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**ORDERED VECTOR VALUED DOUBLE  
SEQUENCE SPACES**

ABSTRACT. In this paper we have introduced an order relation on convergent double sequences and have constructed an ordered vector space, Riesz space, order complete vector space in case of double sequences. We have verified the Archimedean property.

KEY WORDS: double sequence, ordered space, convergence, Riesz space, Archimedean property.

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**1. Introduction**

Vector valued sequence space generated by the elements of ordered vector space or Riesz space is studied by Kaushal [8]. Ordered vector spaces and Riesz spaces are real vector spaces equipped with the algebraic structure of the space and have their origin. In an address by Riesz [12] in the International Congress of Mathematicians held at Bologna in the year 1928 considered the partial ordering on the class of linear functionals defined on the class of functions. In this article we have introduced an order relation and constructed a vector valued sequence space  ${}_2\Lambda(X)$  generated by double sequences. Further we have shown that the space  ${}_2\Lambda(X)$  is an ordered vector space, Riesz space, order complete vector space and it satisfies the Archimedean property.

The notion of cone metric space has been applied by various authors in the recent past. It has been applied for introducing and investigating different new sequence spaces and studying their different algebraic and topological properties by Abdeljawad [1], Beg, Abbas and Nazir [3], Dhanorkar and Salunke [5] and many others. In this article we have investigated different properties of the notion of statically convergence in cone metric space.

Some initial works on double sequence spaces is found in Bromwich [4]. Later on it was investigated by Basarir and Solancan [2], Hardy [6], Moricz

[9], Moricz and Rhoades [10], Patterson [11], Tripathy [13], Tripathy and Dutta [14], Tripathy and Sarma ([15], [16], [17], [18]), Turkmenoglu [19] and many others.

## 2. Definitions and preliminaries

Throughout the article  $w$ ,  $\ell_\infty$ ,  $c$ ,  $c_0$  denote the classes of all, bounded, convergent and null sequences respectively.

A double sequence is denoted by  $X = (x_{nk})$  i.e. a double infinite array of elements  $x_{nk}$ , for all  $n, k \in N$ .

Again, in this article we consider a seminormed space  $X$ , seminormed by  $q$ . The zero element of  $X$  is denoted by  ${}_2\bar{\theta}$ , i.e. a double infinite array of  $\theta$ , the zero element of  $X$

**Definition 1.** *A set  $P$  in a real vector space  $X$  is said to be a cone if it possesses the following properties:*

- (i)  $P + P \subset P$
- (ii)  $\alpha P \subset P$ , for each  $\alpha$  in  $R^+$
- (iii)  $P \cap -P = \{\theta\}$ .

*The presence of a cone  $P$  in a real vector space  $X$  yields the ordered vector space structure of  $X$  for the ordering ' $\geq$ ' defined by  $x \geq y$  if and only if  $x - y \in P$ , for all  $x, y \in X$ .*

**Definition 2.** *A cone  $P$  in a vector space  $E$  determines a transitive and reflexive relation ' $\leq$ ' by  $x \leq y$  if  $y - x \in P$  and the relation ' $\leq$ ' is compatible with the vector structure as:*

- (i) if  $x \geq 0$  and  $y \geq 0$  then  $x + y \geq 0$
- (ii) if  $x \geq 0$  then  $\lambda x \geq 0$  for all  $\lambda \geq 0$ , where  $x, y \in E$ .

*Then the relation ' $\leq$ ' determined by the cone  $P$  is called the vector (partial) ordering of  $E$  and the pair  $(E, P)$  or  $(E, \leq)$  called the ordered vector space.*

**Definition 3.** *Let  $E$  be a partially ordered vector space. If there exists the least upper bound for any two elements  $x, y \in E$ , then  $E$  is called a Riesz space or vector lattice.*

**Definition 4.** *A vector valued sequence space is a vector space whose elements are sequences and the terms of the sequence are chosen from a vector space  $X$ , where  $X$  may be different from  $R$  or  $C$ .*

**Definition 5.** *A directed set is a non-empty set " $A$ " together with a reflexive and transitive binary relation ' $\leq$ ' with the additional property that every pair of elements has an upper bound. In other words, for any  $a$  and  $b$  in  $A$  there must exist  $c$  in  $A$  with  $a \leq c$  and  $b \leq c$ .*

**Definition 6.** An ordered vector space  $X$  is said to be order complete if every directed, majorised subset of  $X$  has a supremum in  $X$ .

**Definition 7.** An ordered vector space  $X$  is said to be Archimedean if  $x \leq \theta$  whenever  $\lambda x \leq y$  for some  $y \in P$  and for all  $\lambda \in R^+$ .

In this article we introduce the following definitions:

**Definition 8.** Let  $X$  be an ordered vector space. Then for any two vector valued double sequences  $\bar{x} = (x_{nk})$  and  $\bar{y} = (y_{nk})$ , the order relation ' $\leq$ ' is defined by  $x_{nk} \leq y_{nk}$ , for all  $n, k \in N$ .

The class of all ordered vector valued double sequences is denoted by  ${}_2\Lambda(X)$ .

**Definition 9.** Let  $E$  be a subset of all vector valued double sequences. Then a relation " $r$ " on  $E$  is said to be antisymmetric if  $(x_{nk})$  is  $r$  related with  $(y_{nk})$  and  $(y_{nk})$  is  $r$  related with  $(x_{nk})$  implies that  $x_{nk} = y_{nk}$  for all values of  $n$  and  $k$ .

**Definition 10.** Let " $r$ " be an order relation on a set  $E$ , where  $E$  is a subset of all vector valued double sequences. Then  $E$  is ordered with respect to " $r$ " if " $r$ " is reflexive, transitive and antisymmetric.

**Definition 11.** A cone  ${}_2\bar{P}$  in the vector space  ${}_2\Lambda(X)$  determines a transitive and reflexive relation ' $\leq$ ' by  $(x_{nk}) \leq (y_{nk})$  if and only if  $(y_{nk} - x_{nk}) \in {}_2\bar{P}$  and the relation ' $\leq$ ' is compatible with the vector structure as follows:

- (i) if  $(x_{nk}) \geq {}_2\bar{\theta}$  and  $(y_{nk}) \geq {}_2\bar{\theta}$  then  $(x_{nk} + y_{nk}) \geq {}_2\bar{\theta}$ .
- (ii) if  $(x_{nk}) \geq {}_2\bar{\theta}$  then  $(\lambda x_{nk}) \geq {}_2\bar{\theta}$ , for all  $\lambda \geq 0$ , where  $(x_{nk}), (y_{nk}) \in {}_2\Lambda(X)$ .

Then the relation ' $\leq$ ' determined by the cone  ${}_2\bar{P}$  is called vector (partial) ordering of  ${}_2\Lambda(X)$  and the pair  $({}_2\Lambda(X), {}_2\bar{P})$  or  $({}_2\Lambda(X), \leq)$  is called ordered vector space in double sequence.

**Definition 12.** An ordered vector space  ${}_2\Lambda(X)$  is said to be Archimedean if  $(x_{nk}) \leq {}_2\bar{\theta}$  whenever  $(\lambda x_{nk}) \leq (y_{nk})$ , for some  $(y_{nk}) \in {}_2\bar{P}$  and all  $\lambda \in R^+$  and  $(x_{nk}) \in {}_2\Lambda(X)$ .

### 3. Main results

The proof of the following result is easy, so omitted.

**Theorem 1.** Let  ${}_2\Lambda(X)$  be the set of all double sequences. Let the vector addition and scalar multiplication on  ${}_2\Lambda(X)$  be defined by

$$(x_{nk}) + (y_{nk}) = (x_{nk} + y_{nk}) \text{ and } \alpha(x_{nk}) = (\alpha x_{nk}),$$

for all  $(x_{nk}), (y_{nk}) \in {}_2\Lambda(X)$  and  $\alpha \in R$ . Then  $({}_2\Lambda(X), +, \cdot)$  is a vector space.

**Theorem 2.** *Let  ${}_2\Lambda(X)$  be the set of all ordered double sequences and  $X$  be an ordered vector space with respect to the cone  $P$ , then  ${}_2\Lambda(X)$  is an ordered vector space with the cone  ${}_2\bar{P} = \{\bar{x} \in {}_2\Lambda(X) : x_{nk} \in P, \text{ for all } n, k \geq 1\}$ .*

**Proof. Reflexivity.** Let  $\bar{x} = (x_{nk})$  be an element in  ${}_2\Lambda(X)$ . Then clearly we have  $\{(n, k) \in N \times N : x_{nk} \neq x_{nk}\}$  contains no element. Therefore  $x_{nk} = x_{nk}$ , for all  $n, k \in N$ . Therefore " $\leq$ " is reflexive.

**Transitivity.** Let  $\bar{x} = (x_{nk})$ ,  $\bar{y} = (y_{nk})$  and  $\bar{z} = (z_{nk})$  be the elements in  ${}_2\Lambda(X)$ . Suppose,  $\bar{x} \leq \bar{y}$  and  $\bar{y} \leq \bar{z}$ , then we have  $x_{nk} \leq y_{nk}$  and  $y_{nk} \leq z_{nk}$ , for all  $n, k \in N$ . It follows that  $x_{nk} \leq z_{nk}$ , for all  $n, k \in N$ . Hence  $\bar{x} \leq \bar{z}$ . Therefore the relation " $\leq$ " is transitive.

**Antisymmetry.** Let  $\bar{x} = (x_{nk})$  and  $\bar{y} = (y_{nk})$  be two elements in  ${}_2\Lambda(X)$ . Suppose  $\bar{x} \leq \bar{y}$  and  $\bar{y} \leq \bar{x}$ . Then we have  $x_{nk} \leq y_{nk}$  and  $y_{nk} \leq x_{nk}$ , for all  $n, k \in N$ . It follows that  $x_{nk} = y_{nk}$ , for all  $n, k \in N$ . Hence " $\leq$ " is antisymmetric.

Let  $\bar{x} \leq \bar{y}$ , then we have  $x_{nk} \leq y_{nk}$ , for all  $n, k \in N$ . Then  $x_{nk} + z_{nk} \leq y_{nk} + z_{nk}$ , for all  $n, k \in N$ . Hence  $\bar{x} + \bar{z} \leq \bar{y} + \bar{z}$ . Next let,  $\bar{x} \leq \bar{y}$ , then we have  $x_{nk} \leq y_{nk}$ , for all  $n, k \in N$ . Hence  $\alpha\bar{x} \leq \alpha\bar{y}$ , for all  $\alpha \geq 0$  and  $n, k \in N$ .

Now we have to show that  ${}_2\bar{P} = \{\bar{x} \in {}_2\Lambda(X) : x_{nk} \in P, \text{ for all } n, k \geq 1\}$  is a cone for  ${}_2\Lambda(X)$ .

Let  $\bar{x} = (x_{nk})$  and  $\bar{y} = (y_{nk})$  be two elements in  ${}_2\bar{P}$ . Then  $x_{nk} \in P$  and  $y_{nk} \in P$ , for all  $n, k \geq 1$ . Hence  $x_{nk} + y_{nk} \in P$ , for all  $n, k \geq 1$  by linearity of  $P$ . Hence  $\bar{x} + \bar{y} \in {}_2\bar{P}$ . Clearly  $\theta \in P$ . Hence  ${}_2\bar{\theta} = (\theta_{nk}) \in {}_2\bar{P}$ . Therefore  ${}_2\bar{P} \neq \{\theta_{nk}\}$ . Let  $a_{nk} (\neq \theta_{nk}) \in {}_2\bar{P}$ . Then clearly  $\bar{a} = (a_{nk}) \in {}_2\Lambda(X)$ .

We have  ${}_2\bar{\theta} \in {}_2\bar{P}$  also  ${}_2\bar{\theta} \in -{}_2\bar{P}$ . Hence  ${}_2\bar{\theta} \in {}_2\bar{P} \cap -{}_2\bar{P}$ . Therefore  ${}_2\bar{P}$  is a cone for  ${}_2\Lambda(X)$ . Hence  ${}_2\Lambda(X)$  is an ordered vector space with the cone  ${}_2\bar{P} = \{\bar{x} \in {}_2\Lambda(X) : x_{nk} \in P, \text{ for all } n, k \geq 1\}$ . ■

**Theorem 3.** *If  $X$  is a Riesz space then  ${}_2\Lambda(X)$  is a Riesz space.*

**Proof.** Let  $\bar{x} = (x_{nk})$  and  $\bar{y} = (y_{nk})$  be any two elements in  ${}_2\Lambda(X)$ . Now,  $\text{Sup}^m\{\bar{x}, \bar{y}\} = \bar{x} \vee \bar{y}$ . Here,  $\bar{x} \vee \bar{y} = (x_{nk} + y_{nk})$ . Now,  $\bar{x}, \bar{y} \leq (x_{nk} + y_{nk})$ . Let  $\bar{a} = (a_{nk}) \in {}_2\Lambda(X)$  be such that  $\bar{x}, \bar{y} \leq \bar{a}$ . Therefore  $(x_{nk} + y_{nk}) \leq \bar{a} \Rightarrow \bar{x} \vee \bar{y} \leq \bar{a}$ . Since  $\bar{x}, \bar{y} \in {}_2\Lambda(X)$ , so,  $x_{nk}, y_{nk} \in X$ . Since  $X$  is a Riesz space, so,  $\text{Sup}^m\{x_{nk}, y_{nk}\} = (x_{nk} + y_{nk}) \in X$ . Therefore  $\bar{x} \vee \bar{y} = (x_{nk} + y_{nk}) \in {}_2\Lambda(X)$ . Hence  ${}_2\Lambda(X)$  is a Riesz space. ■

**Theorem 4.** *If  $X$  is an order complete ordered vector space, then  ${}_2\Lambda(X)$  is also an order complete ordered vector space.*

**Proof.** Suppose,  $X$  be an order complete ordered vector space. Let us consider a directed subset  ${}_2\bar{A}$  of  ${}_2\Lambda(X)$  majorised by  $\bar{a} = (a_{nk})$  i.e.  $\bar{x} \leq \bar{a}$ , for all  $\bar{x} = (x_{nk}) \in {}_2\bar{A}$ .

Then  $(x_{nk}) \leq (a_{nk})$ , for all  $x_{nk} \in X$ , for all  $n, k \in N$ . Hence  $x_{nk} \leq a_{nk}$ , for all  $x_{nk} \in X$ , for all  $n, k \in N$ . Let  $A_{nk} = \{x_{nk} : \bar{x} = (x_{nk}) \in {}_2\bar{A}\}$ , for all  $n, k \in N$ . Then  $A_{nk}$  is a directed subset of  $X$  majorised by  $a_{nk}$ , for each  $n, k \in N$ .

Since  $X$  is order complete so there is an element  $b_{nk}$  in  $X$  such that  $b_{nk} = \sup A_{nk}$ , for each  $n, k \geq 1$ . Let  $\bar{b} = (b_{nk})$ , then  $\bar{b} \in {}_2\Lambda(X)$ . Let  $\bar{c} = (c_{nk})$  be an upper bound of  ${}_2\bar{A}$ . Then  $\bar{x} \leq \bar{c}$ , for each  $\bar{x} \in {}_2\bar{A}$ . Hence  $(x_{nk}) \leq (c_{nk})$ , for each  $n, k$  in  $N$  and  $x_{nk} \in X$ . It follows that  $x_{nk} \leq c_{nk}$ , for each  $n, k$  in  $N$  and  $x_{nk} \in X$ . Since  $b_{nk} \in X$ , so  $b_{nk} \leq c_{nk}$ , for each  $n, k \geq 1$ . Then  $(b_{nk}) \leq (c_{nk})$ , for each  $n, k \geq 1$ . Hence  $\bar{x} \leq \bar{c}$ . Therefore  $\bar{b} = \sup {}_2\bar{A}$  (since  $b_{nk} = \sup A_{nk}$  and  $\bar{b} = (b_{nk}) \in {}_2\Lambda(X)$ ). Thus the directed subset  ${}_2\bar{A}$  of  ${}_2\Lambda(X)$  has a supremum  $\bar{b}$ . Hence  ${}_2\Lambda(X)$  is an order complete ordered vector space. ■

**Theorem 5.** *An ordered vector space  $X$  is Archimedean if and only if  ${}_2\Lambda(X)$  is Archimedean.*

**Proof.** Suppose  $X$  be Archimedean. Let us consider  $\bar{x} = (x_{nk})$  and  $\bar{y} = (y_{nk})$  in  ${}_2\Lambda(X)$  such that  $\lambda\bar{x} \leq \bar{y}$ , for all  $\lambda$  in  $R_+$ . Then  $(\lambda x_{nk}) \leq (y_{nk})$ , for each  $n, k$  in  $N$  and  $\lambda$  in  $R_+$ . Hence  $\lambda x_{nk} \leq y_{nk}$ , for each  $n, k$  in  $N$  and  $\lambda$  in  $R_+$ . Since  $X$  is Archimedean, so  $x_{nk} \leq \theta_{nk}$  whenever  $\lambda x_{nk} \leq y_{nk}$ , for each  $n, k \geq 1$ , where  $\theta_{nk} = \theta$  is the zero element in  $X$ . Then  $(\lambda x_{nk}) \leq (\theta_{nk})$ , for each  $n, k \geq 1$ . Hence  $\bar{x} \leq {}_2\bar{\theta}$  whenever  $\lambda\bar{x} \leq \bar{y}$  in  ${}_2\Lambda(X)$ , for all  $\lambda$  in  $R_+$ . Therefore  ${}_2\Lambda(X)$  is Archimedean.

Conversely, suppose  ${}_2\Lambda(X)$  be Archimedean. Let  $x_{nk} \in X$  and  $y_{nk} \in P$  be such that  $\lambda x_{nk} \leq y_{nk}$ , for each  $n, k$  in  $N$  and  $\lambda$  in  $R_+$ . Then  $(\lambda x_{nk}) \leq (y_{nk})$ , for each  $n, k$  in  $N$  and  $\lambda$  in  $R_+$ .  $\lambda\bar{x} \leq \bar{y}$  in  ${}_2\Lambda(X)$ , for all  $\lambda$  in  $R_+$ . Since  ${}_2\Lambda(X)$  is Archimedean, so  $\bar{x} \leq {}_2\bar{\theta}$  whenever  $\lambda\bar{x} \leq \bar{y}$ , where  ${}_2\bar{\theta} = (\theta_{nk})$  in  ${}_2\Lambda(X)$ . Then  $(\lambda x_{nk}) \leq (\theta_{nk})$ , for each  $n, k \geq 1$ . It follows that  $x_{nk} \leq \theta_{nk}$  whenever  $\lambda x_{nk} \leq y_{nk}$ , for each  $n, k$  in  $N$  and  $\lambda$  in  $R_+$ . Hence  $X$  is Archimedean. ■

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