization of the concept of non compatible mappings yet it requires either completeness of the whole space or any of the range space or continuity of mappings. Recently, the new notion of CLR property (common limit range property) was given by Sintunavarat and Kumam [46] that further relaxes the requirement of the closeness of range subspaces.

Definition 15. *Two mapping f and g of 2-Menger space satisfy the common limit in the range of g (CLRg) property if*

$$\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = gx, \quad \textit{for some} \quad x \in X.$$

Example 2. Let $X = [0, \infty)$ be the usual metric space. Define $f, g : X \to X$ by $fx = x + 1$ and $gx = 2x$ for all $x \in X$. Consider the sequence $\{x_n\} = 1 + \frac{1}{n}$. Since $\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = 2 = g1$, therefore, f and g satisfy the (CLRg) property.

In 2011, Choudhary [2] proved the following fixed point theorem.

Theorem 1 ([2]). *Let (X, F, Δ) be a complete 2-Menger space, where Δ is the 3^{rd} order minimum t-norm and the mapping $T : X \to X$ be a self mapping which satisfies the following inequality for all $x, y, p \in X$,*

$$F_{Tx,Ty,p}(\phi(t)) \geq \Psi\Big(F_{x,Tx,p}(\phi(\frac{t_1}{a})), F_{y,Ty,p}(\phi(\frac{t_2}{b}))\Big)$$

where $t_1, t_2, t > 0$ with $t = t_1 + t_2$, $a, b > 0$ with $0 < a + b < 1$, Ψ is a Ψ-function and ϕ is a Φ-function.

Then the mapping T has a unique fixed point in X.

Now we extend Theorem 1 as follows.

2. Weakly compatible maps

Theorem 2. *Let (X, F, Δ) be a complete 2-Menger space, where Δ is the 3^{rd} order minimum t-norm of Hadzic type and f and $g : X \to X$ be a self mappings which satisfy the following conditions:*

$$f(X) \subseteq g(X) \tag{1}$$

$$\textit{any one of } f(X) \textit{and } g(X) \textit{ is complete} \tag{2}$$

$$F_{fx,fy,p}(\phi(t)) \geq \Psi\Big(F_{gx,fx,p}(\phi(\frac{t_1}{a})), F_{fy,gy,p}(\phi(\frac{t_2}{b}))\Big) \tag{3}$$

$$f \textit{ and } g \textit{ are weakly compatible on } X. \tag{4}$$

where $t_1, t_2, t > 0$ with $t = t_1 + t_2$, $a, b > 0$ with $0 < a + b < 1$, Ψ is a $\Psi-$function and ϕ is a Φ-function.

For any $x_0 \in X$, the sequence $\{y_n\} \in X$ can be constructed as follows:

$$y_n = gx_{n+1} = fx_n, \quad n = 0, 1, 2, \ldots$$

such that $c \in (0, 1)$ and $c = a + b$, the following condition holds:

$$\lim_{n\to\infty} \Delta_{i=n}^{\infty} F_{y_0,y_1,p}\Phi(\frac{1}{c^i}) = 1.$$

Then f and g have a unique common fixed point.

Proof. Let $x_0 \in X$. We now construct a sequence $\{y_n\} \in X$ by $y_n = gx_{n+1} = fx_n$, $n \in N$, where N is the set of all positive integers.

Now for $t, t_1, t_2 > 0$ with $t = t_1 + t_2$, we have

$$\begin{aligned}
F_{y_{n+1},y_n,p}(\phi(t)) &= F_{fx_{n+1},fx_n,p}(\phi(t)) \\
&\geq \Psi\Big(F_{gx_{n+1},fx_{n+1},p}(\phi(\frac{t_1}{a})), F_{fx_n,gx_n,p}(\phi(\frac{t_2}{b}))\Big) \\
&\geq \Psi\Big(F_{y_n,y_{n+1},p}(\phi(\frac{t_1}{a})), F_{y_n,y_{n-1},p}(\phi(\frac{t_2}{b}))\Big).
\end{aligned}$$

$$F_{y_{n+1},y_n,p}(\phi(t)) \geq \Psi\Big(F_{y_{n+1},y_n,p}(\phi(\frac{t_1}{a})), F_{y_n,y_{n-1},p}(\phi(\frac{t_2}{b}))\Big). \tag{5}$$

Let $t_1 = \frac{at}{a+b}$ and $t_2 = \frac{bt}{a+b}$ and $c = a+b$, then obviously we have $0 < c < 1$. From (5), we have

$$F_{y_{n+1},y_n,p}(\phi(t)) \geq \Psi\Big(F_{y_{n+1},y_n,p}(\phi(\frac{t}{c})), F_{y_n,y_{n-1},p}(\phi(\frac{t}{c}))\Big). \tag{6}$$

We now claim that for all $t > 0$,

$$F_{y_{n+1},y_n,p}(\phi(\frac{t}{c})) \geq F_{y_n,y_{n-1},p}(\phi(\frac{t}{c})). \tag{7}$$

If possible, let for some $t > 0$,

$$F_{y_{n+1},y_n,p}(\phi(\frac{t}{c})) < F_{y_n,y_{n-1},p}(\phi(\frac{t}{c})).$$

i.e

$$F_{y_n,y_{n-1},p}(\phi(\frac{t}{c})) > F_{y_{n+1},y_n,p}(\phi(\frac{t}{c})).$$

Now we have from (6)

$$\begin{aligned} F_{y_{n+1},y_n,p}(\phi(t)) &\geq \Psi\Big(F_{y_{n+1},y_n,p}(\phi(\frac{t}{c})), F_{y_{n+1},y_n,p}(\phi(\frac{t}{c}))\Big) \\ &> F_{y_{n+1},y_n,p}(\phi(\frac{t}{c})) \\ &\geq F_{y_{n+1},y_n,p}(\phi(\frac{t}{c})), \end{aligned}$$

which is a contradiction, since $0 < c < 1$, ϕ is strictly increasing and F is non-decreasing.

Therefore for all $t > 0$, we have

$$F_{y_{n+1},y_n,p}(\phi(\frac{t}{c})) \geq F_{y_n,y_{n-1},p}(\phi(\frac{t}{c})).$$

From (6) and (7), we have

$$\begin{aligned} F_{y_{n+1},y_n,p}(\phi(t)) &\geq \Psi\Big(F_{y_{n+1},y_n,p}(\phi(\frac{t}{c})), F_{y_n,y_{n-1},p}(\phi(\frac{t}{c}))\Big) \\ &\geq \Psi\Big(F_{y_n,y_{n-1},p}(\phi(\frac{t}{c})), F_{y_n,y_{n-1},p}(\phi(\frac{t}{c}))\Big) \\ &> F_{y_n,y_{n-1},p}(\phi(\frac{t}{c})) \\ &> F_{y_{n-1},y_{n-2},p}(\phi(\frac{t}{c^2})) \\ &> F_{y_{n-2},y_{n-3},p}(\phi(\frac{t}{c^3})) \\ &\vdots \\ &> F_{y_1,y_0,p}(\phi(\frac{t}{c^n})) \end{aligned} \tag{8}$$

i.e

$$F_{y_{n+1},y_n,p}(\phi(t)) \geq F_{y_1,y_0,p}(\phi(\frac{t}{c^n})). \tag{9}$$

Therefore,

$$\lim_{n\to\infty} F_{y_{n+1},y_n,p}(\phi(t)) = 1 \quad \text{for all} \quad t > 0. \tag{10}$$

By the virtue of property of Φ, we can choose $s > 0$ such that $s > \phi(t)$. Then for all $p \in X$ and $t > 0$, we have

$$\lim_{n\to\infty} F_{y_n,y_{n+1},p}(s) = 1 \quad \text{for all} \quad t > 0. \tag{11}$$

Next we claim that $\{y_n\}$ is a Cauchy sequence.

In view of the condition (i) and (iv) in Definition 8, for all $s > 0$, we can find a positive number t such that $s > \phi(t)$. Now from the paper Dosenovic, we can easily follow $\{y_n\}$ is a Cauchy sequence in X, i.e,

$$\begin{aligned}
F_{y_n,y_{n+m},p}(s) &> F_{y_n,y_{n+m},p}(\phi(t)) = F_{y_n,y_{n+m},p}(\phi(t)) \\
&\geq \underbrace{\Delta(\Delta(\ldots\Delta}_{(m-1)\text{-times}}(F_{y_n,y_{n+1},p}(\phi(t)), F_{y_{n+1},y_{n+2},p}(\phi(t)), \\
&\quad \ldots, F_{y_{n+m-1},y_{n+m},p}(\phi(t)) \\
&\geq \underbrace{\Delta(\Delta(\ldots\Delta}_{(m-1)\text{-times}}(F_{y_0,y_1,p}(\phi(\frac{t}{c^n})), F_{y_0,y_1,p}(\phi(\frac{t}{c^{n+1}})), \\
&\quad \ldots, F_{y_0,y_1,p}(\phi(\frac{t}{c^{n+m-1}}))))) \\
&= \Delta_{i=n}^{n+m-1} F_{y_0,y_1,p}(\phi(\frac{t}{c^i})) \geq \Delta_{i=n}^{\infty} F_{y_0,y_1,p}(\phi(\frac{t}{c^i}))
\end{aligned}$$

i.e

$$F_{y_n,y_{n+m},p}(s) \geq \Delta_{i=n}^{\infty} F_{y_0,y_1,p}(\phi(\frac{t}{c^i})).$$

This implies that $\lim_{n\to\infty} \Delta_{i=n}^{\infty} F_{y_0,y_1,p}(\phi(\frac{t}{c^i})) = 1$ for every $t > 0$. This means that the sequence $\{y_n\}$ is a Cauchy sequence.

Since (X, F, Δ) is a complete 2-Menger space, therefore, we have $\{y_n\}$ is convergent in X for some z in X, i.e

$$\lim_{n\to\infty} y_n = z = \lim_{n\to\infty} gx_{n+1} = \lim_{n\to\infty} fx_n. \tag{12}$$

Let $g(X)$ is complete, there exists a point $z_1 \in X$ such that $gz_1 = z$.

We now show that $fz_1 = z$. If possible, let $0 < F_{z,fz_1,p}(\phi(t)) < 1$ for some $t > 0$. By virtue of the property of ϕ, we can choose ξ_1, ξ_2, t_1, t_2 such that

$$\phi(t) = \xi_1 + \xi_2 + \phi(t_1 + t_2)$$

Again since $0 < b < 1$, we can get $\phi(\frac{t_2}{b}) > \phi(t)$. Then we have,

$$
\begin{aligned}
(13)\quad F_{z,fz_1,p}(\phi(t)) &= F_{z,fz_1,p}\Big(\xi_1 + \xi_2 + \phi(t_1+t_2)\Big)\\
&\geq \Delta\Big(F_{z,fz_1,y_{n+1}(\xi_1),F_{z,y_{n+1},p}(\xi_2)}, F_{y_{n+1},fz_1,p}(\phi(t_1+t_2))\Big)\\
&\geq \Delta\Big(F_{z,y_{n+1},fz_1(\xi_1),F_{z,y_{n+1},p}(\xi_2)}, F_{fx_{n+1},fz_1,p}(\phi(t_1+t_2))\Big)\\
&\geq \Delta\Big(F_{z,y_{n+1},fz_1(\xi_1),F_{z,y_{n+1},p}(\xi_2)},\\
&\qquad \Psi(F_{gx_{n+1},fx_{n+1},p}(\phi(\frac{t_1}{a})), F_{fz_1,gz_1,p}(\phi(\frac{t_2}{b})))\Big)\\
&\geq \Delta\Big(F_{z,y_{n+1},fz_1(\xi_1),F_{z,y_{n+1},p}(\xi_2)}\Big),\\
&\qquad \Psi\Big(F_{y_n,y_{n+1},p}(\phi(\frac{t_1}{a})), F_{fz_1,z,p}(\phi(t))\Big).
\end{aligned}
$$

By (11), (12), (13), there exists a positive integer δ_1 such that

$$
F_{z,y_{n+1},fz_1(\xi_1),F_{z,y_{n+1},p}(\xi_2)}, F_{y_n,y_{n+1},p}(\phi(\frac{t_1}{a})) > F_{fz_1,z,p}(\phi(t))
$$

for all $n > \delta_1$. Then we have from (13), $F_{z,fz_1,p}(\phi(t)) > F_{z,fz_1,p}(\phi(t))$, which is a contradiction, therefore $F_{z,fz_1,p}(\phi(t)) = 1$ for all $t > 0$, which implies $fz_1 = z = gz_1$. Since f and g are weakly compatible, it follows that $fgz_1 = gfz_1$ i.e $fz = gz$.

Now we show that z is a fixed point of f and g. From(3), we have

$$
F_{fz,fx_n,p}(\phi(t)) \geq \Psi\Big(F_{gz,fz,p}(\phi(\frac{t_1}{a})), F_{fx_n,gx_n,p}(\phi(\frac{t_2}{b}))\Big)
$$

where $t = t_1 + t_2$. Then,

$$
F_{fz,fx_n,p}(\phi(t)) \geq \Psi\Big(F_{gz,fz,p}(\phi(\frac{t_1}{a})), F_{y_n,y_{n-1},p}(\phi(\frac{t_2}{b}))\Big).
$$

Taking limit as $n \to \infty$, we have $F_{fz,z,p}(\phi(t)) \geq \Psi(1,1) = 1$ and using property of Φ, we have $fz = z = gz$. Thus z is a fixed point of f and g.

Uniqueness. Let u, v be two fixed points, therefore, for all $t > 0$,

$$
\begin{aligned}
F_{u,v,p}(\phi(t)) &= F_{fu,fv,p}(\phi(t))\\
&\geq \Psi\Big(F_{gu,fu,p}\phi((\frac{t_1}{a})), F_{fv,gv,p}\phi((\frac{t_2}{b}))\Big)\\
&\qquad\qquad \text{for}\quad t_1, t_2 > 0 \quad\text{and}\quad t_1 + t_2 = t\\
&\geq \Psi\Big(F_{u,u,p}\phi((\frac{t_1}{a})), F_{v,v,p}\phi((\frac{t_2}{b}))\Big)\\
&= \Psi(1,1) = 1.
\end{aligned}
$$

Therefore $u = v$. This completeness the proof of the theorem. ■

Now we construct an example with the help of example in [2]

Example 3. Let $X = \{\alpha, \beta, \gamma, \delta\}$, the t-norm Δ is 3^{rd} order minimum t-norm and F be defined as:

$$F_{\alpha,\beta,\gamma}(t) = F_{\alpha,\beta,\delta}(t) = \begin{cases} 0, & \text{if} \quad t \le 0 \\ 0.40, & \text{if} \quad 0 < t < 4 \\ 1, & \text{if} \quad t \ge 4 \end{cases}$$

and

$$F_{\alpha,\gamma,\delta}(t) = F_{\beta,\gamma,\delta}(t) = \begin{cases} 0, & \text{if} \quad t \le 0 \\ 1, & \text{if} \quad t > 0. \end{cases}$$

Then (X, F, Δ) is a complete 2-Menger space. Now we define $f : X \to X$ and $g : X \to X$ as follows:

$$f\alpha = \delta, \quad f\beta = \gamma, \quad f\gamma = \gamma, \quad f\delta = \gamma \quad \text{and} \quad gx = x \quad \text{for all} \quad x \in X.$$

Then the mapping f and g satisfy all the conditions of the theorem 2, where

$$\phi(t) = \begin{cases} \sqrt{t}, & \text{if} \quad t > 0 \\ 0, & \text{if} \quad t \le 0. \end{cases}$$

and γ is the unique fixed point of f and g.

Theorem 3. *Theorem 2 remains true if a weakly compatible property is replaced by any one (retaining the rest of the hypothesis) of the following:*
(i) R-weakly commuting property,
(ii) R-weakly commuting property of type (A_f),
(iii) R-weakly commuting property of type (A_g),
(iv) R-weakly commuting property of type (P),
(v) Weakly commuting property.

Proof. Since all the conditions of Theorem 2 are satisfied, then the existence of coincidence points for the pair (f, g) is insured. Let z be an arbitrary point of coincidence for (f, g), then using R-weakly commuting one gets

$$F(fgz, gfz, p, t) \ge F(fz, gz, p, \frac{t}{R}) = 1$$

which amounts to say that $fgz = gfz$. Thus the pair (f, g) is coincidently commuting. Now applying Theorem 2, one concludes that f and g have a unique common fixed point.

In case (f, g) is an R-weakly commuting property of type A_f, then

$$F(fgz, g^2z, p, t) \ge F(fz, gz, p, \frac{t}{R}) = 1$$

which amounts to say that $fgz = g^2z$ i.e., $fgz = gfz$ (by Theorem 2). Similarly, if the pair is R-weakly commuting mappings of type (A_g) or type

(P) or weakly commuting, then (f, g) also commutes at their point of coincidence. Now in view of Theorem 2, in all four cases f and g have a unique common fixed point. This completes the proof. ■

As an application of Theorem 2, we prove common fixed point theorems for two finite families of mapping which runs as follows:

Theorem 4. *Let* $\{f_1, f_2, \ldots, f_m\}$ *and* $\{g_1, g_2, \ldots, g_n\}$ *be two finite families of self-mappings of a 2-Menger space* (X, F, Δ) *with continuous t-norm of Hadzic type such that* $f = f_1 f_2 \ldots f_m$, $g = g_1 g_2 \ldots g_n$, *satisfy condition (1),(2),(3),(4), then*

(i) f *and* g *have a point of coincidence.*

Moreover, if $f_i f_j = f_j f_i$, $g_r g_s = g_s g_r$, $f_i g_r = g_r f_i$, *for all* $i, j \in I_1 = \{1, 2, \ldots, m\}$, $r, s \in I_2 = \{1, 2, \ldots, n\}$, *then for all* $i \in I_1$, $r \in I_2$, *then* f_i, g_r *have a common fixed point.*

Proof. The conclusion (i) is immediate as f and g satisfy all the conditions of Theorem 3. Now appealing to component wise commutativity of various pairs, one can immediately prove that $fg = gf$ and hence, obviously pair (f, g) are coincidently commuting. Note that all the conditions of Theorem 3 (for mappings f, g) are satisfied ensuring the existence of a unique common fixed point, say z. Now one need to show that z remains the fixed point of all the component maps. For this consider

$$\begin{aligned}
f(f_i z) &= ((f_1, f_2, \ldots, f_m) f_i) z \\
&= (f_1, f_2, \mho, f_{m-1})((f_m f_i) z) \\
&= (f_1, f_2, \ldots, f_{m-1})(f_i f_m z) \\
&= (f_1, f_2, \ldots, f_{m-2})(f_{m-1} f_i (f_m z)) \\
&= (f_1, f_2, \ldots, f_{m-2})(f_i f_{m-1} (f_m z)) \\
&= \ldots \\
&= f_1 f_i (f_2 f_3 f_4, \ldots, f_m z) \\
&= f_i f_1 (f_2 f_3 f_4, \ldots, f_m z) = f_i(fz) = f_i z.
\end{aligned}$$

Similarly, one can show that

$$\begin{aligned}
f(g_r z) &= g_r(fz) = g_r z, \\
g(g_r z) &= g_r(gz) = g_r z, \\
g(f_i z) &= f_i(gz) = f_i z,
\end{aligned}$$

which show that (for all i and r) $f_i z$ and $g_r z$ are other fixed points of the pair (f, g). Now appealing to the uniqueness of common fixed points of pair (f, g), we get

$$z = f_i z = g_r z$$

which shows that z is a common fixed point of f_i, g_r, for all i and r. ■

By setting $f = f_1 = f_2 = \ldots = f_m$, $g = g_1 = g_2 = \ldots = g_n$, one deduces the following for certain iterates of maps, which runs as follows:

Corollary 1. *Let f, g be two self-mappings of a 2-Menger space (X, F, Δ) such thatf^m, g^n, satisfy the conditions (1) and (3). If one of $f^m(X)$, $g^n(X)$ is a closed subset of X, then f and g have a unique common fixed point provided the pair (f, g) commute.*

3. E.A. property

Now we prove a result for weakly compatible mappings along with E.A. property as follows:

Theorem 5. *Let f and g be self mappings of 2-Menger space (X, F, Δ) satisfying (3) and the following:*

$$f \;\; and \;\; g \;\; satisfy\, the\, E.A.\, property \tag{14}$$

$$g(X) \;\; is \;\; a \;\; closed \;\; subspace \;\; of \;\; X \tag{15}$$

$$f \;\; and \;\; g \;\; are \;\; weakly \;\; compatible \;\; on \;\; X, \tag{16}$$

$$provided \;\; 0 < F_{x,y,p}(t) < 1.$$

Then f and g have a unique common fixed point in X.

Proof. Since f and g satisfy the E.A. property, therefore there exists a sequence $\{x_n\}$ in X such that

$$\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = u \in X.$$

As $g(x)$ is a closed subspace of X, therefore, every convergent sequence of points of $g(X)$ has a limit in $g(X)$. Therefore,

$$u = \lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = ga \quad \text{for some} \quad a \in X. \tag{17}$$

This implies $u = ga \in g(X)$. We now show that $fa = u = ga$. If possible, let $0 < F_{u,fa,p}(\phi(t)) < 1$, for some $t > 0$. By virtue of the property of ϕ we can choose ξ_1, ξ_2, t_1, t_2 such that

$$\phi(t) = \xi_1 + \xi_2 + \phi(t_1 + t_2).$$

Again, since $0 < b < 1$, we can get $\phi(\frac{t_2}{b}) > \phi(t)$. Then we have,

$$
\begin{aligned}
(18)\quad F_{u,fa,p}(\phi(t)) &= F_{u,fa,p}\Big(\xi_1 + \xi_2 + \phi(t_1+t_2)\Big) \\
&\geq \Delta\Big(F_{u,fa,fu}(\xi_1), F_{u,fu,p}(\xi_2), F_{fu,fa,p}(\phi(t_1+t_2))\Big) \\
&\geq \Delta\Big(F_{u,fa,fu}(\xi_1), F_{u,fu,p}(\xi_2)\Big), \\
&\qquad \Psi\Big(F_{gu,fu,p}(\phi(\frac{t_1}{a})), F_{fa,ga,p}(\phi(\frac{t_2}{b}))\Big) \\
&\geq \Delta\Big(F_{u,fa,fu}(\xi_1), F_{u,fu,p}(\xi_2)\Big), \\
&\qquad \Psi\Big(F_{gu,fu,p}(\phi(\frac{t_1}{a})), F_{fa,u,p}(\phi(t))\Big).
\end{aligned}
$$

By (18),(17), there exists a positive integer δ_2 such that

$$F_{u,fa,fu}(\xi_1), F_{u,fu,p}(\xi_2)\Big), F_{gu,fu,p}(\phi(\frac{t_1}{a})) > F_{fa,u,p}(\phi(t)) \text{ for all } n > \delta_2.$$

Then we have from (18), $F_{u,fu,p}(\phi(t)) > F_{u,fa,p}(\phi(t))$, which is a contradiction, therefore $F_{u,fa,p}(\phi(t)) = 1$ for all $t > 0$, which implies that $fa = u = ga$. Since f and g are weakly compatible, it follows that $gfa = fga$ i.e $fu = gu$.

Now, we show that u is a fixed point of f and g. From (3), we have

$$F_{fu,fx_n,p}(\phi(t)) \geq \Psi\Big(F_{gu,fu,p}(\phi(\frac{t_1}{a})), F_{fx_n,gx_n,p}(\phi(\frac{t_2}{b}))\Big)$$

where $t = t_1 + t_2$. Taking limit as $n \to \infty$, we have

$$
\begin{aligned}
F_{fu,u,p}(\phi(t)) &\geq \Psi\Big(F_{fu,fu,p}(\phi(\frac{t_1}{a})), F_{u,u,p}(\phi(\frac{t_2}{b}))\Big) \\
F_{fu,u,p}(\phi(t)) &\geq \Psi\Big(1,1\Big)
\end{aligned}
$$

and using of ϕ, we have

$$fu = u = gu.$$

Thus u is a fixed point of f and g.

Uniqueness. Let u, v be two fixed points, therefore for all $t > 0$,

$$
\begin{aligned}
F_{u,v,p}(\phi(t)) &= F_{fu,fv,p}(\phi(t)) \\
&\geq \Psi\Big(F_{gu,fu,p}(\phi(\frac{t_1}{a})), F_{fv,gv,p}(\phi(\frac{t_2}{b}))\Big) \\
&\qquad \text{for} \quad t_1, t_2 > 0 \quad \text{and} \quad t = t_1 + t_2 \\
&\geq \Psi\Big(F_{u,u,p}(\phi(\frac{t_1}{a})), F_{v,v,p}(\phi(\frac{t_2}{b}))\Big) = \Psi\Big(1,1\Big) = 1.
\end{aligned}
$$

Therefore, $u = v$. Thus u is a unique fixed point of f and g. ■

4. (CLRg) property and weakly compatible maps

Now we prove a theorem for weakly compatible mappings along with (CLRg) property as follows:

Theorem 6. *Let f and g be self mappings of 2-Menger space (X, F, Δ) satisfying (3) and the following:*

$$f \textit{ and } g \textit{ satisfy the } (CLRg) \textit{ property} \tag{19}$$

$$f \textit{ and } g \textit{ are weakly compatible on } X, \textit{ provided } 0 < F_{x,y,p}(t) < 1. \tag{20}$$

Then f and g have a unique common fixed point in X.

Proof. Since f and g satisfy the $(CLRg)$ property, therefore there exists a sequence $\{x_n\}$ in X such that

$$\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = gx \in X. \tag{21}$$

We now show that $fx = gx$. If possible, let $0 < F_{fx,gx,p}(\phi(t)) < 1$, for some $t > 0$. By virtue of the property of ϕ we can choose ξ_1, ξ_2, t_1, t_2 such that

$$\phi(t) = \xi_1 + \xi_2 + \phi(t_1 + t_2)$$

Again, since $0 < b < 1$, we can get $\phi(\frac{t_2}{b}) > \phi(t)$. Then we have,

$$\begin{aligned} F_{gx_n,fx,p}(\phi(t)) &= F_{gx_n,fx,p}\Big(\xi_1 + \xi_2 + \phi(t_1 + t_2)\Big) \\ &\geq \Delta\Big(F_{gx_n,fx,fu}(\xi_1), F_{gx_n,fu,p}(\xi_2), F_{fu,fx,p}(\phi(t_1 + t_2))\Big). \end{aligned}$$

for all $n \in N$. Taking $\lim_{n\to\infty}$, we have

$$\begin{aligned} F_{gx,fx,p}(\phi(t)) &\geq \Delta\Big(F_{gx,fx,fu}(\xi_1), F_{gx,fu,p}(\xi_2)\Big) \\ &\qquad \Psi\Big(F_{gu,fu,p}(\phi(\frac{t_1}{a})), F_{fx,gx,p}(\phi(\frac{t_2}{b}))\Big) \\ &\geq \Delta\Big(F_{gx,fx,fu}(\xi_1), F_{gx,fu,p}(\xi_2)\Big) \\ &\qquad \Psi\Big(F_{gu,fu,p}(\phi(\frac{t_1}{a})), F_{fx,gx,p}(\phi(t))\Big) \end{aligned} \tag{22}$$

By (21),(22), there exists a positive integer δ_3 such that

$$F_{gx,fx,fu}(\xi_1), F_{gx,fu,p}(\xi_2), F_{gu,fu,p}(\phi(\frac{t_1}{a})) > F_{fx,gx,p}(\phi(t))$$

for all $n > \delta_3$. Then we have from (22),

$$F_{gx,fx,p}(\phi(t)) > F_{gx,fx,p}(\phi(t)),$$

which is a contradiction. Therefore $F_{gx,fx,p}(\phi(t)) = 1$ for all $t > 0$ which implies that $fx = gx = u$. Now from Theorem 5 we can obtain that u is a unique fixed point of f and g. ■

5. Conclusions

Fixed point theorem for weakly compatible, weakly compatible along with E.A. property, weakly compatible along with CLR property have been proved and that reflects the utility of weakly compatible maps along with E.A. and CLR properties. Now, there arises a natural question: "How fixed point theorems can be improved to the setting of non-complete metric spaces and without continuity of f and g over the whole space X?" We give the partial answer. It seems that fixed point theorems can be improved by using E.A. property. Aamri and El Moutawakil generalized the concept of non compatiblity in metric spaces by defining the notion E.A. property and proved common fixed point theorems under strict contractive conditions. A major benefit of E.A. property is that it ensures convergence of desired sequences without completeness. Further it was pointed out in [1] that E. A. property buys containment of ranges without any continuity requirements besides minimizes the commutativity conditions of the maps to the commutativity at their points of coincidence. Moreover, E.A. property Allows replacing the completeness requirement of the space with a more natural condition of closeness of the range.

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