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SOME MORE PROPERTIES OF γ -s-CLOSED SPACES

ABSTRACT. The concept of γ -s-closed spaces have been introduced and explored by S. Hussain and B. Ahmad [6]. In this paper, some more interesting characterizations have been constructed for the description of such spaces.

KEY WORDS: γ -closed (open), γ -interior (closure), γ -semi-open (closed), γ -s-closed, γ -s- θ -converges, γ -s- θ -accumulate, γ -s- θ -complete accumulation, γ -s- θ -adherence.

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1. Introduction

One of the important areas of mathematics is Topology. Whose study is not only interesting and present new results but also put into context old ones like continuous functions. Therefore, we may find its influence in approximately all fields of mathematics.

S. Kasahara [10] introduced and discussed an operation γ of a topology τ into the power set $P(X)$ of a space X . H. Ogata [15] introduced the concept of γ -open sets and investigated the related topological properties of the associated topology τ_γ and τ by using operation γ .

S. Hussain and B. Ahmad [1-9] continued studying the properties of γ -operations on topological spaces and investigated many interesting results. In 2007-08, they introduced and discussed γ -s-closed spaces and subspaces[5-6]. It is shown that the concept of γ -s-closed spaces generalized s-closed spaces [12]. It is interesting to note that γ -s-closedness is the generalization of γ_0 -compactness (which generalized compactness) defined and investigated in [3].

In this paper, some more interesting characterizations have been constructed for the description of such spaces.

First, we recall some definitions and results used in this paper. Hereafter, we shall write a space in place of a topological space.

2. Preliminaries

Definition 1 ([15]). Let X be a space. An operation $\gamma : \tau \rightarrow P(X)$ is a function from τ to the power set of X such that $V \subseteq V^\gamma$, for each $V \in \tau$, where V^γ denotes the value of γ at V . The operations defined by $\gamma(G) = G$, $\gamma(G) = cl(G)$ and $\gamma(G) = intcl(G)$ are examples of operation γ .

Definition 2 ([15]). Let $A \subseteq X$. A point $x \in A$ is said to be γ -interior point of A , if there exists an open nbd N of x such that $N^\gamma \subseteq A$ and we denote the set of all such points by $int_\gamma(A)$. Thus

$$int_\gamma(A) = \{x \in A : x \in N \in \tau \text{ and } N^\gamma \subseteq A\} \subseteq A.$$

Note that A is γ -open [15] iff $A = int_\gamma(A)$. A set A is called γ -closed [1] iff $X - A$ is γ -open.

Definition 3 ([15]). A point $x \in X$ is called a γ -closure point of $A \subseteq X$, if $U^\gamma \cap A \neq \phi$, for each open nbd U of x . The set of all γ -closure points of A is called γ -closure of A and is denoted by $cl_\gamma(A)$. A subset A of X is called γ -closed, if $cl_\gamma(A) \subseteq A$. Note that $cl_\gamma(A)$ is contained in every γ -closed superset of A .

Definition 4 ([1]). The γ -exterior of A , written $ext_\gamma(A)$ is defined as the γ -interior of $X - A$. That is, $ext_\gamma(A) = int_\gamma(X - A)$.

Definition 5 ([1]). The γ -boundary of A , written $bd_\gamma(A)$ is defined as the set of points which do not belong to the γ -interior or the γ -exterior of A .

Definition 6 ([15]). An operation γ on τ is said be regular, if for any open nbds U, V of $x \in X$, there exists an open nbd W of x such that $U^\gamma \cap V^\gamma \supseteq W^\gamma$.

Definition 7 ([15]). An operation γ on τ is said to be open, if for any open nbd U of each $x \in X$, there exists γ -open set B such that $x \in B$ and $U^\gamma \supseteq B$.

Definition 8 ([15]). A space X is said to be γ - T_2 space, if for each disjoint points x, y of X , there exist open sets U and V such that $x \in U$, $y \in V$ and $U^\gamma \cap V^\gamma = \phi$.

Definition 9 ([5]). A subset A of a space X is called γ -regular open, if $A = int_\gamma(cl_\gamma(A))$. The set of γ -regular open sets is denoted by $RO_\gamma(X)$.

Note that $RO_\gamma(X) \subseteq \tau_\gamma \subseteq \tau$. Where τ_γ denotes the set of all γ -open sets in X .

Definition 10 ([5]). A subset A of a space X is called γ -regular closed, denoted by $RC_\gamma(X)$, if one of the following conditions holds:

- (i) $A = cl_\gamma(int_\gamma(A))$.
- (ii) $(X - A) \in RO_\gamma(X)$.

Clearly A is γ -regular open if and only if $X - A$ is γ -regular closed.

Definition 11 ([3]). A subset A of a space X is said to be γ_0 -compact relative to X , if every cover $\{V_i : i \in I\}$ of X by γ -open sets of X , there exists a finite subset I_0 of I such that $A \subseteq \bigcup_{i \in I_0} cl_\gamma(V_i)$. A space X is γ_0 -compact, if $X = \bigcup_{i \in I_0} cl_\gamma(V_i)$.

Definition 12 ([3]). A space X is said to be a γ -regular space, if for any γ -closed set A and any point $x \notin A$, there exist γ -open sets U and V such that $x \in U$, and $A \subseteq V$ and $U \cap V = \phi$.

Definition 13 ([2]). Let X be a space and $x \in X$. Then x is called a γ -limit point (γ -adherent point) of A if and only if $U^\gamma \cap (A - \{x\}) \neq \phi$, where U is open set in X . The set of all γ -limit points (γ -adherent points) is called a γ -derived set (γ -adherent) and is denoted by A_γ^d (ad_γ).

Definition 14 ([1]). Let $A \subseteq X$. Then A is called γ -dense in itself, if $A \subseteq A_\gamma^d$.

Definition 15 ([9]). A subset A of a space X is said to be a γ -semi-open set, if there exists a γ -open set O such that $O \subseteq A \subseteq cl_\gamma(O)$. The set of all γ -semi-open sets is denoted by $SO_\gamma(X)$. A is γ -semi-closed if and only if $X - A$ is γ -semi-open in X . Note that A is γ -semi-closed if and only if $int_\gamma(cl_\gamma(A)) \subseteq A$ [4].

It is shown that every γ -open sets is γ -semi-open but converse is not true in general [4].

Definition 16 ([4]). Let X be a space and $A \subseteq X$. The intersection of all γ -semi-closed sets containing A is called γ -semi-closure of A and is denoted by $scl_\gamma(A)$. A is γ -semi-closed iff $scl_\gamma(A) = A$.

Definition 17 ([4]). Let X be a space and $A \subseteq X$. The union of γ -semi-open subsets of A is called γ -semi-interior of A and is denoted by $sint_\gamma(A)$.

Definition 18 ([8]). A space X is said to be γ -s-regular, if for any γ -semi-regular set A and $x \notin A$, there exist disjoint γ -open sets U and V such that $A \subseteq U$ and $x \in V$.

Definition 19 ([9]). A subset A of a space X is said to be γ -semi-regular, if it is both γ -semi-open and γ -semi-closed. The class of all γ -semi-regular sets of X is denoted by $SR_\gamma(A)$. If γ is regular, then the union of γ -semi-regular sets is γ -semi-regular.

Definition 20 ([6]). A space X is γ -extremally disconnected space, if $cl_\gamma(U)$ is γ -open set, for every γ -open set U in X .

It is also shown [6] that $cl_\gamma(U) = scl_\gamma(U)$, if X is γ -extremally disconnected.

Proposition 1 ([6]). If $A \in SO_\gamma(X)$, then $scl_\gamma(A) \in SR_\gamma(X)$.

Definition 21 ([6]). A point $x \in X$ is said to be a γ -semi- θ -adherent point of a subset A of X if $scl_\gamma(U) \cap A \neq \phi$, for every $U \in SO_\gamma(X)$. The set of all γ -semi- θ -adherent points of A is called the γ -semi- θ -closure of A and is denoted by $s_\gamma cl_\theta(A)$.

A subset A is called γ -semi- θ -closed if $s_\gamma cl_\theta(A) = A$.

Proposition 2 ([6]). Let A be a subset of a space X . Then we have

- (i) If $A \in SO_\gamma(X)$, then $scl_\gamma(A) = s_\gamma cl_\theta(A)$.
- (ii) If $A \in SR_\gamma(X)$, then A is γ -semi- θ -closed.

Definition 22 ([6]). A filter base Γ on X is said to γ -SR-converges to $x \in X$, if for each $V \in SR_\gamma(X)$, there exists $F \in \Gamma$ such that $F \subseteq V$.

Proposition 3 ([6]). A filter base Γ is said to be γ -SR-accumulate at $x \in X$ if $V \cap F \neq \phi$, for every $V \in SR_\gamma(X)$ and every $F \in \Gamma$.

3. Characterizations of γ -s-closed spaces

Definition 23 ([6]). A space X is said to be γ -s-closed, if for any cover $\{V_\alpha : \alpha \in I\}$ of X by γ -semi-open sets of X , there exists a finite subset I_0 of I such that $X = \bigcup_{\alpha \in I_0} scl_\gamma(V_\alpha)$.

Definition 24 ([6]). A filter base Γ on X is said to γ -s- θ -converges to $x \in X$, if for each $V \in SO_\gamma(X)$ containing x , there exists $A_i \in \Gamma$ such that $A_i \subseteq scl_\gamma(V)$.

Definition 25 ([6]). A filter base Γ is said to be γ -s- θ -accumulate at $x \in X$, if $A_i \cap scl_\gamma(V) \neq \phi$, for every $V \in SO_\gamma(X)$ containing x and each $A_i \in \Gamma$.

Definition 26 ([6]). The γ -s- θ -adherence of filterbase Γ denoted by $s_\gamma ad_\theta(\Gamma)$ and is defined as $\bigcap \{s_\gamma cl_\theta(A) : A \in \Gamma\}$.

The proof of the following theorem follows from the definitions and is thus omitted:

Theorem 1. Let X be a space and Γ be a filterbase on X . Then

- (i) Γ γ -s- θ -accumulates to a point $x \in X$ if and only if it γ -SR-accumulates to x .

- (ii) Γ γ -s- θ -converges to a point $x \in X$ if and only if it γ -SR-converges to x .

Definition 27 ([6]). Let X be a space and $|C|$ denotes the cardinality of $C \subseteq X$. A point $x \in X$ is said to be γ -s- θ -complete accumulation point of $C \subseteq X$, if $|C \cap B| = |C|$ for each $B \in SR_\gamma(X)$.

Proposition 4 ([6]). For any space X , the following are equivalent:

- (i) X is γ -s-closed.
- (ii) Every cover of X by γ -semi-regular sets has a finite sub cover.
- (iii) For every family $\{V_\alpha : \alpha \in I\}$ of γ -semi-regular sets such that $\bigcap\{V_\alpha : \alpha \in I\} = \phi$, there exists a finite subset I_0 of I such that $\bigcap\{V_\alpha : \alpha \in I_0\} = \phi$.
- (iv) Every filter base γ -SR-accumulates at some point of X .
- (v) Every maximal filter base γ -SR-converges to some point of X .

Theorem 2. A space X is γ -s-closed if and only if every infinite subset $A \subseteq X$ has a γ -s- θ -complete accumulation point in X .

Proof. Suppose X is γ -s-closed and A is an infinite subset of X . Let $S = \{x \in X : x \text{ is not a } \gamma\text{-s-}\theta\text{-complete accumulation point of } A\}$. For each $x \in S$, we can find a γ -semi-regular set R_x in X such that $|R_x \cap A| < |A|$. If $S = X$, then $U = \{R_x : x \in S\}$ is a cover of X by γ -semi-regular sets. Since X is γ -s-closed by Proposition 4, we can select a finite subcover say $U_0 = \{R_{x_1}, \dots, R_{x_n}\} \subseteq U$. Then $A \subseteq \bigcup\{R_{x_i} \cap A : i = 1, 2, \dots, n\}$ and $|A| = \max\{|R_{x_i} \cap A| : i = 1, 2, \dots, n\}$, a contradiction. Consequently, $X - A \neq \phi$ implies that A has a γ -s- θ -complete accumulation point.

Conversely, suppose that every infinite subset A of X has a γ -s- θ -complete accumulation point in X , and if possible, let X be not γ -s-closed. Then there is a cover U of X by γ -semi-regular sets, which has no finite subcover. Put $\alpha = \min\{|v| : v \subset U \text{ and } v \text{ is a cover of } X\}$. Fix $v^* \subset U$, for which $|v^*| = \alpha$ and $\bigcup\{U : U \in v^*\} = X$. Obviously, by assumption $\alpha \geq d$ where d is the cardinality of the set of natural numbers. Well order the set v^* by some minimal well ordering $' \prec'$. Let U be any element of v^* , since $' \prec'$ is a minimal well ordering $|\{V : V \in v^* \text{ and } v \prec U\}| < |\{V : V \in v^*\}|$. Obviously v^* can not have any subcover with cardinality less than α and so $\bigcup\{V : V \in v^* \text{ and } V \prec U\} \neq X$ for each $U \in v^*$. Choose a point $x(U) \in X - \bigcup\{V \cup \{x(V)\} : V \in v^* \text{ and } V \prec U\}$ for each $U \in v^*$. This can always be done because otherwise one could select from v^* a cover of small cardinality. If $M = \{x(U) : U \in v^*\}$ then we shall show that M has no γ -s- θ -complete accumulation point in X . Let $y \in X$. Now $y \in U_1$ for some $U_1 \in v^*$, since v^* is a covering of Y . But $x(U) \in U_1$ implies, by the choice of $x(U)$, that $U \prec U_1$. Hence $\{U : U \in v^* \text{ and } x(U) \in U_1\} \subset \{V : V \in v^* \text{ and } V \prec U_1\}$. But $|\{U : U \in v^* \text{ and } U \prec U_1\}| < \alpha$, by the minimality

of \prec . Consequently $|M \cap U_1| < \alpha$. But $|M| < |\alpha| \geq d$ because $V_1 \neq V_2$ and $V_1, V_2 \in v^*$ imply that $x(V_1) \neq x(V_2)$. Thus M has no γ -s- θ -complete accumulation point. ■

Definition 28 ([6]). A function $f : X \rightarrow Y$ is said have a γ -s- θ -subclosed graph, if for each $x \in X$ and each filterbase Γ on $X - \{x\}$ with $\Gamma \rightarrow x$, $f(\Gamma)$ has almost one γ -s- θ -accumulation point viz. $f(x)$. Equivalently f has a γ -s- θ -subclosed graph if for each $x \in X$ and each net $\{S_n : n \in D\}$ on $X - \{x\}$ with $S_n \rightarrow x$, $f(S_n)$ is frequently in every γ -semi-regular set containing $f(x)$.

Definition 29 ([4]). A function $f : X \rightarrow Y$ is a γ -semi-continuous if and only if for any γ -open set B in Y , $f^{-1}(B)$ is γ -semi-open in X .

Theorem 3 ([4]). Let $f : X \rightarrow Y$ be a function and $x \in X$. Then f is γ -semi-continuous if and only if for each γ -open set B containing $f(x)$, there exists $A \in SO_\gamma(X)$ such that $x \in A$ and $f(A) \subseteq B$, where γ is a regular operation.

The proof of the following theorem is obvious.

Theorem 4. Let $f : X \rightarrow Y$ be a function. Then the following are equivalent:

- (i) f is γ -semi-continuous.
- (ii) for each filterbase Γ on X , $f(ad_\gamma(\Gamma)) \subseteq s_\gamma ad_\theta(\Gamma)$.

Theorem 5. Let $f : X \rightarrow Y$ be a function from space X to Y such that each filterbase on Y with atmost one γ -s- θ -adherent point is γ -s- θ -convergent. Then f is γ -semi-continuous if f has a γ -s- θ -subclosed graph.

Proof. Let f has a γ -s- θ -subclosed graph. To show that f is γ -semi-continuous it suffices to show that $f(ad_\gamma(\Gamma)) \subseteq s_\gamma ad_\theta(\Gamma)$ for any filterbase Γ on X . Let $y \in f(ad_\gamma(\Gamma))$. Then there exists a $x \in ad_\gamma(\Gamma)$ such that $y = f(x)$. Let $\Gamma^* = \{(V \cap F) - \{x\}, V \in \tau_\gamma(x) \text{ and } F \in \Gamma\}$, where $\tau_\gamma(x)$ denotes the class of all γ -open sets containing x . We consider following two cases:

Case 1. Γ^* is not a filterbase; then, for some $F \in \Gamma$, $V \cap (F - \{x\}) = \Phi$, i.e. $x \in F$. We demand that $x \in F$, for each $F \in \Gamma$; infact, if possible, $x \notin F$ for some $F \in \Gamma$ say F_1 ; then there exists a F_2 for which $F \cap F_1 \supseteq F_2$. But $V \cap F_2 - \{x\} \subseteq V \cap F - \{x\} = \phi$ which gives $x \in F_2 \subseteq F_1$, a contradiction. Thus $x \in F$ for each $F \in \Gamma$ and so $f(x) \in f(F)$ for each $F \in \Gamma$. Consequently $f(x) \in s_\gamma ad_\theta f(\Gamma)$.

Case 2. Suppose Γ^* is a filterbase on $X - \{x\}$. Then $\Gamma^* \rightarrow x$ and by definition of γ -s- θ -subclosed graph, $f(\Gamma^*)$ has atmost one γ -s- θ -accumulation point viz. $f(x)$. Hence by the given condition, $f(\Gamma^*)$ is γ -s- θ -convergent and it γ -s- θ -converges to $f(x) = y$. Thus $y \in s_\gamma ad_\theta f(\Gamma^*) \subseteq s_\gamma ad_\theta f(\Gamma)$. Thus f is γ -semi-continuous. ■

Theorem 6. *Let $f : X \rightarrow Y$ be a function from space X to Y such that every filterbase on Y has at most one γ -s- θ -accumulation point. If f is γ -semi-continuous, then the graph of f is γ -s- θ -subclosed.*

Proof. Let $f : X \rightarrow Y$ be γ -semi-continuous and let Γ be a filterbase on $X - \{x\}$ converging to x . Since f is γ -semi-continuous, $f(ad(\Gamma)) \subseteq s_\gamma ad_\theta(\Gamma)$. That is, $f(x)$ is a γ -s- θ -adherent point of $f(\Gamma)$. By the given condition $f(x)$ is the only γ -s- θ -adherent point of $f(\Gamma)$. Hence the graph of f is γ -s- θ -subclosed. ■

Theorem 7. *Let X be a γ -s-closed space, then every filterbase on X with at most one γ -s- θ -accumulation point is γ -s- θ -convergent.*

Proof. Let X be a γ -s-closed. Then by Proposition 4 and Theorem 1, every filterbase has at least one γ -s- θ -accumulation point. Let x be the only γ -s- θ -accumulation point of Γ . If possible, let Γ do not γ -s- θ -converge to $x \in X$. Then for some $R \in SR_\gamma(x)$, $F \cap (X - R) \neq \phi$ for each $F \in \Gamma$ so that $\Gamma^* = \{F \cap (X - R) : F \in \Gamma\}$ is a filterbase on X and hence has a γ -s- θ -accumulation point say x_0 . Since the γ -semi-regular set R containing x has empty intersection with each member of Γ^* , $x \neq x_0$. But every set belongs to $SR_\gamma(x_0)$ intersects each member of Γ^* and hence each member of Γ . Thus Γ γ -s- θ -accumulates to x_0 which is a contradiction. ■

Theorem 8. *Let $f : X \rightarrow Y$ be a function from space X to Y . If f is γ -semi-continuous whenever f has a γ -s- θ -subclosed graph. Then Y is γ -s-closed.*

Proof. To prove that Y is γ -s-closed, it is sufficient to show by virtue of Proposition 4 and Theorem 2 that every filterbase on Y has a γ -s- θ -accumulation point. If not, then suppose that there exists a filterbase Γ on Y such that $s_\gamma ad_\theta(\Gamma) = \phi$. Choose $x_0 \in Y$. Define $f : (Y, (x_0, \Gamma)) \rightarrow (Y, \tau)$, where τ is the given topology on Y , as follows: $f(y) = y$ for each $y \in Y$. Let $p \in Y$ and let Γ^* be a filterbase on $Y - \{p\}$ such that $\Gamma^* \rightarrow p$ in $(Y, (x_0, \Gamma))$. Then $p = x_0$ and so Γ^* is a filterbase on $Y - \{x_0\}$ and $f(\Gamma^*) = s_\gamma ad_\theta(\Gamma^*) \subseteq s_\gamma ad_\theta(\Gamma)$ (since $\Gamma \subseteq \Gamma^*$) = $\phi \subseteq \{f(x_0)\}$. This shows that f has a γ -s- θ -subclosed graph. But f is not γ -semi-continuous. In fact, $ad_\gamma(\Gamma) = \{x_0\}$ and $f(x_0) = x_0$; but $s_\gamma ad_\theta(\Gamma) = \phi$ and hence we can not say that $f(ad_\gamma(\Gamma)) \subseteq s_\gamma ad_\theta(f(\Gamma))$. But this leads to a contradiction. Therefore every filterbase on Y has a γ -s- θ -accumulation point and hence Y is γ -s-closed. ■

Theorem 9. *A space Y is γ -s-closed if and only if every filterbase on Y with almost one γ -s- θ -accumulation point is γ -s- θ -convergent.*

Proof. *Necessity.* This follows from Theorem 7.

Sufficiency. Let X be any space and suppose $f : X \rightarrow Y$ have γ -s- θ -sub-closed graph. By Theorem 5, f is γ -semi-continuous. Thus γ -s-closedness of Y assures from Theorem 8. ■

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