DE GRUYTER OPEN
F A S C I C U L I   M A T H E M A T I C I

Nr 55 2015

DOI:10.1515/fascmath-2015-0013

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INEQUALITIES OF JENSEN TYPE FOR φ-CONVEX FUNCTIONS

Abstract. Some inequalities of Jensen type for φ-convex functions defined on real intervals are given.

Key words: convex functions, integral inequalities, h-convex functions.

AMS Mathematics Subject Classification: 26D15, 25D10.

1. Introduction

We recall here some concepts of convexity that are well known in the literature.

Let I be an interval in $\mathbb{R}$.

Definition 1 ([38]). *We say that $f : I \to \mathbb{R}$ is a Godunova-Levin function or that f belongs to the class $Q(I)$ if f is non-negative and for all $x, y \in I$ and $t \in (0,1)$ we have*

$$f(tx + (1-t)y) \leq \frac{1}{t} f(x) + \frac{1}{1-t} f(y). \tag{1}$$

Some further properties of this class of functions can be found in [29], [30], [32], [44], [47] and [48]. Among others, its has been noted that non-negative monotone and non-negative convex functions belong to this class of functions.

Definition 2 ([32]). *We say that a function $f : I \to \mathbb{R}$ belongs to the class $P(I)$ if it is nonnegative and for all $x, y \in I$ and $t \in [0,1]$ we have*

$$f(tx + (1-t)y) \leq f(x) + f(y). \tag{2}$$

Obviously $Q(I)$ contains $P(I)$ and for applications it is important to note that also $P(I)$ contain all nonnegative monotone, convex and *quasi convex functions,* i. e. nonnegative functions satisfying

$$f(tx + (1-t)y) \leq \max\{f(x), f(y)\} \tag{3}$$

for all $x, y \in I$ and $t \in [0,1]$.

For some results on P-functions see [32] and [45] while for quasi convex functions, the reader can consult [31].

Definition 3 ([7])**.** *Let s be a real number, $s \in (0,1]$. A function $f : [0,\infty) \to [0,\infty)$ is said to be s-convex (in the second sense) or Breckner s-convex if*

$$f(tx + (1-t)y) \le t^s f(x) + (1-t)^s f(y)$$

for all $x, y \in [0,\infty)$ and $t \in [0,1]$.

For some properties of this class of functions see [1], [2], [7], [8], [27], [28], [39], [41] and [50].

In order to unify the above concepts for functions of real variable, S. Varošanec introduced the concept of h-convex functions as follows.

Assume that I and J are intervals in $\mathbb{R}$, $(0,1) \subseteq J$ and functions h and f are real non-negative functions defined in J and I, respectively.

Definition 4 ([53])**.** *Let $h : J \to [0,\infty)$ with h not identical to 0. We say that $f : I \to [0,\infty)$ is an h-convex function if for all $x, y \in I$ we have*

$$f(tx + (1-t)y) \le h(t) f(x) + h(1-t) f(y) \tag{4}$$

for all $t \in (0,1)$.

For some results concerning this class of functions see [53], [6], [42], [51], [49] and [52].

We can introduce now another class of functions.

Definition 5. *We say that the function $f : I \to [0,\infty)$ is of s-Godunova-Levin type, with $s \in [0,1]$, if*

$$f(tx + (1-t)y) \le \frac{1}{t^s} f(x) + \frac{1}{(1-t)^s} f(y), \tag{5}$$

for all $t \in (0,1)$ and $x, y \in I$.

We observe that for $s = 0$ we obtain the class of P-functions while for $s = 1$ we obtain the class of Godunova-Levin. If we denote by $Q_s(I)$ the class of s-Godunova-Levin functions defined on I, then we obviously have

$$P(I) = Q_0(I) \subseteq Q_{s_1}(I) \subseteq Q_{s_2}(I) \subseteq Q_1(I) = Q(I)$$

for $0 \le s_1 \le s_2 \le 1$.

The following inequality holds for any convex function f defined on $\mathbb{R}$

$$(b-a) f\left(\frac{a+b}{2}\right) < \int_a^b f(x)dx < (b-a)\frac{f(a)+f(b)}{2}, \quad a, b \in \mathbb{R}. \tag{6}$$

It was firstly discovered by Ch. Hermite in 1881 in the journal *Mathesis* (see [43]). But this result was nowhere mentioned in the mathematical literature and was not widely known as Hermite's result.

E. F. Beckenbach, a leading expert on the history and the theory of convex functions, wrote that this inequality was proven by J. Hadamard in 1893 [5]. In 1974, D. S. Mitrinović found Hermite's note in *Mathesis* [43]. Since (6) was known as Hadamard's inequality, the inequality is now commonly referred as the Hermite-Hadamard inequality. For related results, see [10]-[19], [22]-[26], [33]-[36] and [46].

The following inequality of Hermite-Hadamard type for h-convex function holds [49].

Theorem 1. *Assume that the function* $f : I \to [0, \infty)$ *is an* h*-convex function with* $h \in L[0,1]$*. Let* $y, x \in I$ *with* $y \neq x$ *and assume that the mapping* $[0,1] \ni t \mapsto f[(1-t)x + ty]$ *is Lebesgue integrable on* $[0,1]$*. Then*

$$\frac{1}{2h\left(\frac{1}{2}\right)} f\left(\frac{x+y}{2}\right) \le \frac{1}{y-x}\int_x^y f(u)\,du \le [f(x)+f(y)]\int_0^1 h(t)\,dt. \tag{7}$$

If we write (7) for $h(t) = t$, then we get the classical Hermite-Hadamard inequality for convex functions

$$f\left(\frac{x+y}{2}\right) \le \frac{1}{y-x}\int_x^y f(u)\,du \le \frac{f(x)+f(y)}{2}. \tag{8}$$

If we write (7) for the case of P-type functions $f : I \to [0, \infty)$, i.e., $h(t) = 1$, $t \in [0,1]$, then we get the inequality

$$\frac{1}{2} f\left(\frac{x+y}{2}\right) \le \frac{1}{y-x}\int_x^y f(u)\,du \le f(x)+f(y), \tag{9}$$

that has been obtained for functions of real variable in [32].

If f is Breckner s-convex on I, for $s \in (0,1)$, then by taking $h(t) = t^s$ in (7) we get

$$2^{s-1} f\left(\frac{x+y}{2}\right) \le \frac{1}{y-x}\int_x^y f(u)\,du \le \frac{f(x)+f(y)}{s+1}, \tag{10}$$

that was obtained for functions of a real variable in [27].

If $f : I \to [0, \infty)$ is of s-Godunova-Levin type, with $s \in [0,1)$, then

$$\frac{1}{2^{s+1}} f\left(\frac{x+y}{2}\right) \le \frac{1}{y-x}\int_x^y f(u)\,du \le \frac{f(x)+f(y)}{1-s}. \tag{11}$$

We notice that for $s = 1$ the first inequality in (11) still holds, i.e.

$$\frac{1}{4} f\left(\frac{x+y}{2}\right) \le \int_0^1 f[(1-t)x + ty]\,dt. \tag{12}$$

The case for functions of real variables was obtained for the first time in [32].

2. φ-convex functions

We introduce the following class of h-convex functions.

Definition 6. *Let $\varphi : (0,1) \to (0,\infty)$ a measurable function. We say that the function $f : I \to [0,\infty)$ is a φ-convex function on the interval I if for all $x, y \in I$ we have*

$$f(tx + (1-t)y) \le t\varphi(t) f(x) + (1-t)\varphi(1-t) f(y) \tag{13}$$

for all $t \in (0,1)$.

If we denote $\ell(t) = t$, the identity function, then it is obvious that f is h-convex with $h = \ell\varphi$. Also, all the examples from the introduction can be seen as φ-convex functions with appropriate choices of φ.

If we take $\varphi(t) = \frac{1}{t^{s+1}}$ with $s \in [0,1]$, then we get the class of s-Godunova-Levin functions. Also, if we put $\varphi(t) = t^{s-1}$ with $s \in (0,1)$, then we get the concept of Breckner s-convexity. We notice that for all these examples we have

$$\varphi_+(0) := \lim_{t\to 0+} \varphi(t) = \infty.$$

The case of convex functions, i.e. when $\varphi(t) = 1$ is the only example from above for which $\varphi_+(0)$ is finite, namely $\varphi_+(0) = 1$.

Consider the family of functions, for $p > 1$ and $k > 0$

$$\delta(p,k) : [0,1] \to \mathbb{R}_+, \quad \delta(p,k)(t) = k(1-t)^p + 1. \tag{14}$$

We observe that $\delta_+(p,k)(0) = \delta(p,k)(0) = k+1$, $\delta(p,k)$ is strictly decreasing on $[0,1]$ and $\delta(p,k)(t) \ge \delta(p,k)(1) = 1$.

Definition 7. *We say that the function $f : I \to [0,\infty)$ is a $\delta(p,k)$-convex function on the interval I if for all $x, y \in I$ we have*

$$f(tx + (1-t)y) \le t\left[k(1-t)^p + 1\right] f(x) + (1-t)(kt^p + 1) f(y) \tag{15}$$

for all $t \in (0,1)$.

It is obvious that any nonnegative convex function is a $\delta^{(p,k)}$-convex function for any $p > 1$ and $k > 0$.

For $m > 0$ we consider the family of functions

$$\eta(m) : [0,1] \to \mathbb{R}_+, \quad \eta(m)(t) := \exp[m(1-t)].$$

We observe that $\eta_+(m)(0) = \eta(m)(0) = \exp(m)$, $\eta(m)$ is strictly decreasing on $[0,1]$ and $\eta(m)(t) \ge \eta(m)(1) = 1$.

Definition 8. *We say that the function* $f : I \to [0, \infty)$ *is a* $\eta(m)$*-convex function on the interval* I *if for all* $x, y \in I$ *we have*

$$f(tx + (1-t)y) \le t \exp[m(1-t)] f(x) + (1-t)\exp(mt) f(y) \tag{16}$$

for all $t \in (0,1)$.

It is obvious that any nonnegative convex function is a $\eta(m)$-convex function for any $m > 0$.

There are many other examples one can consider. In fact any continuos function $\varphi : [0,1] \to [1,\infty)$ can generate a class of φ-convex function that contains the class of nonnegative convex functions.

Utilising Theorem 1 we can state the following result.

Theorem 2. *Assume that the function* $f : I \to [0,\infty)$ *is a* φ*-convex function with* $\ell\varphi \in L[0,1]$. *Let* $y, x \in I$ *with* $y \neq x$ *and assume that the mapping* $[0,1] \ni t \mapsto f[(1-t)x + ty]$ *is Lebesgue integrable on* $[0,1]$. *Then*

$$\frac{1}{\varphi\left(\frac{1}{2}\right)} f\left(\frac{x+y}{2}\right) \le \frac{1}{y-x}\int_x^y f(u)\,du \le [f(x) + f(y)] \int_0^1 t\varphi(t)\,dt. \tag{17}$$

The proof follows from (7) by taking $h(t) = t\varphi(t)$, $t \in (0,1)$.

Remark 1. We notice that, since $\int_0^1 t\varphi(t)\,dt$ can be seen as the expectation of a random variable X with the density function φ, the inequality (17) provides a connection to Probability Theory and motivates the introduction of φ-convex function as a natural concept, having available many examples of density functions φ that arise in applications.

For different inequalities related to these classes of functions, see [1]-[4], [6], [9]-[37], [40]-[42] and [45]-[52].

A function $h : J \to \mathbb{R}$ is said to be *supermultiplicative* if

$$h(ts) \ge h(t)h(s) \quad \text{for any} \quad t, s \in J. \tag{18}$$

If the inequality (18) is reversed, then h is said to be *submultiplicative.* If the equality holds in (18) then h is said to be a multiplicative function on J.

In [53] it has been noted that if $h : [0,\infty) \to [0,\infty)$ with $h(t) = (x+c)^{p-1}$, then for $c = 0$ the function h is multiplicative. If $c \ge 1$, then for $p \in (0,1)$ the function h is supermultiplicative and for $p > 1$ the function is submultiplicative.

We observe that, if h, g are nonnegative and supermultiplicative, the same is their product. In particular, if h is supermultiplicative then its product with a power function $\ell_r(t) = t^r$ is also supermultiplicative.

The case of h-convex function with h supermultiplicative is of interest due to several Jensen type inequalities one can derive.

The following results were obtained in [53] for functions of a real variable.

Theorem 3. *Let $h : J \to [0, \infty)$ be a supermultiplicative function on J. If the function $f : I \to [0, \infty)$ is h-convex on the interval I, then for any $w_i \geq 0$, $x_i \in I$, $i \in \{1, \ldots, n\}$, $n \geq 2$ with $W_n := \sum_{i=1}^{n} w_i > 0$ we have*

$$f\left(\frac{1}{W_n}\sum_{i=1}^{n} w_i x_i\right) \leq \sum_{i=1}^{n} h\left(\frac{w_i}{W_n}\right) f(x_i). \tag{19}$$

In particular, we have the unweighted inequality

$$f\left(\frac{1}{n}\sum_{i=1}^{n} x_i\right) \leq h\left(\frac{1}{n}\right)\sum_{i=1}^{n} f(x_i). \tag{20}$$

Let $h(z) = \sum_{n=0}^{\infty} a_n z^n$ be a power series with complex coefficients and convergent on the open disk $D(0, R) \subset \mathbb{C}$, $R > 0$. We have the following examples

$$\begin{aligned} h(z) &= \sum_{n=1}^{\infty} \frac{1}{n} z^n = \ln \frac{1}{1-z}, \quad z \in D(0,1); \\ h(z) &= \sum_{n=0}^{\infty} \frac{1}{(2n)!} z^{2n} = \cosh z, \quad z \in \mathbb{C}; \\ h(z) &= \sum_{n=0}^{\infty} \frac{1}{(2n+1)!} z^{2n+1} = \sinh z, \quad z \in \mathbb{C}; \\ h(z) &= \sum_{n=0}^{\infty} z^n = \frac{1}{1-z}, \quad z \in D(0,1). \end{aligned} \tag{21}$$

Other important examples of functions as power series representations with nonnegative coefficients are:

$$\begin{aligned} h(z) &= \sum_{n=0}^{\infty} \frac{1}{n!} z^n = \exp(z), \quad z \in \mathbb{C}, \\ h(z) &= \sum_{n=1}^{\infty} \frac{1}{2n-1} z^{2n-1} = \frac{1}{2}\ln\left(\frac{1+z}{1-z}\right), \quad z \in D(0,1); \\ h(z) &= \sum_{n=0}^{\infty} \frac{\Gamma\left(n+\frac{1}{2}\right)}{\sqrt{\pi}(2n+1)n!} z^{2n+1} = \sin^{-1}(z), \quad z \in D(0,1); \end{aligned} \tag{22}$$

and

$$h(z) = \sum_{n=1}^{\infty} \frac{1}{2n-1} z^{2n-1} = \tanh^{-1}(z), \quad z \in D(0,1) \tag{23}$$

$$h(z) = {}_2F_1(\alpha,\beta,\gamma,z) = \sum_{n=0}^{\infty} \frac{\Gamma(n+\alpha)\Gamma(n+\beta)\Gamma(\gamma)}{n!\Gamma(\alpha)\Gamma(\beta)\Gamma(n+\gamma)} z^n, \alpha,\beta,\gamma > 0, \quad z \in D(0,1);$$

where Γ is *Gamma function.*

The following result may provide many examples of supemultiplicative functions.

Lemma 1. *Let* $h(z) = \sum_{n=0}^{\infty} a_n z^n$ *be a power series with complex coefficients and convergent on the open disk* $D(0,R) \subset \mathbb{C}$*,* $R > 0$*. Assume that* $0 < r < R$ *and define* $h_r : [0,1] \to [0,\infty)$*,* $h_r(t) := \frac{h(rt)}{h(r)}$*. Then* h_r *is supemultiplicative on* $[0,1]$*.*

Proof. We use the Čebyšev inequality for synchronous (the same monotonicity) sequences $(c_i)_{i\in\mathbb{N}}, (b_i)_{i\in\mathbb{N}}$ and nonnegative weights $(p_i)_{i\in\mathbb{N}}$:

$$\sum_{i=0}^{n} p_i \sum_{i=0}^{n} p_i c_i b_i \geq \sum_{i=0}^{n} p_i c_i \sum_{i=0}^{n} p_i b_i, \tag{24}$$

for any $n \in \mathbb{N}$.

Let $t, s \in (0,1)$ and define the sequences $c_i := t^i$, $b_i := s^i$.. These sequences are decreasing and if we apply Čebyšev's inequality for these sequences and the weights $p_i := a_i r^i \geq 0$ we get

$$\sum_{i=0}^{n} a_i r^i \sum_{i=0}^{n} a_i (rts)^i \geq \sum_{i=0}^{n} a_i (rt)^i \sum_{i=0}^{n} a_i (rs)^i \tag{25}$$

for any $n \in \mathbb{N}$.

Since the series

$$\sum_{i=0}^{\infty} a_i r^i, \quad \sum_{i=0}^{\infty} a_i (rts)^i, \quad \sum_{i=0}^{\infty} a_i (rt)^i \text{ and } \sum_{i=0}^{\infty} a_i (rs)^i$$

are convergent, then by letting $n \to \infty$ in (25) we get

$$h(r)\, h(rts) \geq h(rt)\, h(rs)$$

i.e.

$$h_r(ts) \geq h_r(t)\, h_r(s).$$

This inequality is also obviously satisfied at the end points of the interval $[0,1]$ and the proof is completed. ∎

Remark 2. Utilising the above theorem, we then conclude that the functions

$$h_r : [0,1] \to [0,\infty), \quad h_r(t) := \frac{1-r}{1-rt}, \quad r \in (0,1)$$

and

$$h_r : [0,1] \to [0,\infty), \quad h_r(t) := \exp[-r(1-t)], \ r > 0$$

are supermultiplicative.

We say that the function $f : I \to [0,\infty)$ is r-resolvent convex with r fixed in $(0,1)$, if f is h-convex with $h(t) = \frac{1-r}{1-rt}$, i.e.

$$f(tx + (1-t)y) \le (1-r)\left[\frac{1}{1-rt}f(x) + \frac{1}{1-r+rt}f(y)\right] \tag{26}$$

for any $x, y \in I$ and $t \in [0,1]$.

In particular, for $r = \frac{1}{2}$ we have $\frac{1}{2}$-resolvent convex functions defined by the condition

$$f(tx + (1-t)y) \le \frac{1}{2-t}f(x) + \frac{1}{1+t}f(y) \tag{27}$$

for any $t \in [0,1]$ and $x, y \in I$.

Since

$$t < \frac{1}{2-t} < \frac{1}{t} \ \text{ and } \ 1-t < \frac{1}{1+t} < \frac{1}{1-t} \ \text{ for } \ t \in (0,1)$$

it follows that any nonnegative convex function is $\frac{1}{2}$-resolvent convex which, in its turn, is of Godunova-Levin type.

We say that the function $f : I \to [0,\infty)$ is r-exponential convex with r fixed in $(0,\infty)$, if f is h-convex with $h(t) = \exp[-r(1-t)]$, i.e.

$$f(tx + (1-t)y) \le \exp[-r(1-t)]\,f(x) + \exp(-rt)\,f(y) \tag{28}$$

for any $t \in [0,1]$ and $x, y \in C$.

Since

$$t \le \exp[-r(1-t)] \ \text{ and } \ 1-t \le \exp(-rt) \ \text{ for } \ t \in [0,1]$$

it follows that any nonnegative convex function is r-exponential convex with $r \in (0,\infty)$.

Corollary 1. *Let $h(z) = \sum_{n=0}^{\infty} a_n z^n$ be a power series with complex coefficients and convergent on the open disk $D(0,R) \subset \mathbb{C}$, $R > 0$. Assume that $0 < r < R$ and define $h_r : [0,1] \to [0,\infty)$, $h_r(t) := \frac{h(rt)}{h(r)}$. If the function $f : I \to [0,\infty)$ is h_r-convex on the on the interval I, namely*

$$f(tx + (1-t)y) \le \frac{1}{h(r)}\left[h(rt)\,f(x) + h(r(1-t))\,f(y)\right] \tag{29}$$

for any $t \in [0,1]$ *and* $x, y \in I$, *then for any* $x_i \in I$, $w_i \geq 0$, $i \in \{1, \ldots, n\}$, $n \geq 2$ *with* $W_n := \sum_{i=1}^n w_i > 0$ *we have*

$$f\left(\frac{1}{W_n}\sum_{i=1}^n w_i x_i\right) \leq \frac{1}{h(r)}\sum_{i=1}^n h\left(r\frac{w_i}{W_n}\right) f(x_i). \tag{30}$$

Remark 3. If the function $f : I \to [0, \infty)$ is $\frac{1}{2}$-resolvent convex on I, then for any $x_i \in I$, $w_i \geq 0$, $i \in \{1, \ldots, n\}$, $n \geq 2$ with $W_n := \sum_{i=1}^n w_i > 0$ we have

$$f\left(\frac{1}{W_n}\sum_{i=1}^n w_i x_i\right) \leq W_n \sum_{i=1}^n \frac{1}{2W_n - w_i} f(x_i).$$

If the function $f : I \to [0, \infty)$ is r-exponential convex with r fixed in $(0, \infty)$, then for any $x_i \in I$, $w_i \geq 0$, $i \in \{1, \ldots, n\}$, $n \geq 2$ with $W_n := \sum_{i=1}^n w_i > 0$ we have

$$f\left(\frac{1}{W_n}\sum_{i=1}^n w_i x_i\right) \leq \sum_{i=1}^n \exp\left[-r\left(1 - \frac{w_i}{W_n}\right)\right] f(x_i).$$

We have the following Jensen type inequality for φ-convex functions.

Corollary 2. *Let* $\varphi : J \to [0, \infty)$ *be a supermultiplicative function on* J. *If the function* $f : I \to [0, \infty)$ *is* φ*-convex on the interval* I, *then for any* $w_i \geq 0$, $x_i \in I$, $i \in \{1, \ldots, n\}$, $n \geq 2$ *with* $W_n := \sum_{i=1}^n w_i > 0$ *we have*

$$f\left(\frac{1}{W_n}\sum_{i=1}^n w_i x_i\right) \leq \frac{1}{W_n}\sum_{i=1}^n w_i \varphi\left(\frac{w_i}{W_n}\right) f(x_i). \tag{31}$$

In particular, we have the unweighted inequality

$$f\left(\frac{1}{n}\sum_{i=1}^n x_i\right) \leq \varphi\left(\frac{1}{n}\right)\frac{1}{n}\sum_{i=1}^n f(x_i). \tag{32}$$

The proof follows by Theorem 3 for the supermultiplicative function $h(t) = t\varphi(t)$, $t \in J$.

The inequality (31) will be used further to obtain an integral Jensen type inequality.

3. Some results for differentiable functions

If we assume that the function $f : I \to [0, \infty)$ is differentiable on the interior of I, denoted by $\mathring{I}$, then we have the following *"gradient inequality"* that will play an essential role in the following.

Lemma 2. *Let $\varphi:(0,1)\to(0,\infty)$ be a measurable function and such that the right limit $\varphi_+(0)$ exists and is finite, the left limit $\varphi_-(1)=1$ and the left derivative in 1 denoted $\varphi'_-(1)$ exists and is finite. If the function $f:I\to[0,\infty)$ is differentiable on $\mathring{I}$ and φ-convex, then*

$$\varphi_+(0)f(x)-\left[\varphi'_-(1)+1\right]f(y)\geq f'(y)(x-y) \tag{33}$$

for any $x,y\in\mathring{I}$ with $x\neq y$.

Proof. Since f is φ-convex on I, then

$$t\varphi(t)f(x)+(1-t)\varphi(1-t)f(y)\geq f(tx+(1-t)y)$$

for any $t\in(0,1)$ and for any $x,y\in\mathring{I}$, which is equivalent to

$$t\varphi(t)f(x)+[(1-t)\varphi(1-t)-1]f(y)\geq f(tx+(1-t)y)-f(y)$$

and by dividing by $t>0$ we get

$$\varphi(t)f(x)+\left[\frac{(1-t)\varphi(1-t)-1}{t}\right]f(y)\geq\frac{f(tx+(1-t)y)-f(y)}{t} \tag{34}$$

for any $t\in(0,1)$.

Now, since f is differentiable on $y\in\mathring{I}$, then we have

$$\begin{aligned}\lim_{t\to0+}\frac{f(tx+(1-t)y)-f(y)}{t}&=\lim_{t\to0+}\frac{f(y+t(x-y))-f(y)}{t}\\&=(x-y)\lim_{t\to0+}\frac{f(y+t(x-y))-f(y)}{t(x-y)}=(x-y)f'(y)\end{aligned} \tag{35}$$

for any $x\in\mathring{I}$ with $x\neq y$.

Also since $\varphi_-(1)=1$ and $\varphi'_-(1)$ exists and is finite, we have

$$\begin{aligned}\lim_{t\to0+}\frac{(1-t)\varphi(1-t)-1}{t}&=\lim_{s\to1-}\frac{s\varphi(s)-1}{1-s}=-\lim_{s\to1-}\frac{s\varphi(s)-1}{s-1}\\&=-\lim_{s\to1-}\frac{s(\varphi(s)-\varphi(1))+s-1}{s-1}\\&=-\varphi'_-(1)-1.\end{aligned} \tag{36}$$

Taking the limit over $t\to0+$ in (34) and utilizing (35) and (36) we get the desired result (33). ■

Remark 4. If we assume that

$$\varphi_+(0)\geq\varphi'_-(1)+1, \tag{37}$$

then the inequality (33) also holds for $x = y$.

There are numerous examples of such functions, for instance, if, as above we take $\varphi(t) = k(1-t)^p + 1$, $t \in [0,1]$ $(p > 1, k > 0)$ then $\varphi_+(0) = k+1$, $\varphi_-(1) = 1$ and $\varphi'_-(1) = 0$, which satisfy the condition (37).

If we take $\varphi(t) = \exp[m(1-t)]$ $(m > 0)$, then $\varphi_+(0) = \exp m$, $\varphi_-(1) = 1$ and $\varphi'_-(1) = -m$. We have

$$\varphi_+(0) - \varphi_-(1) - \varphi'_-(1) = e^m - 1 + m > 0$$

for $m > 0$.

The following result holds:

Theorem 4. *Let $\varphi : (0,1) \to (0,\infty)$ a measurable function and such that the right limit $\varphi_+(0)$ exists and is finite, the left limit $\varphi_-(1) = 1$ and the left derivative in 1 denoted $\varphi'_-(1)$ exists and is finite. Assume also that $\varphi'_-(1) > -1$. If the function $f : I \to [0,\infty)$ is differentiable on $\mathring{I}$ and φ-convex, then*

$$\frac{\varphi_+(0)}{\varphi'_-(1)+1}\frac{f(x)+f(y)}{2} \geq \frac{1}{y-x}\int_x^y f(u)\,du \geq \frac{\varphi'_-(1)+1}{\varphi_+(0)} f\left(\frac{x+y}{2}\right) \tag{38}$$

for any $x, y \in I$.

Remark 5. It has been shown in [25] that the inequalities (17) and (38) are not comparable, meaning that some time one is better then the other, depending on the φ-convex function involved.

The following discrete Jensen type inequality holds:

Theorem 5. *Let $\varphi : (0,1) \to (0,\infty)$ be a measurable function and such that the right limit $\varphi_+(0)$ exists and is finite, the left limit $\varphi_-(1) = 1$ and the left derivative in 1 denoted $\varphi'_-(1)$ exists and is finite. Assume also that*

$$\varphi_+(0) \geq \varphi'_-(1) + 1 > 0. \tag{39}$$

If the function $f : I \to [0,\infty)$ is differentiable on $\mathring{I}$ and φ-convex, then for any $w_i \geq 0$, $x_i \in \mathring{I}$, $i \in \{1,\ldots,n\}$, $n \geq 2$ with $W_n := \sum_{i=1}^n w_i > 0$ we have

$$\frac{\varphi_+(0)}{\varphi'_-(1)+1}\frac{1}{W_n}\sum_{i=1}^n w_i f(x_j) \geq f\left(\frac{1}{W_n}\sum_{i=1}^n w_i x_i\right). \tag{40}$$

If $\frac{1}{W_n}\sum_{i=1}^n w_i x_i \neq x_j$ for any $j \in \{1,\ldots,n\}$, then the first condition in (39) can be dropped.

Proof. From (33) we have

$$\varphi_+(0) f(x_j) - \left[\varphi'_-(1) + 1\right] f\left(\frac{1}{W_n}\sum_{i=1}^{n} w_i x_i\right) \geq f'\left(\frac{1}{W_n}\sum_{i=1}^{n} w_i x_i\right)\left(x_j - \frac{1}{W_n}\sum_{i=1}^{n} w_i x_i\right) \tag{41}$$

for any $j \in \{1, \dots, n\}$.

If we multiply (41) by $w_i \geq 0$ and sum over j from 1 to n we get

$$\varphi_+(0)\sum_{j=1}^{n} w_j f(x_j) - \left[\varphi'_-(1) + 1\right]\sum_{j=1}^{n} w_j f\left(\frac{1}{W_n}\sum_{i=1}^{n} w_i x_i\right) \geq f'\left(\frac{1}{W_n}\sum_{i=1}^{n} w_i x_i\right)\sum_{j=1}^{n} w_j\left(x_j - \frac{1}{W_n}\sum_{i=1}^{n} w_i x_i\right) = 0,$$

which proves the desired result (40). ■

4. Integral inequalities

We have the following Jensen inequality for the Riemann integral:

Theorem 6. *Let $u : [a,b] \to [m,M]$ be a Riemann integrable function. Suppose that $\varphi : J \to [0,\infty)$ is a supermultiplicative function on J and the function $f : [m,M] \to [0,\infty)$ is φ-convex and continuous on the interval $[m,M]$. If the right limit $\varphi_+(0)$ exists and is finite, then*

$$f\left(\frac{1}{b-a}\int_a^b u(t)\,dt\right) \leq \varphi_+(0)\frac{1}{b-a}\int_a^b f(u(t))\,dt. \tag{42}$$

Proof. Consider the sequence of divisions

$$d_n : \ x_i^{(n)} = a + \frac{i}{n}(b-a), \ i \in \{0, \dots, n\}$$

and the intermediate points

$$\xi_i^{(n)} = a + \frac{i}{n}(b-a), \ i \in \{0, \dots, n\}.$$

We observe that the norm of the division $\Delta_n := \max_{i\in\{0,\dots,n-1\}}(x_{i+1}^{(n)} - x_i^{(n)}) = \frac{b-a}{n} \to 0$ as $n \to \infty$ and since u is Riemann integrable on $[a,b]$, then

$$\int_a^b u(t)\,dt = \lim_{n\to\infty}\sum_{i=0}^{n-1} u\left(\xi_i^{(n)}\right)\left[x_{i+1}^{(n)} - x_i^{(n)}\right] = \lim_{n\to\infty}\frac{b-a}{n}\sum_{i=0}^{n-1} u\left(a + \frac{i}{n}(b-a)\right).$$

Also, since $f:[m,M]\to[0,\infty)$ is Riemann integrable, then $f\circ u$ is Riemann integrable and

$$\int_a^b f(u(t))\,dt=\lim_{n\to\infty}\frac{b-a}{n}\sum_{i=0}^{n-1}f\left[u\left(a+\frac{i}{n}(b-a)\right)\right].$$

Utilising the inequality (31) for $w_i:=\frac{b-a}{n}$ and $x_i:=u\left(a+\frac{i}{n}(b-a)\right)$ we have

$$\begin{aligned}(43)\qquad & f\left(\frac{1}{b-a}\frac{b-a}{n}\sum_{i=0}^{n-1}u\left(a+\frac{i}{n}(b-a)\right)\right)\\ &\le\frac{1}{b-a}\frac{b-a}{n}\sum_{i=0}^{n-1}\varphi\left(\frac{1}{n}\right)f\left(u\left(a+\frac{i}{n}(b-a)\right)\right)\\ &=\frac{1}{b-a}\varphi\left(\frac{1}{n}\right)\frac{b-a}{n}\sum_{i=0}^{n-1}f\left(u\left(a+\frac{i}{n}(b-a)\right)\right)\end{aligned}$$

for any $n\ge1$.

Since f is continuous, then

$$\lim_{n\to\infty}f\left(\frac{1}{b-a}\frac{b-a}{n}\sum_{i=0}^{n-1}u\left(a+\frac{i}{n}(b-a)\right)\right)=f\left(\frac{1}{b-a}\int_a^b u(t)\,dt\right).$$

Also

$$\lim_{n\to\infty}\varphi\left(\frac{1}{n}\right)=\varphi_+(0)<\infty.$$

Therefore, taking the limit over $n\to\infty$ in the inequality (43) we deduce the desired result (42). ■

We have the following Hermite-Hadamard type inequality:

Corollary 3. *Suppose that $\varphi:J\to[0,\infty)$ is a supermultiplicative function on J and the function $f:I\to[0,\infty)$ is φ-convex and continuous on the interval I. If the right limit $\varphi_+(0)$ exists and is finite with $\varphi_+(0)>0$, then for any $x,y\in I$ with $x\ne y$ we have*

$$(44)\qquad \frac{1}{\varphi_+(0)}f\left(\frac{x+y}{2}\right)\le\frac{1}{y-x}\int_x^y f(u(t))\,dt.$$

Remark 6. If the function $f:[m,M]\to[0,\infty)$ is a $\delta(p,k)$-convex and continuous function on the interval $[m,M]$ ($p>1$ and $k>0$, see

Definition 7) then for any $u:[a,b]\to[m,M]$ a Riemann integrable function on $[a,b]$ we have

$$\frac{1}{k+1}f\left(\frac{1}{b-a}\int_a^b u(t)\,dt\right)\le\frac{1}{b-a}\int_a^b f(u(t))\,dt. \tag{45}$$

If the function $f:[m,M]\to[0,\infty)$ is a $\eta(s)$-convex and continuous function on the interval $[m,M]$ ($s>0$, see Definition 8) then for any $u:[a,b]\to[m,M]$ a Riemann integrable function on $[a,b]$ we have

$$\frac{1}{e^s}f\left(\frac{1}{b-a}\int_a^b u(t)\,dt\right)\le\frac{1}{b-a}\int_a^b f(u(t))\,dt. \tag{46}$$

Let $(\Omega,\mathcal{A},\mu)$ be a measurable space consisting of a set Ω, a $\sigma-$ algebra $\mathcal{A}$ of parts of Ω and a countably additive and positive measure μ on $\mathcal{A}$ with values in $\mathbb{R}\cup\{\infty\}$. For a $\mu-$measurable function $w:\Omega\to\mathbb{R}$, with $w(x)\ge0$ for $\mu-$ a.e.(almost every) $x\in\Omega$, consider the Lebesgue space

$$L_w(\Omega,\mu):=\{f:\Omega\to\mathbb{R},\ f\text{is }\mu\text{-measurable and }\int_\Omega w(x)\,|f(x)|\,d\mu(x)<\infty\}.$$

For simplicity of notation we write everywhere in the sequel $\int_\Omega wd\mu$ instead of $\int_\Omega w(x)\,d\mu(x)$.

Theorem 7. *Let $\varphi:(0,1)\to(0,\infty)$ be a measurable function and such that the right limit $\varphi_+(0)$ exists and is finite, the left limit $\varphi_-(1)=1$ and the left derivative in 1 denoted $\varphi'_-(1)$ exists and is finite. Assume also that*

$$\varphi_+(0)\ge\varphi'_-(1)+1>0. \tag{47}$$

If the function $f:I\to[0,\infty)$ is differentiable on $\mathring{I}$ and φ-convex, then for any $u:\Omega\to[m,M]\subset\mathring{I}$ so that $f\circ u,\ u\in L_w(\Omega,\mu)$, where $w\ge0$ μ-a.e. (almost everywhere) on Ω with $\int_\Omega wd\mu=1$ we have

$$\frac{\varphi_+(0)}{\varphi'_-(1)+1}\int_\Omega w(f\circ u)\,d\mu\ge f\left(\int_\Omega wud\mu\right). \tag{48}$$

If $\int_\Omega wud\mu\ne u(x)$ for μ-a.e. $x\in\Omega$, then we can drop the first condition in (47).

Proof. From (33) and since $\int_\Omega wud\mu\in[m,M]\subset\mathring{I}$ we have

$$\begin{aligned}\varphi_+(0)\,f(u(x))-\left[\varphi'_-(1)+1\right]f\left(\int_\Omega wud\mu\right)\\ \ge f'\left(\int_\Omega wud\mu\right)\left(u(x)-\int_\Omega wud\mu\right),\text{ for any }x\in\Omega.\end{aligned} \tag{49}$$

If we multiply (49) by $w \geq 0$ μ-a.e. on Ω and integrate over the positive measure μ we get

$$\varphi_{+}(0) \int_{\Omega} w(x) f(u(x)) d\mu(x) - \left[\varphi_{-}'(1) + 1\right] f\left(\int_{\Omega} wud\mu\right) \int_{\Omega} w(x) d\mu(x)$$
$$\geq f'\left(\int_{\Omega} wud\mu\right) \int_{\Omega} w(x) \left(u(x) - \int_{\Omega} wud\mu\right) d\mu(x) = 0,$$

which produces the desired result (48). ∎

Remark 7. If the function $f : [m, M] \to [0, \infty)$ is a $\delta(p, k)$-convex and continuous function on the interval $[m, M]$, then for any $u : \Omega \to [m, M] \subset \mathring{I}$ so that $f \circ u,\ u \in L_w(\Omega, \mu)$, where $w \geq 0$ μ-a.e. on Ω with $\int_{\Omega} wd\mu = 1$ we have

$$\int_{\Omega} w(f \circ u) d\mu \geq \frac{1}{k+1} f\left(\int_{\Omega} wud\mu\right). \tag{50}$$

If the function $f : [m, M] \to [0, \infty)$ is a $\eta(s)$-convex and continuous function on the interval $[m, M]$ then for any $u : \Omega \to [m, M] \subset \mathring{I}$ so that $f \circ u,\ u \in L_w(\Omega, \mu)$, where $w \geq 0$ μ-a.e. on Ω with $\int_{\Omega} wd\mu = 1$ we have

$$\int_{\Omega} w(f \circ u) d\mu \geq \frac{1}{e^s} f\left(\int_{\Omega} wud\mu\right). \tag{51}$$

These results generalize the inequalities (45) and (46).

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Received on 03.03.2015 and, in revised form, on 08.05.2015.