DE GRUYTER OPEN
FASCICULI MATHEMATICI
Nr 54 2015
DOI:10.1515/fascmath-2015-0008

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SOME FIXED POINT THEOREMS USING wt-DISTANCE IN b-METRIC SPACES

Abstract. In this paper we establish some common fixed point theorems by using the concept of wt-distance in a b-metric space. Our results extend and generalize several well known comparable results in the existing literature. Finally, some examples are provided to illustrate our results.

Key words: b-metric space, wt-distance, expansive mapping, common fixed point.

AMS Mathematics Subject Classification: 54H25, 47H10.

1. Introduction

The study of metric fixed point theory has been at the centre of vigorous research activity and it has a wide range of applications in applied mathematics and sciences. Over the past two decades a considerable amount of research work for the development of fixed point theory have executed by several mathematicians. There has been a number of generalizations of the usual notion of a metric space. One such generalization is a b-metric space introduced and studied by Bakhtin [5] and Czerwik [11]. After that a series of articles have been dedicated to the improvement of fixed point theory in b-metric spaces. Recently, Hussain et.al.[16] introduced a new concept of wt-distance on b-metric spaces, which is a b-metric version of the w-distance of Kada et.al.[20] and proved some fixed point results in a partially ordered b-metric space by using the wt-distance. In this work, we prove some common fixed point theorems for a pair of self mappings by using the wt-distance. Further, our results are used to obtain several important fixed point theorems in b-metric spaces. Finally, some examples are provided to examine the strength of the hypothesis of the main result.

2. Preliminaries

In this section we need to recall some basic notations, definitions, and necessary results from existing literature.

Definition 1 ([11])**.** *Let X be a nonempty set and $s \geq 1$ be a given real number. A function $d : X \times X \to \mathbb{R}^+$ is said to be a b-metric on X if the following conditions hold:*

(i) $d(x,y) = 0$ if and only if $x = y$;
(ii) $d(x,y) = d(y,x)$ for all $x, y \in X$;
(iii) $d(x,y) \leq s\,(d(x,z) + d(z,y))$ for all $x, y, z \in X$.

The pair (X,d) is called a b-metric space.

Observe that if $s = 1$, then the ordinary triangle inequality in a metric space is satisfied, however it does not hold true when $s > 1$. Thus the class of b-metric spaces is effectively larger than that of the ordinary metric spaces. That is, every metric space is a b-metric space, but the converse need not be true. The following examples illustrate the above remarks.

Example 1. Let $X = \{-1, 0, 1\}$. Define $d : X \times X \to \mathbb{R}^+$ by $d(x,y) = d(y,x)$ for all $x, y \in X$, $d(x,x) = 0$, $x \in X$ and $d(-1,0) = 3$, $d(-1,1) = d(0,1) = 1$. Then (X,d) is a b-metric space, but not a metric space since the triangle inequality is not satisfied. Indeed, we have that

$$d(-1,1) + d(1,0) = 1 + 1 = 2 < 3 = d(-1,0).$$

It is easy to verify that $s = \frac{3}{2}$.

Example 2 ([16])**.** Let $X = \mathbb{R}$ and $d : X \times X \to \mathbb{R}^+$ be such that

$$d(x,y) = \mid x - y \mid^2 \text{ for any } x, y \in X.$$

Then (X,d) is a b-metric space with $s = 2$, but not a metric space.

Definition 2 ([9])**.** *Let (X,d) be a b-metric space, $x \in X$ and (x_n) be a sequence in X. Then*

(i) (x_n) converges to x if and only if $\lim\limits_{n\to\infty} d(x_n, x) = 0$. We denote this by $\lim\limits_{n\to\infty} x_n = x$ or $x_n \to x (n \to \infty)$.
(ii) (x_n) is Cauchy if and only if $\lim\limits_{n,m\to\infty} d(x_n, x_m) = 0$.
(iii) (X,d) is complete if and only if every Cauchy sequence in X is convergent.

Definition 3. *Let (X,d) be a b-metric space and let $T : X \to X$ be a given mapping. We say that T is continuous at $x_0 \in X$ if for every sequence (x_n) in X, we have $x_n \to x_0$ as $n \to \infty \Longrightarrow T(x_n) \to T(x_0)$ as $n \to \infty$. If T is continuous at each point $x_0 \in X$, then we say that T is continuous on X.*

Theorem 1 ([1])**.** *Let (X,d) be a b-metric space and suppose that (x_n) and (y_n) converge to $x, y \in X$, respectively. Then, we have*

$$\frac{1}{s^2} d(x,y) \leq \liminf_{n\to\infty} d(x_n, y_n) \leq \limsup_{n\to\infty} d(x_n, y_n) \leq s^2 d(x,y).$$

In particular, if $x = y$*, then* $\lim_{n\to\infty} d(x_n, y_n) = 0$*. Moreover, for each* $z \in X$*, we have*

$$\frac{1}{s}d(x,z) \leq \liminf_{n\to\infty} d(x_n,z) \leq \limsup_{n\to\infty} d(x_n,z) \leq sd(x,z).$$

Definition 4 ([16])**.** *Let* (X,d) *be a* b*-metric space with constant* $s \geq 1$*. Then a function* $p : X \times X \to [0,\infty)$ *is called a* wt*-distance on* X *if the following conditions are satisfied:*

(*i*) $p(x,z) \leq s\,(p(x,y) + p(y,z))$ *for any* $x,y,z \in X$;

(*ii*) *for any* $x \in X$, $p(x,.) : X \to [0,\infty)$ *is* s*-lower semi-continuous*;

(*iii*) *for any* $\epsilon > 0$ *there exists* $\delta > 0$ *such that* $p(z,x) \leq \delta$ *and* $p(z,y) \leq \delta$ *imply* $d(x,y) \leq \epsilon$.

Let us recall that a real valued function f defined on a b-metric space X is said to be s-lower semi-continuous at a point x_0 in X if $\liminf_{x_n\to x_0} f(x_n) = \infty$ or $f(x_0) \leq \liminf_{x_n\to x_0} sf(x_n)$, whenever $x_n \in X$ for each $n \in \mathbb{N}$ and $x_n \to x_0$ [18].

Lemma 1 ([16])**.** *Let* (X,d) *be a* b*-metric space with constant* $s \geq 1$ *and let* p *be a* wt*-distance on* X*. Let* (x_n) *and* (y_n) *be sequences in* X*, let* (α_n) *and* (β_n) *be sequences in* $[0,\infty)$ *converging to* 0*, and let* $x,y,z \in X$*. Then the following hold:*

(*i*) *If* $p(x_n,y) \leq \alpha_n$ *and* $p(x_n,z) \leq \beta_n$ *for any* $n \in \mathbb{N}$*, then* $y = z$*. In particular, if* $p(x,y) = 0$ *and* $p(x,z) = 0$*, then* $y = z$;

(*ii*) *if* $p(x_n,y_n) \leq \alpha_n$ *and* $p(x_n,z) \leq \beta_n$ *for any* $n \in \mathbb{N}$*, then* (y_n) *converges to* z;

(*iii*) *if* $p(x_n,x_m) \leq \alpha_n$ *for any* $n,m \in \mathbb{N}$ *with* $m > n$*, then* (x_n) *is a Cauchy sequence*;

(*iv*) *if* $p(y,x_n) \leq \alpha_n$ *for any* $n \in \mathbb{N}$*, then* (x_n) *is a Cauchy sequence.*

Example 3 ([16])**.** Let (X,d) be a b-metric space. Then d is a wt-distance on X.

Example 4 ([16])**.** Let $X = \mathbb{R}$ and $d(x,y) = (x-y)^2$. Then the function $p : X \times X \to [0,\infty)$ defined by $p(x,y) =\mid x \mid^2 + \mid y \mid^2$ for every $x,y \in X$ is a wt-distance on X.

Example 5 ([16])**.** Let $X = \mathbb{R}$ and $d(x,y) = (x-y)^2$. Then the function $p : X \times X \to [0,\infty)$ defined by $p(x,y) =\mid y \mid^2$ for every $x,y \in X$ is a wt-distance on X.

Definition 5. *Let* (X,d) *be a* b*-metric space with constant* $s \geq 1$*. A mapping* $T : X \to X$ *is called expansive if there exists a real constant* $k > s$ *such that*

$$d(T(x),T(y)) \geq k\,d(x,y) \quad \text{for all} \quad x,y \in X.$$

3. Main results

In this section, we present our new results.

Theorem 2. *Let p be a wt-distance on a complete b-metric space (X,d) with constant $s \geq 1$. Let T_1, T_2 be mappings from X into itself. Suppose that there exists $r \in [0, \frac{1}{s})$ such that*

$$\max \left\{ \begin{array}{l} p(T_1(x), T_2T_1(x)), \\ \\ p(T_2(x), T_1T_2(x)) \end{array} \right\} \leq r \min \{ p(x, T_1(x)), p(x, T_2(x))\} \tag{1}$$

for every $x \in X$ and that

$$\inf \{p(x,y) + \min\{p(x,T_1(x)),\, p(x,T_2(x))\} : x \in X\} > 0 \tag{2}$$

for every $y \in X$ with y is not a common fixed point of T_1 and T_2. Then T_1 and T_2 have a common fixed point in X. Moreover, if $v = T_1(v) = T_2(v)$, then $p(v,v) = 0$.

Proof. Let $u_0 \in X$ be arbitrary and define a sequence (u_n) by

$$u_n = \begin{cases} T_1(u_{n-1}), & \text{if } n \text{ is odd} \\ T_2(u_{n-1}), & \text{if } n \text{ is even.} \end{cases}$$

If $n \in \mathbb{N}$ is odd, then by using (1)

$$\begin{aligned} p(u_n, u_{n+1}) &= p(T_1(u_{n-1}), T_2(u_n)) \\ &= p(T_1(u_{n-1}), T_2T_1(u_{n-1})) \\ &\leq \max\{p(T_1(u_{n-1}), T_2T_1(u_{n-1})), \\ &\qquad\qquad p(T_2(u_{n-1}), T_1T_2(u_{n-1}))\} \\ &\leq r\min\{p(u_{n-1}, T_1(u_{n-1})), p(u_{n-1}, T_2(u_{n-1}))\} \\ &\leq rp(u_{n-1}, T_1(u_{n-1})) \\ &= rp(u_{n-1}, u_n). \end{aligned}$$

If n is even,then by (1), we have

$$\begin{aligned} p(u_n, u_{n+1}) &= p(T_2(u_{n-1}), T_1(u_n)) \\ &= p(T_2(u_{n-1}), T_1T_2(u_{n-1})) \\ &\leq \max\{p(T_2(u_{n-1}), T_1T_2(u_{n-1})), \\ &\qquad\qquad p(T_1(u_{n-1}), T_2T_1(u_{n-1}))\} \\ &\leq r\min\{p(u_{n-1}, T_2(u_{n-1})), p(u_{n-1}, T_1(u_{n-1}))\} \\ &\leq rp(u_{n-1}, T_2(u_{n-1})) \\ &= rp(u_{n-1}, u_n). \end{aligned}$$

Thus for any positive integer n, we obtain

$$p(u_n, u_{n+1}) \leq r\, p(u_{n-1}, u_n). \tag{3}$$

By repeated application of (3), we get

$$p(u_n, u_{n+1}) \leq r^n\, p(u_0, u_1). \tag{4}$$

For $m, n \in \mathbb{N}$ with $m > n$, we have by repeated use of (4)

$$\begin{aligned} p(u_n, u_m) &\leq s\,[p(u_n, u_{n+1}) + p(u_{n+1}, u_m)] \\ &\leq sp(u_n, u_{n+1}) + s^2 p(u_{n+1}, u_{n+2}) + \ldots \\ &\quad + s^{m-n-1}\,[p(u_{m-2}, u_{m-1}) + p(u_{m-1}, u_m)] \\ &\leq \left[sr^n + s^2 r^{n+1} + \ldots + s^{m-n-1} r^{m-2} + s^{m-n-1} r^{m-1}\right] p(u_0, u_1) \\ &\leq \left[sr^n + s^2 r^{n+1} + \cdots + s^{m-n-1} r^{m-2} + s^{m-n} r^{m-1}\right] p(u_0, u_1) \\ &= sr^n \left[1 + sr + (sr)^2 + \cdots + (sr)^{m-n-2} + (sr)^{m-n-1}\right] p(u_0, u_1) \\ &\leq \frac{sr^n}{1 - sr}\, p(u_0, u_1). \end{aligned}$$

By Lemma 1(*iii*), (u_n) is a Cauchy sequence in X. Since X is complete, (u_n) converges to some point $z \in X$. Let $n \in \mathbb{N}$ be fixed. Then since (u_m) converges to z and $p(u_n, .)$ is s-lower semi-continuous, we have

$$p(u_n, z) \leq \liminf_{m \to \infty}\, s\, p(u_n, u_m) \leq \frac{s^2 r^n}{1 - sr}\, p(u_0, u_1).$$

Assume that z is not a common fixed point of T_1 and T_2. Then by hypothesis

$$\begin{aligned} 0 &< \inf\{p(x, z) + min\,\{\, p(x, T_1(x)), p(x, T_2(x))\} : x \in X\} \\ &\leq \inf\{p(u_n, z) + min\,\{p(u_n, T_1(u_n)), p(u_n, T_2(u_n))\} : n \in \mathbb{N}\} \\ &\leq \inf\left\{\frac{s^2 r^n}{1 - sr} p(u_0, u_1) + p(u_n, u_{n+1}) : n \in \mathbb{N}\right\} \\ &\leq \inf\left\{\frac{s^2 r^n}{1 - sr} p(u_0, u_1) + r^n\, p(u_0, u_1) : n \in \mathbb{N}\right\} \\ &= 0 \end{aligned}$$

which is a contradiction. Therefore, $z = T_1(z) = T_2(z)$.

If $v = T_1(v) = T_2(v)$ for some $v \in X$, then

$$\begin{aligned} p(v, v) &= \max\{p(T_1(v), T_2 T_1(v)), p(T_2(v), T_1 T_2(v))\} \\ &\leq r \min\{p(v, T_1(v)), p(v, T_2(v))\} \\ &= r \min\{p(v, v), p(v, v)\} \\ &= rp(v, v) \end{aligned}$$

which gives that, $p(v, v) = 0$. ■

Corollary 1. *Let (X,d) be a complete b-metric space with constant $s \geq 1$, let p be a wt-distance on X and let T be a mapping from X into itself. Suppose that there exists $r \in [0, \frac{1}{s})$ such that*

$$p(T(x), T^2(x)) \leq r\, p(x, T(x))$$

for every $x \in X$ and that

$$\inf \{p(x,y) + p(x, T(x)) : x \in X\} > 0$$

for every $y \in X$ with $y \neq T(y)$. Then T has a fixed point in X. Moreover, if $v = T(v)$, then $p(v,v) = 0$.

Proof. The result follows from Theorem 2 by taking $T_1 = T_2 = T$. ∎

As an application of Corollary 1, we have the following results.

Theorem 3. *Let (X,d) be a complete b-metric space with constant $s \geq 1$, let p be a wt-distance on X and let T be a continuous mapping from X into itself. Suppose that there exists $r \in [0, \frac{1}{s})$ such that*

$$p(T(x), T^2(x)) \leq rp(x, T(x))$$

for every $x \in X$. Then T has a fixed point in X. Moreover, if $v = T(v)$, then $p(v,v) = 0$.

Proof. If possible, suppose there exists $y \in X$ with $y \neq T(y)$ and

$$\inf \{p(x,y) + p(x, T(x)) : x \in X\} = 0.$$

Then there exists a sequence (x_n) in X such that

$$\lim_{n\to\infty} \{p(x_n, y) + p(x_n, T(x_n))\} = 0$$

which gives that $p(x_n, y) \to 0$ and $p(x_n, T(x_n)) \to 0$. By using Lemma 1, it follows that $T(x_n) \to y$. We also have

$$\begin{aligned} p(x_n, T^2(x_n)) &\leq s\left[p(x_n, T(x_n)) + p(T(x_n), T^2(x_n))\right] \\ &\leq s(1+r)p(x_n, T(x_n)) \longrightarrow 0. \end{aligned}$$

Therefore, $(T^2(x_n))$ converges to y. But $T : X \to X$ being continuous, we have

$$T(y) = T\left(\lim_{n\to\infty} T(x_n)\right) = \lim_{n\to\infty} T^2(x_n) = y$$

which contradicts the fact that $y \neq T(y)$. Thus, if $y \neq T(y)$, then

$$\inf \{p(x,y) + p(x, T(x)) : x \in X\} > 0.$$

By applying Corollary 1, we obtain the desired conclusion. ∎

Theorem 4. *Let* (X, d) *be a complete* b*-metric space with constant* $s \geq 1$ *and let* $T : X \to X$ *be such that*

$$d(T(x), T(y)) \leq \alpha d(x, y) + \beta d(x, T(x)) + \gamma d(y, T(y)) \tag{5}$$

for every $x, y \in X$*, where* $\alpha, \beta, \gamma \geq 0$ *with* $\alpha + \beta + \gamma < \frac{1}{s}$*. Then* T *has a unique fixed point in* X*.*

Proof. We treat the b-metric d as a wt-distance on X. From (5), we have

$$d(T(x), T^2(x)) \leq \alpha d(x, T(x)) + \beta d(x, T(x)) + \gamma d(T(x), T^2(x))$$

which gives that

$$d(T(x), T^2(x)) \leq \frac{\alpha + \beta}{1 - \gamma} d(x, T(x)). \tag{6}$$

Let us put $r = \frac{\alpha+\beta}{1-\gamma}$. Then $r \in [0, \frac{1}{s})$ since $s(\alpha + \beta) + \gamma \leq s(\alpha + \beta + \gamma) < 1$. Therefore, (6) becomes

$$d(T(x), T^2(x)) \leq r d(x, T(x))$$

for every $x \in X$.

Suppose there exists $y \in X$ with $y \neq T(y)$ and

$$\inf \{d(x, y) + d(x, T(x)) : x \in X\} = 0.$$

Then there exists a sequence (x_n) in X such that

$$\lim_{n \to \infty} \{d(x_n, y) + d(x_n, T(x_n))\} = 0.$$

So, we get $d(x_n, y) \to 0$ and $d(x_n, T(x_n)) \to 0$. By Lemma 1, it follows that $T(x_n) \to y$. We also have

$$\begin{aligned} d(y, T(y)) &\leq s\,[d(y, T(x_n)) + d(T(x_n), T(y))] \\ &\leq s\,[d(y, T(x_n)) + \alpha d(x_n, y) + \beta d(x_n, T(x_n)) + \gamma d(y, T(y))] \end{aligned}$$

for any $n \in \mathbb{N}$ and hence

$$d(y, T(y)) \leq s\gamma d(y, T(y)).$$

Therefore, $d(y, T(y)) = 0$ i.e., $y = T(y)$. This is a contradiction. Hence, if $y \neq T(y)$, then

$$\inf \{d(x, y) + d(x, T(x)) : x \in X\} > 0.$$

By applying Corollary 1, we obtain a fixed point of T in X. Clearly, T has unique fixed point in X. ■

Theorem 5. *Let (X, d) be a complete b-metric space with constant $s \geq 1$ and let $T : X \to X$ be such that*

$$d(T(x), T(y)) \leq \alpha d(x, T(y)) + \beta d(y, T(x)) \tag{7}$$

for every $x, y \in X$, where $\alpha, \beta \geq 0$ with $\alpha s < \frac{1}{1+s}$ or $\beta s < \frac{1}{1+s}$. Then T has a fixed point in X. Moreover, if $\alpha + \beta < 1$, then T has a unique fixed point in X.

Proof. We treat the b-metric d as a wt-distance on X. From (7), we have

$$\begin{aligned} d(T(x), T^2(x)) &\leq \alpha d(x, T^2(x)) + \beta d(T(x), T(x)) \\ &\leq \alpha s[d(x, T(x)) + d(T(x), T^2(x))] \end{aligned}$$

which gives that

$$d(T(x), T^2(x)) \leq \frac{\alpha s}{1 - \alpha s} d(x, T(x)). \tag{8}$$

Let us put $r = \frac{\alpha s}{1-\alpha s}$. Then $r \in [0, \frac{1}{s})$. Therefore, (8) becomes

$$d(T(x), T^2(x)) \leq r d(x, T(x))$$

for every $x \in X$. Suppose there exists $y \in X$ with $y \neq T(y)$ and

$$\inf \{d(x, y) + d(x, T(x)) : x \in X\} = 0.$$

Then there exists a sequence (x_n) in X such that

$$\lim_{n\to\infty} \{d(x_n, y) + d(x_n, T(x_n))\} = 0.$$

So, we get $d(x_n, y) \to 0$ and $d(x_n, T(x_n)) \to 0$. By Lemma 1, it follows that $T(x_n) \to y$. We also have

$$\begin{aligned} d(y, T(y)) &\leq s\,[d(y, T(x_n)) + d(T(x_n), T(y))] \\ &\leq s\,[d(y, T(x_n)) + \alpha d(x_n, T(y)) + \beta d(y, T(x_n))] \\ &\leq s\,[d(y, T(x_n)) + \alpha s d(x_n, y) + \alpha s d(y, T(y)) + \beta d(y, T(x_n))] \end{aligned}$$

for any $n \in \mathbb{N}$ and hence

$$d(y, T(y)) \leq s^2 \alpha d(y, T(y)).$$

Therefore, $d(y, T(y)) = 0$ i.e., $y = T(y)$. This is a contradiction. Hence, if $y \neq T(y)$, then

$$\inf \{d(x, y) + d(x, T(x)) : x \in X\} > 0.$$

By applying Corollary 1, we obtain a fixed point of T in X.

Now suppose that $\alpha+\beta<1$. Assume that there are $u, v \in X$ such that $T(u)=u$ and $T(v)=v$. Then

$$d(u, v)=d(T(u), T(v)) \leq \alpha d(u, v)+\beta d(v, u)=(\alpha+\beta) d(u, v).$$

This shows that $d(u, v)=0$ i.e., $u=v$. Therefore, T has a unique fixed point in X. ■

Theorem 6. *Let (X, d) be a complete b-metric space with constant $s \geq 1$ and let T be a mapping from X into itself. Suppose there exists $r \in [0, \frac{1}{s})$ such that*

$$d(T(x), T(y)) \leq r \max\{d(x, y), d(x, T(x)), d(y, T(y)), d(y, T(x))\} \tag{9}$$

for every $x, y \in X$. Then T has a unique fixed point in X.

Proof. We treat the b-metric d as a wt-distance on X. From (9), we have

$$\begin{aligned} d(T(x), T^2(x)) &\leq r \max\left\{\begin{array}{l} d(x, T(x)), d(x, T(x)), \\ d(T(x), T^2(x)), d(T(x), T(x)) \end{array}\right\} \\ &= r \max\{d(x, T(x)), d(T(x), T^2(x))\}. \end{aligned} \tag{10}$$

Without loss of generality, we assume that $T(x) \neq T^2(x)$. For, otherwise, T has a fixed point. Since $r<\frac{1}{s}$, we obtain from (10) that

$$d(T(x), T^2(x)) \leq r d(x, T(x))$$

for every $x \in X$. Assume that there exists $y \in X$ with $y \neq T(y)$ and

$$\inf\{d(x, y)+d(x, T(x)): x \in X\}=0.$$

Then there exists a sequence (x_n) in X such that

$$\lim_{n \rightarrow \infty}\{d(x_n, y)+d(x_n, T(x_n))\}=0.$$

So, we get $d(x_n, y) \rightarrow 0$ and $d(x_n, T(x_n)) \rightarrow 0$. By Lemma 1, it follows that $T(x_n) \rightarrow y$. We also have

$$\begin{aligned} d(y, T(y)) &\leq s\left[d(y, T(x_n))+d(T(x_n), T(y))\right] \\ &\leq s\, d(y, T(x_n)) \\ &\quad + sr \max\{d(x_n, y), d(x_n, T(x_n)), d(y, T(y)), d(y, T(x_n))\} \end{aligned}$$

for any $n \in \mathbb{N}$ and hence

$$d(y, T(y)) \le srd(y, T(y)).$$

Therefore, $d(y, T(y)) = 0$ i.e., $y = T(y)$. This is a contradiction. Hence, if $y \neq T(y)$, then

$$\inf \{d(x, y) + d(x, T(x)) : x \in X\} > 0.$$

By applying Corollary 1, we obtain a fixed point of T in X. Clearly, fixed point of T is unique. ∎

Theorem 7. *Let p be a wt-distance on a complete b-metric space (X, d) with constant $s \ge 1$. Let T_1, T_2 be mappings from X onto itself. Suppose that there exists $r > s$ such that*

$$\min \left\{ \begin{array}{l} p(T_2T_1(x), T_1(x)), \\ p(T_1T_2(x), T_2(x)) \end{array} \right\} \ge r \max \{p(T_1(x), x), p(T_2(x), x)\} \tag{11}$$

for every $x \in X$ and that

$$\inf \{p(x, y) + \min \{p(T_1(x), x), p(T_2(x), x)\} : x \in X\} > 0 \tag{12}$$

for every $y \in X$ with y is not a common fixed point of T_1 and T_2. Then T_1 and T_2 have a common fixed point in X. Moreover, if $v = T_1(v) = T_2(v)$, then $p(v, v) = 0$.

Proof. Let $u_0 \in X$ be arbitrary. Since T_1 is onto, there is an element u_1 satisfying $u_1 \in T_1^{-1}(u_0)$. Since T_2 is also onto, there is an element u_2 satisfying $u_2 \in T_2^{-1}(u_1)$. Proceeding in the same way, we can find $u_{2n+1} \in T_1^{-1}(u_{2n})$ and $u_{2n+2} \in T_2^{-1}(u_{2n+1})$ for $n = 1, 2, 3, \ldots$.

Therefore, $u_{2n} = T_1(u_{2n+1})$ and $u_{2n+1} = T_2(u_{2n+2})$ for $n = 0, 1, 2, \ldots$.

If $n = 2m$, then using (11)

$$\begin{aligned} p(u_{n-1}, u_n) &= p(u_{2m-1}, u_{2m}) \\ &= p(T_2(u_{2m}), T_1(u_{2m+1})) \\ &= p(T_2T_1(u_{2m+1}), T_1(u_{2m+1})) \\ &\ge \min \{p(T_2T_1(u_{2m+1}), T_1(u_{2m+1})), \\ &\qquad p(T_1T_2(u_{2m+1}), T_2(u_{2m+1}))\} \\ &\ge r \max \{p(T_1(u_{2m+1}), u_{2m+1}), p(T_2(u_{2m+1}), u_{2m+1})\} \\ &\ge rp(T_1(u_{2m+1}), u_{2m+1}) \\ &= rp(u_{2m}, u_{2m+1}) \\ &= rp(u_n, u_{n+1}). \end{aligned}$$

If $n = 2m+1$, then by (11), we have

$$\begin{aligned} p(u_{n-1}, u_n) &= p(u_{2m}, u_{2m+1}) \\ &= p(T_1(u_{2m+1}), T_2(u_{2m+2})) \\ &= p(T_1T_2(u_{2m+2}), T_2(u_{2m+2})) \\ &\geq \min\{p(T_2T_1(u_{2m+2}), T_1(u_{2m+2})), \\ &\qquad p(T_1T_2(u_{2m+2}), T_2(u_{2m+2}))\} \\ &\geq r \max\{p(T_1(u_{2m+2}), u_{2m+2}), p(T_2(u_{2m+2}), u_{2m+2})\} \\ &\geq rp(T_2(u_{2m+2}), u_{2m+2}) \\ &= rp(u_{2m+1}, u_{2m+2}) \\ &= rp(u_n, u_{n+1}). \end{aligned}$$

Thus for any positive integer n, we obtain

$$p(u_{n-1}, u_n) \geq rp(u_n, u_{n+1})$$

which implies that,

$$p(u_n, u_{n+1}) \leq \frac{1}{r} p(u_{n-1}, u_n) \leq \ldots \leq \left(\frac{1}{r}\right)^n p(u_0, u_1). \tag{13}$$

Let $\alpha = \frac{1}{r}$, then $0 < \alpha < \frac{1}{s}$ since $r > s$.

Now, (13) becomes

$$p(u_n, u_{n+1}) \leq \alpha^n p(u_0, u_1).$$

So, if $m > n$, then

$$\begin{aligned} p(u_n, u_m) &\leq s[p(u_n, u_{n+1}) + p(u_{n+1}, u_m)] \\ &\leq sp(u_n, u_{n+1}) + s^2 p(u_{n+1}, u_{n+2}) + \ldots \\ &\quad + s^{m-n-1}[p(u_{m-2}, u_{m-1}) + p(u_{m-1}, u_m)] \\ &\leq \left[s\alpha^n + s^2\alpha^{n+1} + \ldots + s^{m-n-1}\alpha^{m-2} + s^{m-n-1}\alpha^{m-1}\right] p(u_0, u_1) \\ &\leq \left[s\alpha^n + s^2\alpha^{n+1} + \ldots + s^{m-n-1}\alpha^{m-2} + s^{m-n}\alpha^{m-1}\right] p(u_0, u_1) \\ &= s\alpha^n \left[1 + s\alpha + (s\alpha)^2 + \ldots + (s\alpha)^{m-n-2} + (s\alpha)^{m-n-1}\right] p(u_0, u_1) \\ &\leq \frac{s\alpha^n}{1 - s\alpha} p(u_0, u_1). \end{aligned}$$

By Lemma 1(iii), (u_n) is a Cauchy sequence in X. Since X is complete, (u_n) converges to some point $z \in X$. Let $n \in \mathbb{N}$ be fixed. Then since (u_m) converges to z and $p(u_n, .)$ is s-lower semi-continuous, we have

$$p(u_n, z) \leq \liminf_{m\to\infty} sp(u_n, u_m) \leq \frac{s^2\alpha^n}{1 - s\alpha} p(u_0, u_1). \tag{14}$$

Assume that z is not a common fixed point of T_1 and T_2. Then by hypothesis

$$\begin{aligned} 0 &< \inf\{p(x,z)+\min\{p(T_1(x),x),p(T_2(x),x)\}:x\in X\} \\ &\leq \inf\{p(u_n,z)+min\{p(T_1(u_n),u_n),p(T_2(u_n),u_n)\}:n\in\mathbb{N}\} \\ &\leq \inf\left\{\frac{s^2\alpha^n}{1-s\alpha}p(u_0,u_1)+p(u_{n-1},u_n):n\in\mathbb{N}\right\} \\ &\leq \inf\left\{\frac{s^2\alpha^n}{1-s\alpha}p(u_0,u_1)+\alpha^{n-1}p(u_0,u_1):n\in\mathbb{N}\right\} \\ &= 0 \end{aligned}$$

which is a contradiction. Therefore, $z = T_1(z) = T_2(z)$.

If $v = T_1(v) = T_2(v)$ for some $v \in X$, then

$$\begin{aligned} p(v,v) &= \min\{p(T_2T_1(v),T_1(v)),p(T_1T_2(v),T_2(v))\} \\ &\geq r\max\{p(T_1(v),v),p(T_2(v),v)\} \\ &= r\max\{p(v,v),p(v,v)\} \\ &= rp(v,v) \end{aligned}$$

which gives that, $p(v,v) = 0$. ■

Corollary 2. *Let p be a wt-distance on a complete b-metric space (X,d) with constant $s \geq 1$ and let $T : X \to X$ be an onto mapping. Suppose that there exists $r > s$ such that*

(15) $$p(T^2(x),T(x)) \geq rp(T(x),x)$$

for every $x \in X$ and that

(16) $$\inf\{p(x,y)+p(T(x),x):x\in X\}>0$$

for every $y \in X$ with $y \neq T(y)$. Then T has a fixed point in X. Moreover, if $v = T(v)$, then $p(v,v) = 0$.

Proof. Taking $T_1 = T_2 = T$ in Theorem 7, we have the desired result. ■

As an application of Corollary 2, we have the following results.

Theorem 8. *Let (X,d) be a complete b-metric space with constant $s \geq 1$ and let $T : X \to X$ be an onto continuous mapping. Suppose there exists $r > s$ such that*

$$d(T^2(x),T(x)) \geq rd(T(x),x)$$

for every $x \in X$. Then T has a fixed point in X.

Proof. We consider d as a wt-distance on X. Then d satisfies condition (15) of Corollary 2.

Assume that there exists $y \in X$ with $y \neq T(y)$ and

$$\inf\{d(x,y) + d(T(x),x) : x \in X\} = 0.$$

Then there exists a sequence (x_n) such that

$$\lim_{n\to\infty} \{d(x_n,y) + d(T(x_n),x_n)\} = 0.$$

So, we have $d(x_n,y) \to 0$ and $d(T(x_n),x_n) \to 0$ as $n \to \infty$.

Now,

$$d(T(x_n),y) \leq d(T(x_n),x_n) + d(x_n,y) \to 0 \text{ as } n \to \infty.$$

Since T is continuous, we have

$$T(y) = T\left(\lim_{n\to\infty} x_n\right) = \lim_{n\to\infty} T(x_n) = y.$$

This is a contradiction. Hence if $y \neq T(y)$, then

$$\inf\{d(x,y) + d(T(x),x) : x \in X\} > 0,$$

which is condition (16) of Corollary 2. By Corollary 2, there exists $z \in X$ such that $z = T(z)$. ■

Theorem 9. *Let (X,d) be a complete b-metric space with constant $s \geq 1$ and let $T : X \to X$ be an onto continuous mapping. If there is a real number r with $r > s$ satisfying*

$$d(T(x),T(y)) \geq r \min\{d(x,T(x)), d(T(y),y), d(x,y)\} \tag{17}$$

for every $x,y \in X$, then T has a fixed point in X.

Proof. We consider d as a wt-distance on X. Replacing y by $T(x)$ in (17), we have

$$d(T(x),T^2(x)) \geq r \min\{d(x,T(x)), d(T^2(x),T(x)), d(x,T(x))\} \tag{18}$$

for every $x \in X$. Without loss of generality, we may assume that $T(x) \neq T^2(x)$. For, otherwise, T has a fixed point. Since $r > s \geq 1$, it follows from (18) that

$$d(T^2(x),T(x)) \geq rd(T(x),x)$$

for every $x \in X$. By the argument similar to that used in Theorem 8, we can prove that, if $y \neq T(y)$, then

$$\inf\{d(x,y) + d(T(x),x) : x \in X\} > 0.$$

So, Corollary 2 applies to obtain a fixed point of T. ■

Remark 1. The class of mappings satisfying condition (17) is strictly larger than that of expansive mappings. For, if $T : X \to X$ is expansive, then there exists $r > s$ such that

$$d(T(x), T(y)) \geq r\, d(x, y) \geq r \min\{d(x, T(x)), d(T(y), y), d(x, y)\}$$

for all $x, y \in X$. On the otherhand, the identity mapping satisfies condition (17) but it is not expansive.

We now supplement Theorem 2 by examination of conditions (1) and (2) in respect of their independence. We furnish Examples 6 and 7 below to show that these two conditions are independent in the sense that Theorem 2 shall fall through by dropping one in favour of the other.

Example 6. Let $X = \{0\} \cup \left\{\frac{1}{3^n} : n \geq 1\right\}$ and $d(x, y) =\mid x - y \mid^2$ for all $x, y \in X$. Then (X, d) is a complete b-metric space with constant $s = 2$. Define $T : X \to X$ by $T(0) = \frac{1}{3}$ and $T\left(\frac{1}{3^n}\right) = \frac{1}{3^{n+1}}$ for $n \geq 1$. Clearly, T has no fixed point in X. It is easy to verify that $d(T(x), T^2(x)) \leq \frac{1}{9} d(x, T(x))$ for all $x \in X$. Therefore, condition (1) holds for $T_1 = T_2 = T$. On the other hand, $T(y) \neq y$ for all $y \in X$ and so

$$\begin{aligned} &\inf\{d(x, y) + d(x, T(x)) : x, y \in X \text{ with } y \neq T(y)\} \\ &\quad = inf\,\{d(x, y) + d(x, T(x)) : x, y \in X\} = 0. \end{aligned}$$

Thus, condition (2) is not satisfied for $T_1 = T_2 = T$. We note that Theorem 2 does not hold without condition (2).

Example 7. Let $X = [3, \infty) \cup \{1, 2\}$ and $d(x, y) =\mid x - y \mid^2$ for all $x, y \in X$. Then (X, d) is a complete b-metric space with constant $s = 2$. Define $T : X \to X$ where

$$T(x) = \begin{cases} 1, & \text{for} \quad x \in (X \setminus \{1\}) \\ 2, & \text{for} \quad x = 1. \end{cases}$$

Clearly, T possesses no fixed point in X.

Now,

$$\begin{aligned} &\inf\{d(x, y) + d(x, T(x)) : x, y \in X \text{ with } y \neq T(y)\} \\ &\quad = \inf\{d(x, y) + d(x, T(x)) : x, y \in X\} > 0. \end{aligned}$$

Thus, condition (2) is satisfied for $T_1 = T_2 = T$. But, for $x = 1$, we find that $d(T(x), T^2(x)) = 1 > r d(x, T(x))$ for any $r \in [0, \frac{1}{s})$. So, condition (1) does not hold for $T_1 = T_2 = T$. In this case we observe that Theorem 2 does not work without condition (1).

Note. In examples above we treat the b-metric d as a wt-distance on X in reference to Theorem 2.

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Received on 05.05.2014 and, in revised form, on 10.04.2015.