DE GRUYTER OPEN
FASCICULI MATHEMATICI

Nr 54 2015

DOI:10.1515/fascmath-2015-0009

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CONTINUITY VIA Λ_I-OPEN SETS

Abstract. Noiri and Keskin [8] introduced the notions of Λ_I-sets and Λ_I-closed sets using ideals on topological spaces. In this work we use sets that are complements of Λ_I-closed sets, which are called Λ_I-open, to characterize new variants of continuity namely Λ_I-continuous, quasi-Λ_I-continuous and Λ_I-irresolute functions.

Key words: 54C08, 54D05.

AMS Mathematics Subject Classification: local function, Λ_I-open sets, Λ_I-irresolute functions.

1. Introduction

The theory of ideal on topological spaces has been the subject of many studies in recent years. It was the works of Janković and Hamlet [5, 6], Abd El-Monsef, Lashien and Nasef [1] and Hatir and Noiri [3] which motivated the research in applying topological ideals to generalize the most basic properties in general topology. In 1992, Janković and Hamlet [6] introduced the notion of I-open sets in topological spaces. Later, Abd El-Monsef, Lashien and Nasef [1] investigated I-open sets and I-continuous functions. Quite recently, Noiri and Keskin [8] have introduced the notions of Λ_I-sets and Λ_I-closed sets to obtain characterizations of two low separation axioms, namely I-T_1 and I-$T_{1/2}$ spaces. In this article we introduce the notion of Λ_I-open sets in order to characterize new variants of continuity in ideal topological spaces.

2. Preliminaries

Throughout this paper, $P(X)$, Cl(A) and Int(A) denote the power set of X, the closure of A and the interior of A, respectively. An ideal I on a topological space (X, τ) is a nonempty collection of subsets of X which satisfies the following two properties:

(i) $A \in I$ and $B \subset A$ implies $B \in I$;

* Research Partially Suported by Consejo de Investigación UDO.

(*ii*) $A \in I$ and $B \in I$ implies $A \cup B \in I$.

A topological space (X, τ) with an ideal I on X is called an ideal topological space and is denoted by (X, τ, I). Given an ideal topological space (X, τ, I), a set operator $(.)^\star : P(X) \to P(X)$, called a local function [7] of A with respect to τ and I, is defined as follows: for $A \subset X$, $A^\star(I, \tau) = \{x \in X : U \cap A \notin I$ for every $U \in \tau(x)\}$, where $\tau(x) = \{U \in \tau : x \in U\}$. When there is no chance for confusion, we will simply write $A^\star$ for $A^\star(I, \tau)$. In general, $X^\star$ is a proper subset of X. The hypothesis $X = X^\star$ [4] is equivalent to the hypothesis $\tau \cap I = \emptyset$ [9]. According to [3], we call the ideal topological spaces which satisfy this hypothesis Hayashi-Samuels spaces (briefly H.S.S.). Note that $\mathrm{Cl}^\star(A) = A \cup A^\star(I, \tau)$ defines a Kuratowski closure for a topology $\tau^\star(I)$ (also denoted $\tau^\star$ when there is no chance for confusion), finer than τ. A basis $\beta(I, \tau)$ for $\tau^\star(I, \tau)$ can be described as follows: $\beta(I, \tau) = \{V - J : V \in \tau$ and $J \in I\}$. The elements of $\tau^\star$ are called $\tau^\star$-open and the complement of a $\tau^\star$-open is called $\tau^\star$-closed. It is well known that a subset A of an ideal topological space (X, τ, I) is $\tau^\star$-closed if and only if $A^\star \subset A$ [5].

Definition 1. *A subset A of an ideal topological space (X, τ, I) is said to be I-open [6] if $A \subset Int(A^\star)$. The complement of an I-open set is said to be I-closed. The family of all I-open sets of an ideal topological space (X, τ, I) is denoted by* $\mathrm{IO}(X, \tau)$.

The following three definitions has been introduced by Noiri and Keskin [8].

Definition 2. *Let A be a subset of an ideal topological space (X, τ, I). A subset $\Lambda_I(A)$ is defined as follows: $\Lambda_I(A) = \cap\{U : A \subset U,\ U \in \mathrm{IO}(X, \tau)\}$.*

Definition 3. *Let (X, τ, I) be an ideal topological space. A subset A of X is said to be:*

(*i*) *Λ_I-set if $A = \Lambda_I(A)$.*

(*ii*) *Λ_I-closed if $A = U \cap F$, where U is a Λ_I-set and F is an $\tau^\star$-closed set.*

In [8] the following implications are shown:

$$I\text{-open} \Longrightarrow \Lambda_I\text{-set} \Longrightarrow \Lambda_I\text{-closed}.$$

Lemma 1 (Noiri and Keskin [8]). *For an H.S.S. (X, τ, I), we take $\tau^{\Lambda_I} = \{A : A$ is a Λ_I-set of $(X, \tau, I)\}$. Then the pair (X, τ^{Λ_I}) is an Alexandroff space.*

Remark 1. According to Lemma 1, a subset A of an H.H.S. (X, τ, I) is open in (X, τ^{Λ_I}), if A is a Λ_I-set of (X, τ, I). Furthermore, when we mention the pair (X, τ^{Λ_I}), it will be understood that (X, τ, I) is an H.S.S.

Definition 4. *A subset A of an ideal topological space (X, τ, I) is called Λ_I-open if $X - A$ is a Λ_I-closed set.*

Lemma 2. *If (X, τ, I) is a H.S.S., the every $\tau^\star$-open set is Λ_I-open.*

Proof. This follows from Proposition 6 of [8]. ■

Lemma 3. *Let $\{B_\alpha : \alpha \in \Delta\}$ be a family of subsets of the ideal topological space (X, τ, I). If B_α is Λ_I-open for each $\alpha \in \Delta$, then $\bigcup\{B_\alpha : \alpha \in \Delta\}$ is Λ_I-open.*

Proof. The proof is an immediate consequence from Theorem 5 of [8]. ■

Definition 5. *A function $f : (X, \tau, I) \to (Y, \sigma, J)$ is said to be I-irresolute [2], if $f^{-1}(V)$ is an I-open set in (X, τ, I) for each J-open set V of (Y, σ, J).*

Theorem 1. *If a function $f : (X, \tau, I) \to (Y, \sigma, J)$ is I-irresolute, then $f : (X, \tau^{\Lambda_I}) \to (Y, \sigma^{\Lambda_J})$ is continuous.*

Proof. Let V be any Λ_J-set of (Y, σ, J), that is $V \in \sigma^{\Lambda_J}$, then $V = \Lambda_J(V) = \cap\{W : V \subset W$ and W is J-open in $(Y, \sigma, J)\}$. Since f is I-irresolute, $f^{-1}(W)$ is an I-open set in (X, τ, I) for each W, hence we have

$$\begin{aligned}\Lambda_I(f^{-1}(V)) &= \cap\{U : f^{-1}(V) \subset U \text{ and } U \in \mathrm{IO}(X, \tau)\}\\ &\subset \cap\{f^{-1}(W) : f^{-1}(V) \subset f^{-1}(W) \text{ and } W \in \mathrm{JO}(Y, \sigma)\}\\ &= f^{-1}(V).\end{aligned}$$

On the other hand, always we have $f^{-1}(V) \subset \Lambda_I(f^{-1}(V))$ and so $f^{-1}(V) = \Lambda_I(f^{-1}(V))$. Therefore, $f^{-1}(V) \in \tau^{\Lambda_I}$ and $f : (X, \tau^{\Lambda_I}) \to (Y, \sigma^{\Lambda_J})$ is continuous. ■

3. New variants of continuity

In this section we use the notions of open, Λ_I-open and $\tau^\star$-open sets in order to introduce new forms of continuous functions called Λ_I-continuous, quasi-Λ_I-continuous and Λ_I-irresolute. We study the relationships between these classes of functions and also obtain some properties and characterizations of them.

Definition 6. *A function $f : (X, \tau, I) \to (Y, \sigma, J)$ is called:*

(*i*) *Λ_I-continuous, if $f^{-1}(V)$ is a Λ_I-open set in (X, τ, I) for each open set V of (Y, σ, J).*

(*ii*) *Quasi-Λ_I-continuous, if $f^{-1}(V)$ is a Λ_I-open set in (X, τ, I) for each $\sigma^\star$-open set V of (Y, σ, J).*

(iii) Λ_I*-irresolute, if* $f^{-1}(V)$ *is a* Λ_I*-open set in* (X,τ,I) *for each* Λ_J*-open set* V *of* (Y,σ,J).

Theorem 2. *If* $f:(X,\tau,I)\to(Y,\sigma,J)$ *is* Λ_I*-irresolute function and* (Y,σ,J) *is an H.S.S., then* f *is quasi-*Λ_I*-continuous.*

Proof. Let V be a $\sigma^\star$-open set of (Y,σ,J), then by Lemma 2, we have V is a Λ_J-open set of (Y,σ,J) and since f is Λ_I-irresolute, $f^{-1}(V)$ is a Λ_I-open set of (X,τ,I). Therefore, f is quasi-Λ_I-continuous. ■

The following example shows a function quasi-Λ_I-continuous which is not Λ_I-irresolute.

Example 1. Let $X=\{a,b,c\}$, $\tau=\{\emptyset,\{a,c\},X\}$, $\sigma=\{\emptyset,\{a\},\{a,b\},\{a,c\},X\}$, $I=\{\emptyset,\{c\}\}$ and $J=\{\emptyset,\{b\}\}$. The collection of the Λ_I-open sets of (X,τ,I) is $\{\emptyset,\{a,b\},\{a,c\},\{a\},X\}$, the collection of the $\sigma^\star$-open sets of (X,σ,J) is $\{\emptyset,\{a\},\{a,c\},\{a,b\},X\}$ and the collection of the Λ_J-open sets of (X,σ,J) is $\{\{\emptyset\},\{a\},\{b\},\{c\},\{a,b\},\{a,c\},\{b,c\},X\}$. The identity function $f:(X,\tau,I)\to(X,\sigma,J)$ is quasi-Λ_I-continuous, but is not Λ_I-irresolute, since $f^{-1}(\{b\})=\{b\}$, $f^{-1}(\{c\})=\{c\}$ and $f^{-1}(\{b,c\})=\{b,c\}$ are not Λ_I-open sets.

The following example shows that the condition that (Y,σ,J) let be an H.S.S. is necessary in the above theorem.

Example 2. Let $X=\{a,b,c\}$, $\tau=\{\emptyset,\{a\},\{b\},\{c\},\{a,b\},\{a,c\},\{b,c\},X\}$ and $I=\{\emptyset,\{c\}\}$. Note that X is not an *H.S.S.* since $\tau\cap I=\{\emptyset,\{c\}\}$. Furthermore, the collection of the Λ_I-open sets of (X,τ,I) is $\{\emptyset,\{a,c\},\{b,c\},\{c\},X\}$, the collection of the $\tau^\star$-open sets of (X,σ,J) is $\{\emptyset,\{a\},\{b\},\{c\},\{a,b\},\{b,c\}\{a,c\},X\}$. The identity function $f:(X,\tau,I)\to(X,\tau,I)$ is Λ_I-irresolute, but is not quasi-Λ_I-continuous, since $f^{-1}(\{a\})=\{a\}$, $f^{-1}(\{b\})=\{b\}$ and $f^{-1}(\{a,b\})=\{a,b\}$ are not Λ_I-open sets.

Theorem 3. *If* $f:(X,\tau,I)\to(Y,\sigma,J)$ *is quasi-*Λ_I*-continuous function, then* f *is* Λ_I*-continuous.*

Proof. Let V be an open set of (Y,σ,J), then V is $\sigma^\star$-open set of (Y,σ,J) and since f is quasi-Λ_I-continuous, $f^{-1}(V)$ is a Λ_I-open set of (X,τ,I). This shows that f is Λ_I-continuous. ■

The following example shows a function Λ_I-continuous which is not quasi-Λ_I-continuous.

Example 3. Let $X=\{a,b,c\}$, $\tau=\{\emptyset,\{c\},\{a,c\},\{b,c\},X\}$, $\sigma=\{\emptyset,\{a,c\},X\}$, $I=\{\emptyset,\{b\}\}$ and $J=\{\emptyset,\{c\}\}$. The collection of the Λ_I-open sets of (X,τ,I) is $\{\emptyset,\{b,c\},\{a,c\},\{c\},X\}$ and the collection of $\sigma^\star$-open sets of (X,σ,J) is $\{\emptyset,\{a\},\{a,c\},\{a,b\},X\}$. The identity function $f:(X,\tau,I)\to$

(X, σ, J) is Λ_I-continuous, but is not quasi-Λ_I-continuous, because $f^{-1}(\{a\}) = \{a\}$ and $f^{-1}(\{a, b\}) = \{a, b\}$ are not Λ_I-open sets.

Corollary. *If $f : (X, \tau, I) \to (Y, \sigma, J)$ is a Λ_I-irresolute function and (Y, σ, J) is an H.S.S., then f is Λ_I-continuous.*

Proof. This is an immediate consequence of Theorems 2 and 3. ■

By the above results, for an H.S.S. we have the following diagram and none of these implications is reversible:

$$\Lambda_I\text{-irresolute} \Longrightarrow \text{quasi-}\Lambda_I\text{-continuous} \Longrightarrow \Lambda_I\text{-continuous}.$$

Proposition 1. *Let $f : (X, \tau, I) \to (Y, \sigma, J)$ and $g : (Y, \sigma, J) \to (Z, \theta, K)$ be two functions, where I, J, K are ideals on X, Y, Z respectively. Then:*
(*i*) *$g \circ f$ is Λ_I-irresolute, if f is Λ_I-irresolute and g is Λ_J-irresolute.*
(*ii*) *$g \circ f$ is Λ_I-continuous, if f is Λ_I-irresolute and g is Λ_J-continuous.*
(*iii*) *$g \circ f$ is Λ_I-continuous, if f is Λ_I-continuous and g is continuous.*
(*iv*) *$g \circ f$ is quasi-Λ_I-continuous, if f is Λ_I-irresolute and g is quasi-Λ_J -continuous.*

Proof. (*i*) Let V be a Λ_K-open set in (Z, θ, K). Since g is Λ_J-irresolute, then $g^{-1}(V)$ is a Λ_J-open set in (Y, σ, J), using that f is Λ_I-irresolute, we obtain that $f^{-1}(g^{-1}(V))$ is a Λ_I-open set in (X, τ, I). But $(g \circ f)^{-1}(V) = (f^{-1} \circ g^{-1})(V) = f^{-1}(g^{-1}(V))$ and hence, $(g \circ f)^{-1}(V)$ is a Λ_I-open set in (X, τ, I). This shows that $g \circ f$ is Λ_I-irresolute.
The proofs of (*ii*), (*iii*) and (*iv*) are similar to the case (*i*). ■

In the next three theorems, we characterize Λ_I-continuous, quasi-Λ_I-continuous and Λ_I-irresolute functions, respectively.

Theorem 4. *For a function $f : (X, \tau, I) \to (Y, \sigma)$, the following statements are equivalent:*
(*i*) *f is Λ_I-continuous.*
(*ii*) *$f^{-1}(B)$ is a Λ_I-closed set in (X, τ, I) for each closed set B in (Y, σ).*
(*iii*) *For each $x \in X$ and each open set V in (Y, σ) containing $f(x)$ there exists a Λ_I-open set U in (X, τ, I) containing x such that $f(U) \subset V$.*

Proof. (*i*) ⇒ (*ii*) Let B be any closed set in (Y, σ), then $V = Y - B$ is an open set in (Y, σ) and since f is Λ_I-continuous, $f^{-1}(V)$ is a Λ_I-open subset in (X, τ, I), but $f^{-1}(V) = f^{-1}(Y - B) = f^{-1}(Y) - f^{-1}(B) = X - f^{-1}(B)$ and hence, $f^{-1}(B)$ is a Λ_I-closed set in (X, τ, I).
(*ii*) ⇒ (*i*) Let V be any open set in (Y, σ), then $B = Y - V$ is a closed set in (Y, σ). By hypothesis, we have $f^{-1}(B)$ is a Λ_I-closed set in (X, τ, I), but $f^{-1}(B) = f^{-1}(Y - V) = f^{-1}(Y) - f^{-1}(V) = X - f^{-1}(V)$ and so, $f^{-1}(V)$ is a Λ_I-open set in (X, τ, I). This shows that f is Λ_I-continuous.

$(i) \Rightarrow (iii)$ Let $x \in X$ and V any open set in (Y, σ) such that $f(x) \in V$, then $x \in f^{-1}(V)$ and since f is a Λ_I-continuous function, $f^{-1}(V)$ is a Λ_I-open set in (X, τ, I). If $U = f^{-1}(V)$, then U is a Λ_I-open set in (X, τ, I) containing x such that $f(U) = f(f^{-1}(V)) \subset V$.

$(iii) \Rightarrow (i)$ Let V be any open set in (Y, σ) and $x \in f^{-1}(V)$, then $f(x) \in V$ and by (3) there exists a Λ_I-open set U_x in (X, τ, I) such that $x \in U_x$ and $f(U_x) \subset V$. Thus, $x \in U_x \subset f^{-1}(f(U_x)) \subset f^{-1}(V)$ and hence $f^{-1}(V) = \bigcup\{U_x : x \in f^{-1}(V)\}$. By Lemma 3, we have $f^{-1}(V)$ is a Λ_I-open set in (X, τ, I) and so f is Λ_I-continuous. ■

Theorem 5. *For a function $f : (X, \tau, I) \to (Y, \sigma, J)$, the following statements are equivalent:*

(i) *f is quasi-Λ_I-continuous.*

(ii) *$f^{-1}(B)$ is a Λ_I-closed set in (X, τ, I) for each $\sigma^\star$-closed set B in (Y, σ, J).*

(iii) *For each $x \in X$ and each $\sigma^\star$-open set V in (Y, σ, J) containing $f(x)$ there exists a Λ_I-open set U in (X, τ, I) containing x such that $f(U) \subset V$.*

Proof. The proof is similar to Theorem 4. ■

Theorem 6. *For a function $f : (X, \tau, I) \to (Y, \sigma, J)$, the following statements are equivalent:*

(i) *f is Λ_I-irresolute.*

(ii) *$f^{-1}(B)$ is a Λ_I-closed set in (X, τ, I) for each Λ_J-closet set B in (Y, σ, J).*

(iii) *For each $x \in X$ and each Λ_J-open set V in (Y, σ, J) containing $f(x)$ there exists a Λ_I-open set U in (X, τ, I) containing x such that $f(U) \subset V$.*

Proof. The proof is similar to Theorem 4. ■

4. Λ_I-compactness and Λ_I-connectedness

In this section, new notions of compactness and connectedness are introduced in terms of Λ_I-open sets and I-open sets, in order to study their behavior under the direct images of the new forms of continuity defined in the previous section.

Definition 7. *An ideal topological space (X, τ, I) is said to be:*

(i) *Λ_I-compact if every cover of X by Λ_I-open sets has a finite subcover.*

(ii) *$\tau^\star$-compact if every cover of X by $\tau^\star$-open sets has a finite subcover.*

(iii) *I-compact if every cover of X by I-open sets has a finite subcover.*

Theorem 7. *Let (X,τ,I) be an ideal topological space, the following properties hold:*

(i) (X,τ,I) is Λ_I-compact if and only if for every collection $\{A_\alpha : \alpha \in \Delta\}$ of Λ_I-closed sets in (X,τ,I) satisfying $\bigcap\{A_\alpha : \alpha \in \Delta\} = \emptyset$, there is a finite subcollection $A_{\alpha_1}, A_{\alpha_2}, \ldots, A_{\alpha_n}$ with $\bigcap\{A_{\alpha_k} : k = 1,\ldots,n\} = \emptyset$.

(ii) (X,τ,I) is $\tau^\star$-compact if and only if for every collection $\{A_\alpha : \alpha \in \Delta\}$ of $\tau^\star$-closed sets in (X,τ,I) satisfying $\bigcap\{A_\alpha : \alpha \in \Delta\} = \emptyset$, there is a finite subcollection $A_{\alpha_1}, A_{\alpha_2}, \ldots, A_{\alpha_n}$ with $\bigcap\{A_{\alpha_k} : k = 1,\ldots,n\} = \emptyset$.

(iii) (X,τ,I) is I-compact if and only if for every collection $\{A_\alpha : \alpha \in \Delta\}$ of I-closed sets in (X,τ,I) satisfying $\bigcap\{A_\alpha : \alpha \in \Delta\} = \emptyset$, there is a finite subcollection $A_{\alpha_1}, A_{\alpha_2}, \ldots, A_{\alpha_n}$ with $\bigcap\{A_{\alpha_k} : k = 1,\ldots,n\} = \emptyset$.

Proof. (i) Let $\{A_\alpha : \alpha \in \Delta\}$ be a collection of Λ_I-closed sets such that $\bigcap\{A_\alpha : \alpha \in \Delta\} = \emptyset$, then $\{X - A_\alpha : \alpha \in \Delta\}$ is a collection of Λ_I-open sets such that

$$X = X - \emptyset = X - \bigcap\{A_\alpha : \alpha \in \Delta\} = \bigcup\{X - A_\alpha : \alpha \in \Delta\},$$

that is, $\{X - A_\alpha : \alpha \in \Delta\}$ is a cover of X by Λ_I-open sets. Since (X,τ,I) is Λ_I-compact, there exists a finite subcollection $X - A_{\alpha_1}, X - A_{\alpha_2}, \ldots, X - A_{\alpha_n}$ such that

$$X = \bigcup\{X - A_{\alpha_k} : k = 1,\ldots,n\} = X - \bigcap\{A_{\alpha_k} : k = 1,\ldots,n\}.$$

This shows that $\bigcap\{A_{\alpha_k} : k = 1,\ldots,n\} = \emptyset$. Conversely, suppose that $\{U_\alpha : \alpha \in \Delta\}$ is a cover of X by Λ_I-open sets, then $\{X - U_\alpha : \alpha \in \Delta\}$ is a collection of Λ_I-closed sets such that $\bigcap\{X - U_\alpha : \alpha \in \Delta\} = X - \bigcup\{U_\alpha : \alpha \in \Delta\} = X - X = \emptyset$. By hypothesis, there exists a finite subcollection $X - U_{\alpha_1}, X - U_{\alpha_2}, \ldots, X - U_{\alpha_n}$ such that $\bigcap\{X - U_{\alpha_k} : k = 1,\ldots,n\} = \emptyset$. Follows $X = X - \emptyset = X - \bigcap\{X - U_{\alpha_k} : k = 1,\ldots,n\} = X - (X - \bigcup\{U_{\alpha_k} : k = 1,\ldots,n\}) = \bigcup\{U_{\alpha_k} : k = 1,\ldots,n\}$. This shows that (X,τ,I) is Λ_I-compact.

The proofs of (ii) and (iii) are similar to the case (i). ■

Theorem 8. *Let (X,τ,I) be an ideal topological space, the following properties hold:*

(i) If (X,τ^{Λ_I}) is compact, then (X,τ,I) is I-compact.

(ii) If (X,τ,I) is an H.S.S. Λ_I-compact, then (X,τ,I) is $\tau^\star$-compact.

(iii) If (X,τ,I) is an H.S.S. Λ_I-compact, then (X,τ,I) is compact.

Proof. (i) Let $\{U_\alpha : \alpha \in \Delta\}$ any cover of X by I-open sets, then every $\alpha \in \Delta$, U_α is a Λ_I-set and hence, $U_\alpha \in \tau^{\Lambda_I}$ for each $\alpha \in \Delta$. Since (X,τ^{Λ_I}) is compact, there exists a finite subset Δ_0 of Δ such that $X = \bigcup\{U_\alpha : \alpha \in \Delta_0\}$. This shows that (X,τ) is I-compact.

(ii) Let $\{F_\alpha : \alpha \in \Delta\}$ be a collection of $\tau^\star$-closed sets of X such that $\bigcap\{F_\alpha : \alpha \in \Delta\} = \emptyset$. Since every $\tau^\star$-closed set is Λ_I-closed, then $\{F_\alpha : \alpha \in \Delta\}$ is a collection of Λ_I-closed sets and (X, τ, I) is Λ_I-compact. By Theorem 7(1), there exists a finite subset Δ_0 of Δ such that $\bigcap\{F_\alpha : \alpha \in \Delta_0\} = \emptyset$ and by Theorem 7(2), we conclude that (X, τ, I) is $\tau^\star$-compact.

(iii) Follows from (2) and the fact that $\tau \subset \tau^\star$. ■

Theorem 9. *If $f : (X, \tau, I) \to (Y, \sigma, J)$ is a surjective function, the following properties hold:*

(i) *If f is Λ_I-irresolute and (X, τ, I) is Λ_I-compact, then (Y, σ, J) is Λ_J-compact.*

(ii) *If f is I-irresolute and (X, τ, I) is I-compact, then (Y, σ, J) is J-compact.*

(iii) *If f is quasi-Λ_I-continuous and (X, τ, I) is Λ_I-compact, then (Y, σ, J) is $\sigma^\star$-compact.*

(iv) *If f is Λ_I-continuous and (X, τ, I) is Λ_I-compact, then (Y, σ, J) is compact.*

Proof. (i) Let $\{V_\alpha : \alpha \in \Delta\}$ be a cover of Y by Λ_J-open sets. Since f is Λ_I-irresolute, $\{f^{-1}(V_\alpha) : \alpha \in \Delta\}$ is a cover of X by Λ_I-open sets and by the Λ_I-compactnes of (X, τ, I), there exists a finite subset Δ_0 of Δ such that $X = \bigcup\{f^{-1}(V_\alpha) : \alpha \in \Delta_0\}$. Since f is surjective, then $Y = f(X) = f(\bigcup\{f^{-1}(V_\alpha) : \alpha \in \Delta_0\}) = \bigcup\{f(f^{-1}(V_\alpha)) : \alpha \in \Delta_0\} = \{V_\alpha : \alpha \in \Delta_0\}$ and this shows that (Y, θ, J) is Λ_J-compact.

The proofs of (ii), (iii) and (iv) are similar to case (1). ■

Definition 8. *An ideal topological space (X, τ, I) is said to be:*

(i) *Λ_I-connected if X cannot be written as a disjoint union of two non-empty Λ_I-open sets.*

(ii) *$\tau^\star$-connected if X cannot be written as a disjoint union of two non-empty $\tau^\star$-open sets.*

(iii) *I-connected if X cannot be written as a disjoint union of two non-empty I-open sets.*

Theorem 10. *Let (X, τ, I) be an ideal topological space, the following properties hold:*

(i) *If (X, τ^{Λ_I}) is connected, then $(X, \tau; I)$ is I-connected.*

(ii) *If (X, τ, I) is an H.S.S. Λ_I-connected, then (X, τ, I) is $\tau^\star$-connected.*

(iii) *If (X, τ, I) is an H.S.S. Λ_I-connected, then (X, τ, I) is connected.*

Proof. (i) Suppose that (X, τ, I) is not I-connected, then there exist non-empty I-open sets A and B such that $A \cap B = \emptyset$ and $A \cup B = X$. By Lemma 6(b) of [8], A and B are Λ_I-sets and hence, (X, τ^{Λ_I}) is not connected.

(ii) Suppose that (X, τ, I) is not $\tau^\star$-connected, then there exist non-empty $\tau^\star$-open sets A and B such that $A \cap B = \emptyset$ and $A \cup B = X$. By Lemma 2, we have A and B are Λ_I-open sets and so, (X, τ, I) is not Λ_I-connected.

(iii) Follows from (2) and the fact that $\tau \subset \tau^\star$. ■

Theorem 11. *For an ideal topological space (X, τ, I), the following statements are equivalent:*

(i) (X, τ, I) is Λ_I-connected.

(ii) $\emptyset$ and X are the only subsets of X which are both Λ_I-open and Λ_I-closed.

(iii) Every Λ_I-continuous function of X into a discrete space Y with at least two points, is a constant function.

Proof. $(i) \Rightarrow (ii)$ Let V be a subset of X which is both Λ_I-open and Λ_I-closed, then $X - V$ is both Λ_I-open and Λ_I-closed, so $X = V \cup (X - V)$. Since (X, τ, I) is Λ_I-connected, then one of those sets is $\emptyset$. Therefore, $V = \emptyset$ or $V = X$.

$(ii) \Rightarrow (i)$ Suppose that (X, τ, I) is not Λ_I-connected and let $X = U \cup V$, where U and V are disjoint nonempty Λ_I-open sets in (X, τ, I), then $U = X - V$ is both Λ_I-open and Λ_I-closed. By hypothesis, $U = \emptyset$ or $U = X$, which is a contradiction. Therefore, (X, τ, I) is Λ_I-connected.

$(ii) \Rightarrow (iii)$ Let $f : (X, \tau, I) \to Y$ be a Λ_I-continuous function, where Y is a topological space with the discrete topology and contains at least two points, then X can be cover by a collection of sets which are both Λ_I-open and Λ_I-closed of the form $\{f^{-1}(y) : y \in Y\}$, from these, we conclude that there exists a $y_0 \in Y$ such that $f^{-1}(\{y_0\}) = X$ and so, f is a constant function.

$(iii) \Rightarrow (ii)$ Let W be a subset of (X, τ, I) which is both Λ_I-open and Λ_I-closed. Suppose that $W \neq \emptyset$ and let $f : (X, \tau, I) \to Y$ be the function defined by $f(W) = \{y_1\}$ and $f(X - W) = \{y_2\}$ for $y_1, y_2 \in Y$ and $y_1 \neq y_2$. Then f is Λ_I-continuous, since the inverse image de each open set in Y is Λ_I-open in X. Therefore, by (3), f must be the constant map. It follows that $X = W$. ■

Theorem 12. *If $f : (X, \tau, I) \to (Y, \sigma, J)$ is a surjective function, the following properties hold:*

(i) If f is a Λ_I-irresolute and (X, τ, I) is Λ_I-connected, then (Y, σ, J) is Λ_J-connected.

(ii) If f is a I-irresolute function and (X, τ, I) is I-connected, then (Y, σ, J) is J-connected.

(iii) If f is a quasi-Λ_I-continuous function and (X, τ, I) is Λ_I-connected, then (Y, σ, J) is $\sigma^\star$-connected.

(iv) If f is a Λ_I-continuous function and (X, τ, I) is Λ_I-connected, then (Y, σ) is connected.

Proof. (i) Suppose that (Y, σ, J) is not Λ_J-connected, then there exist nonempty Λ_J-open sets H, G in (Y, σ, J) such that $G \cap H = \emptyset$ and $G \cup H = Y$. Hence, we have $f^{-1}(G) \cap f^{-1}(H) = \emptyset$, $f^{-1}(G) \cup f^{-1}(H) = X$ and moreover, $f^{-1}(G)$ and $f^{-1}(H)$ are nonempty Λ_I-open sets in (X, τ, I). This shows that (X, τ, I) is not Λ_I-connected.

The proofs of (ii), (iii) and (iv) are similar to case (i). ■

Open problems. The Theorems 8 and 10 have been proved using the fact that every I-open set is Λ_I-open and that every $\tau^\star$-open set of an H.S.S. is Λ_I-open. But until today, we dont have any contra example in order to shows that the converse of such Theorems are not true.

In that sense we write the following questions.

(i) Does there exists an ideal topological space (X, τ, I) which is I-compact (resp. I-connected) but (X, τ^{Λ_I}) is not a compact (resp. connected) space?

(ii) Does there exists an ideal topological space (X, τ, I) which is $\tau^\star$-compact (resp. $\tau^\star$-connected) but (X, τ) is not Λ_I-compact (resp. Λ_I-connected) space?

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Received on 10.04.2014 and, in revised form, on 09.01.2015.