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ON GENERALIZED ORLICZ SEQUENCE SPACES DEFINED BY DOUBLE SEQUENCES *

Abstract. S.D. Parashar and B. Choudhary defined in 1994 certain paranorms for some Orlicz sequence spaces. Their ideas are applied later for topologization of various generalized Orlicz sequence spaces. The author determines in 2011 some alternative F-seminorms (which are also paranorms) for such spaces. In this paper these results are extended to generalized Orlicz sequence spaces defined via double sequences..

Key words: double sequence, F-seminorm, matrix method, modulus function, Orlicz function, paranorm, φ-function, sequence space, SM method, strong summability.

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1. Introduction

Let $\mathbb{N} = \{1, 2, \dots\}$ and let $\mathbb{K}$ be the field of real numbers $\mathbb{R}$ or complex numbers $\mathbb{C}$. We specify the domains of indices only if they are different from $\mathbb{N}$. By the symbol $\imath$ we denote the identity mapping $\imath(z) = z$. The superposition of two mappings f and g is denoted by fg, i.e., $(fg)(z) = f(g(z))$. For an arbitrary sequence $\mathbf{z} = (z_k)$, by $\mathbf{z}^{(2)} = (z_{ki}^{(2)})$ we denote the corresponding double sequence with the elements $z_{ki}^{(2)} = z_k$. If $\mathbf{w} = (w_k)$ is an another sequence, then by $\mathbf{zw}$ we denote the sequence $(z_k w_k)$ provided that $z_k w_k$ is determined for all $k \in \mathbb{N}$.

Let X be a vector (or linear) space over $\mathbb{K}$. A functional $g : X \to \mathbb{R}$ is called an *F-norm*, if (see, for example, [11])
(N1) $g(0) = 0$;
(N2) $g(x + y) \leq g(x) + g(y) \quad (x, y \in X)$;
(N3) $|\alpha| \leq 1 \ (\alpha \in \mathbb{K}),\ x \in X \implies g(\alpha x) \leq g(x)$;

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(N4) $\lim_n \alpha_n = 0$ $(\alpha_n \in \mathbb{K})$, $x \in X \implies \lim_n g(\alpha_n x) = 0$;
(N5) $g(x) = 0 \implies x = 0$.
A functional g with axioms (N1)–(N4) is called an F-*seminorm.* A *paranorm* g on X is defined by axioms (N1), (N2) and
(N6) $g(-x) = g(x) \quad (x \in X)$;
(N7) $\lim_n \alpha_n = \alpha$ $(\alpha_n, \alpha \in \mathbb{K})$, $\lim_n g(x_n - x) = 0$ $(x_n, x \in X)$
$\implies \lim_n g(\alpha_n x_n - \alpha x) = 0$.
A *seminorm* g on X is determined by axioms (N1), (N2) and
(N8) $g(\alpha x) = |\alpha| g(x) \quad (\alpha \in \mathbb{K},\ x \in X)$.
An F-seminorm (paranorm, seminorm) g is called *total* if (N5) holds. So, an F-norm (norm) is a total F-seminorm (seminorm).

In the following, unlike the module $|\cdot|$, the seminorm of an element $x \in X$ is often denoted by $\dot{|}x\dot{|}$.

Let $\mathbf{X}$ be a sequence of seminormed linear spaces $\left(X_k, \dot{|}\cdot\dot{|}_k\right)$ over $\mathbb{K}$. Thereby, the set $s(\mathbf{X})$ of all sequences $\mathbf{x} = (x_k)$, $x_k \in X_k$, and the set $s^2(\mathbf{X})$ of all double sequences $\mathbf{x}^2 = (x_{ki})$, $x_{ki} \in X_k$, together with coordinatewise addition and scalar multiplication are linear spaces (over $\mathbb{K}$). Any linear subspace of $s^2(\mathbf{X})$ is called a *generalized double sequence space* (GDS space) and any linear subspace of $s(\mathbf{X})$ is called a *generalized sequence space* (GS space). If $\left(X_k, \dot{|}\cdot\dot{|}_k\right) = \left(X, \dot{|}\cdot\dot{|}\right)$, then we write X instead of $\mathbf{X}$. In the case $X = \mathbb{K}$ we omit the symbol X in notation. So, for example, s^2 and s denote the linear spaces of all $\mathbb{K}$-valued double sequences $\mathbf{u}^2 = (u_{ki})$ and single sequences $\mathbf{u} = (u_k)$, respectively. As usual, linear subspaces of s^2 are called *double sequence spaces* (DS spaces) and linear subspaces of s are called *sequence spaces.* Well-known sequence spaces are the sets ℓ_∞, c, c_0, ℓ_p, $(0 < p < \infty)$ and w_0 of all bounded, convergent, convergent to zero, absolutely p-summable and strongly summable to zero number sequences, respectively. Examples of DS spaces are

$$
\begin{aligned}
uc_0 &= \left\{\mathbf{u}^2 \in s^2 : \lim_k u_{ki} = 0 \text{ uniformly in } i\right\}, \\
uw_0 &= \left\{\mathbf{u}^2 \in s^2 : \left(1/k \sum_{j=1}^{k} u_{ji}\right) \in uc_0\right\}
\end{aligned}
$$

and

$$U\lambda = \left\{\mathbf{u}^2 \in s^2 : \widetilde{\mathbf{u}} = (\widetilde{u}_k) \in \lambda\right\}$$

with $\widetilde{u}_k = \sup_i |u_{ki}|$ and $\lambda \in \{s, \ell_\infty, c_0, \ell_p, w_0\}$.

Let $\mathbb{R}^+ = [0, \infty)$. By a *$\varphi$-function* we mean a continuous and non-decreasing function $\phi : \mathbb{R}^+ \to \mathbb{R}^+$ such that $\phi(t) = 0$ if and only if $t = 0$ (cf. [16], p. 4). A φ-function ϕ is called a *modulus function* (or, briefly, a *modulus*) if

$$\phi(t+u) \le \phi(t) + \phi(u), \quad t, u \in \mathbb{R}^+, \tag{1}$$

and an *Orlicz function* if (1) is replaced by the condition of convexity

$$\phi(\alpha t + (1-\alpha)u) \le \alpha\phi(t) + (1-\alpha)\phi(u) \quad t, u \in \mathbb{R}^+, \;\; 0 \le \alpha \le 1.$$

It is not difficult to see that every φ-function ϕ satisfies the condition

$$\phi(\alpha t + (1-\alpha)u) \le \phi(t) + \phi(u), \quad t, u \in \mathbb{R}^+, \;\; 0 \le \alpha \le 1. \tag{2}$$

Let λ be a sequence space and let Λ be a double sequence space. Assume that $T : s_T(\mathbf{X}) \to s^2(\mathbf{X})$ is a linear operator with $T\mathbf{x} = (T_{ki}(\mathbf{x}))$, where $s_T(\mathbf{X})$ is a linear subspace of $s(\mathbf{X})$. For a sequence $\Phi = (\phi_k)$ of φ-functions, and for the sequences $\mathbf{z} = (z_k) \in s(\mathbf{X})$ and $\mathbf{z}^2 = (z_{ki}) \in s^2(\mathbf{X})$, we write

$$\Phi(\mathbf{z}) = \left(\phi_k\left(\left|z_k\right|_k\right)\right), \quad \Phi(\mathbf{z}^2) = \left(\phi_k\left(\left|z_{ki}\right|_k\right)\right)$$

and define the sets of sequences

$$\begin{aligned} \lambda^{\exists}(\Phi, \mathbf{X}) &= \left\{\mathbf{x} \in s(\mathbf{X}) : (\exists\rho > 0)\Phi(\rho^{-1}\mathbf{x}) \in \lambda\right\}, \\ \Lambda(\Phi, T, \mathbf{X}) &= \left\{\mathbf{x} \in s_T(\mathbf{X}) : \Phi(T\mathbf{x}) \in \Lambda\right\}, \\ \Lambda^{\exists}(\Phi, T, \mathbf{X}) &= \left\{\mathbf{x} \in s_T(\mathbf{X}) : (\exists\rho > 0)\Phi(\rho^{-1}T\mathbf{x}) \in \Lambda\right\}. \end{aligned}$$

If Φ is a constant sequence with $\phi_k = \phi$, then we write ϕ instead of Φ.

Let ψ be an Orlicz function. As usual, by an *Orlicz sequence space* we mean the Banach sequence space $\ell^{\exists}(\psi)$ with the norm (see [15] or [16])

$$\|\mathbf{u}\|_\psi = \inf\left\{\rho > 0 : \sum_k \psi(|\rho^{-1}u_k|) \le 1\right\}.$$

Woo [23] showed that if $\Psi = (\psi_k)$ is a sequence of Orlicz functions, then the set $\ell^{\exists}(\Psi)$ is also a Banach sequence space with the norm

$$\|\mathbf{u}\|_\Psi = \inf\left\{\rho > 0 : \sum_k \psi_k(|\rho^{-1}u_k|) \le 1\right\}.$$

In the mathematical literature we may find a series of papers which deal with various generalizations and modifications of $\ell^{\exists}(\phi)$ and $\ell^{\exists}(\Phi)$, where the space ℓ is replaced by different sequence spaces including the spaces of Maddox type, domains of various summability methods and also some sequence spaces defined via double sequences. For example, Nuray and Gülcü [17] considered generalized Orlicz sequence spaces

$$uw_0{}^{\exists}(\imath^{\mathbf{p}}\psi, I_\sigma) = \left\{\mathbf{u} \in s : (\exists\rho > 0)\imath^{\mathbf{p}}\left(\psi\left(\rho^{-1}I_\sigma\mathbf{u}\right)\right) \in uw_0\right\},$$
$$Uw_\infty{}^{\exists}(\imath^{\mathbf{p}}\psi, I_\sigma) = \left\{\mathbf{u} \in s : (\exists\rho > 0)\imath^{\mathbf{p}}\left(\psi\left(\rho^{-1}I_\sigma\mathbf{u}\right)\right) \in U\ell_\infty\right\},$$

where ψ is an Orlicz function, $\mathbf{p} = (p_k)$ is a bounded sequence of positive numbers, $I_\sigma\mathbf{u} = (u_{\sigma^k(i)})$, σ is a one-to-one mapping of $\mathbb{N}$ into itself and $\imath^{\mathbf{p}}$ denotes the sequence of φ-functions $\imath^{p_k}(t) = t^{p_k}$. Denoting $M = \max\{1, \sup_k p_k\}$ and using the ideas of Parashar and Choudhary [18], on $uw_0{}^{\exists}(\imath^{\mathbf{p}}\psi, I_\sigma)$ the paranorm

$$h_0(\mathbf{u}) = \inf_{\rho>0, n\in\mathbb{N}}\left\{\rho^{p_n/M} : \sup_{m,i\in\mathbb{N}}\left(m^{-1}\sum_{k=1}^{m}\left(\psi\left(\left|\rho^{-1}u_{\sigma^k(i)}\right|\right)\right)^{p_k}\right)^{1/M} \leq 1\right\}.$$

We note that h_0 is also an F-seminorm in view of [12, Remark 1]. Moreover, it seems that by the definition of h_0 the authors implicitly assume the validity of the inclusion $uw_0{}^{\exists}(\imath^{\mathbf{p}}\psi, I_\sigma) \subset Uw_\infty{}^{\exists}(\imath^{\mathbf{p}}\psi, I_\sigma)$ or, equivalently, the equality $uw_0{}^{\exists}(\imath^{\mathbf{p}}\psi, I_\sigma) = Uw_0{}^{\exists}(\imath^{\mathbf{p}}\psi, I_\sigma)$.

The idea to topologize different generalized Orlicz sequence spaces by the paranorms of type h_0 is used later by many authors (see, for example, [1], [3] - [10], [17], [19], [20], [22]). We determine alternative F-seminorms (which are also paranorms) on similar generalized Orlicz sequence spaces defined by means of a DS space Λ and a linear operator T which maps single sequences to double sequences. Applications of main theorems are considered in the case if Λ is the strong summability domain of a non-negative matrix method and T is determined by a sequence of summability matrices.

2. Main theorems

Let $\lambda \subset s$ be a sequence space, $\Lambda \subset s^2$ be a DS space, and let $\mathbf{e}_k = (e_j^k)_{j\in\mathbb{N}}$ be the sequences, where $e_j^k = 1$ if $j = k$ and $e_j^k = 0$ otherwise. Then, by our notation, $\mathbf{e}_k^{(2)}$ denotes, for any $k \in \mathbb{N}$, the double sequence of elements $e_{ji}^k = e_j^k$.

Recall that a GS space $\lambda(\mathbf{X}) \subset s(\mathbf{X})$ is called *solid* if $(y_k) \in \lambda(\mathbf{X})$ whenever $(x_k) \in \lambda(\mathbf{X})$ and $\dot{|}y_k\dot{|}_k \leq \dot{|}x_k\dot{|}_k$ for any $k \in \mathbb{N}$. Similarly is defined the solidity of a GDS space $\Lambda(\mathbf{X}) \subset s^2(\mathbf{X})$. For example, it is not difficult to see that $\lambda^{\exists}(\Phi, \mathbf{X})$ is solid whenever λ is solid.

An F-seminormed space (λ, g) is called an AK-*space*, if λ contains the sequences $\mathbf{e}_k$ and for any $\mathbf{u} = (u_k) \in \lambda$ we have $\lim_n \mathbf{u}^{[n]} = \mathbf{u}$, where

$\mathbf{u}^{[n]} = \sum_{k=1}^{n} u_k \mathbf{e}_k$. An F-seminormed DS space (Λ, g) is called a *double* AK-*space* (see, for example, [13]), if Λ contains the sequences $\mathbf{e}_k^{(2)}$ and for any $\mathbf{u}^2 = (u_{ki}) \in \Lambda$ we have $\lim_n \mathbf{u}^{2[n]} = \mathbf{u}^2$, where $\mathbf{u}^{2[n]} = \sum_{k=1}^{n} \mathbf{u}_k \mathbf{e}_k^{(2)}$ with $\mathbf{u}_k = (u_{ki})_{i\in\mathbb{N}}$ and $\mathbf{u}_k \mathbf{e}_k^{(2)} = (u_{ki} e_{ji}^k)_{j,i\in\mathbb{N}}$.

An F-seminorm g on a GS space $\lambda(\mathbf{X}) \subset s(\mathbf{X})$ is said to be *absolutely monotone* if for all $\mathbf{x} = (x_k)$ and $\mathbf{y} = (y_k)$ from $\lambda(\mathbf{X})$ with $|y_k|_k \le |x_k|_k$ $(k \in \mathbb{N})$ we have $g(\mathbf{y}) \le g(\mathbf{x})$. Analogously, an F-seminorm g on a GDS space $\Lambda(\mathbf{X}) \subset s^2(\mathbf{X})$ is said to be *absolutely monotone* if for all $\mathbf{x}^2 = (x_{ki})$ and $\mathbf{y}^2 = (y_{ki})$ from $\Lambda(\mathbf{X})$ with $|y_{ki}|_k \le |x_{ki}|_k$ $(k, i \in \mathbb{N})$ we have $g(\mathbf{y}^2) \le g(\mathbf{x}^2)$.

Our first theorem deals with the linearity of the set $\Lambda^{\exists}(\Phi, T, \mathbf{X})$.

Theorem 1. *Let Λ be a solid DS space and let $T : s_T(\mathbf{X}) \to s^2(\mathbf{X})$ be a linear operator with $T\mathbf{x} = (T_{ki}(\mathbf{x}))$. If $\Phi = (\phi_k)$ is a sequence of φ-functions, then $\Lambda^{\exists}(\Phi, T, \mathbf{X})$ is a GS space. At that, it is solid if*

$$(3) \qquad |x_k|_k \le |y_k|_k \quad (k \in \mathbb{N}) \implies |T_{ki}(\mathbf{x})|_k \le |T_{ki}(\mathbf{y})|_k \quad (k, i \in \mathbb{N}).$$

Proof. Let $\mathbf{x} \in \Lambda^{\exists}(\Phi, T, \mathbf{X})$ with $\Phi(\rho^{-1}T\mathbf{x}) \in \Lambda$, $\rho > 0$. It is clear that $0\mathbf{x} \in \Lambda^{\exists}(\Phi, T, \mathbf{X})$. If $0 \ne \alpha \in \mathbb{K}$, then by $(\rho|\alpha|)^{-1}T(\alpha\mathbf{x}) = \rho^{-1}T\mathbf{x}$ we see that $\alpha\mathbf{x}$ also belongs to $\lambda^{\exists}(\Phi, T, \mathbf{X})$.

Now, if $\mathbf{x}, \mathbf{y} \in \Lambda^{\exists}(\Phi, T, \mathbf{X})$, then there exist positive numbers ρ, σ such that $\Phi(\rho^{-1}T\mathbf{x})$ and $\Phi(\sigma^{-1}T\mathbf{y})$ are in Λ. Because any ϕ_k satisfies (2), for $\theta = \max\{2\rho, 2\sigma\}$ and all $k, i \in \mathbb{N}$ we have that

$$\begin{aligned} \phi_k\left(|\theta^{-1}T_{ki}(\mathbf{x}+\mathbf{y})|_k\right) &\le \phi_k\left(|(2\rho)^{-1}T_{ki}(\mathbf{x})|_k + |(2\sigma)^{-1}T_k(\mathbf{y}|_k\right) \\ &\le \phi_k\left(|\rho^{-1}T_{ki}(\mathbf{x})|_k\right) + \phi_k\left(|\sigma^{-1}T_{ki}(\mathbf{y})|_k\right). \end{aligned}$$

Since Λ is solid, $\Phi(\theta^{-1}T(\mathbf{x}+\mathbf{y}))$ must be in Λ and, consequently, $\mathbf{x}+\mathbf{y} \in \Lambda^{\exists}(\Phi, T, \mathbf{X})$.

Finally, if (3) holds, then $|y_k|_k \le |x_k|_k$ $\quad(k \in \mathbb{N})$ implies

$$(4) \qquad \phi_k\left(|\rho^{-1}T_{ki}(\mathbf{y})|_k\right) \le \phi_k\left(|\rho^{-1}T_{ki}(\mathbf{x})|_k\right), \quad k, i \in \mathbb{N}$$

and the solidity of $\Lambda^{\exists}(\Phi, T, \mathbf{X})$ follows from the solidity of Λ. ■

The following two theorems about the topologization of GS spaces $\Lambda^{\exists}(\Phi, T, \mathbf{X})$ are proved similarly to [12, Theorem 2], by using some standard arguments of the theory of modular spaces (see [16, proof of Theorem 1.5]).

Theorem 2. *Let Φ be a sequence of Orlicz functions and let $T : s_T(\mathbf{X}) \to s^2(\mathbf{X})$ be a linear operator. If the solid DS space Λ is topologized by an absolutely monotone F-seminorm g, then*

$$\widehat{h}(\mathbf{x}) = \inf\left\{\rho > 0 : g\left(\Phi\left(\rho^{-1}T\mathbf{x}\right)\right) \le \rho\right\}$$

is an F-seminorm on $\Lambda^{\exists}(\Phi, T, \mathbf{X})$. *Moreover,* $\widehat{h}$ *is absolutely monotone if* (3) *holds.*

Proof. For any $\mathbf{x}$ from $\Lambda^{\exists}(\Phi, T, \mathbf{X})$ there exists a number $\sigma > 0$ such that $g(\Phi(\sigma T\mathbf{x})) < \infty$. Defining $\rho = \max\{\sigma^{-1}, g(\Phi(\sigma T\mathbf{x}))\}$, by $\rho^{-1} \leq \sigma$ we get

$$g(\Phi(\rho^{-1}T\mathbf{x})) \leq g(\Phi(\sigma T\mathbf{x})) \leq \rho.$$

Thus, the functional $\widehat{h}$ is determined on $\Lambda^{\exists}(\Phi, T, \mathbf{X})$.

It is clear that $\widehat{h}$ satisfies (N1). Futher, let $|\alpha| \leq 1$ and $\mathbf{x} \in \Lambda^{\exists}(\Phi, T, \mathbf{X})$ with $\Phi(\rho^{-1}T\mathbf{x})$ in Λ. Then $\Phi(\rho^{-1}T(\alpha\mathbf{x})) \in \Lambda$ and

$$g(\Phi(\rho^{-1}T(\alpha\mathbf{x}))) \leq g(|\alpha|\Phi(\rho^{-1}T\mathbf{x})) \leq g(\Phi(\rho^{-1}T\mathbf{x})) \tag{5}$$

which implies

$$\{\rho > 0 : g(\Phi(\rho^{-1}T\mathbf{x})) \leq \rho\} \subset \{\rho > 0 : g(\Phi(\rho^{-1}T(\alpha\mathbf{x}))) \leq \rho\}.$$

Consequently, (N3) holds for $\widehat{h}$.

To prove (N2), we arbitrarily fix $\mathbf{x}$, $\mathbf{y} \in \Lambda^{\exists}(\Phi, T, \mathbf{X})$ and $\varepsilon > 0$. If $s = \widehat{h}(\mathbf{x}) + \varepsilon$, $t = \widehat{h}(\mathbf{y}) + \varepsilon$, then

$$g(\Phi(s^{-1}T\mathbf{x})) \leq s, \qquad g(\Phi(t^{-1}T\mathbf{y})) \leq t,$$

and so,

$$\begin{aligned} g\left(\Phi\left(\frac{T(\mathbf{x}+\mathbf{y})}{s+t}\right)\right) &\leq g\left(\Phi\left(\frac{s}{s+t}\,\frac{T\mathbf{x}}{s} + \frac{t}{s+t}\,\frac{T\mathbf{y}}{t}\right)\right) \\ &\leq g(\Phi(s^{-1}T\mathbf{x})) + g(\Phi(t^{-1}T\mathbf{y})) \leq s + t. \end{aligned}$$

Hence $\widehat{h}(\mathbf{x}+\mathbf{y}) \leq \widehat{h}(\mathbf{x}) + \widehat{h}(\mathbf{y}) + 2\varepsilon$, and we obtain

$$\widehat{h}(\mathbf{x}+\mathbf{y}) \leq \widehat{h}(\mathbf{x}) + \widehat{h}(\mathbf{y}).$$

Now we prove that $\widehat{h}$ satisfies (N4). Let $\lim_n \alpha_n = 0$ and $\mathbf{x} \in \Lambda^{\exists}(\Phi, T, \mathbf{X})$ with $\Phi(\rho^{-1}T\mathbf{x}) \in \Lambda$. Fix $\varepsilon > 0$, we can choose an index n_0 such that $\varepsilon^{-1}|\alpha_n| \leq \min\{1, \delta^{-1}\}$ for $n \geq n_0$. Then, for all $n \geq n_0$,

$$g(\Phi(\varepsilon^{-1}T(\alpha_n\mathbf{x}))) \leq g(\Phi(\varepsilon^{-1}|\alpha_n|T\mathbf{x})) \leq g(\Phi(\delta^{-1}T\mathbf{x})) < \infty$$

and, in view of

$$g(\Phi(\varepsilon^{-1}T(\alpha_n\mathbf{x}))) \leq g(\varepsilon^{-1}|\alpha_n|\Phi(T\mathbf{x})),$$

we get

$$\lim_n g(\Phi(\varepsilon^{-1}T(\alpha_n\mathbf{x}))) = 0.$$

Thus

$$g(\Phi(\varepsilon^{-1}T(\alpha_n\mathbf{x}))) \leq \varepsilon$$

for sufficiently large n. Hence $\lim_n \widehat{h}(\alpha_n\mathbf{x}) = 0$.

Finally, suppose that (3) holds and $\dot{|}y_k\dot{|}_k \leq \dot{|}x_k\dot{|}_k$ $(k \in \mathbb{N})$. Then (4) is true and, because g is absolutely monotone, we get

$$g(\Phi(\rho^{-1}T\mathbf{y})) \leq g(\Phi(\rho^{-1}T\mathbf{x})).$$

Consequently,

$$\{\rho > 0 : g(\Phi(\rho^{-1}T\mathbf{x})) \leq 1\} \subset \{\rho > 0 : g(\Phi(\rho^{-1}T\mathbf{y})) \leq 1\}$$

which shows that $\widehat{h}(\mathbf{y}) \leq \widehat{h}(\mathbf{x})$. ■

For fixed $m, n \in \mathbb{N}$ let $\mathbf{e}_{mn} = (e_{ki})$ be the double sequence, where $e_{ki} = 1$ if $k = m$, $i = n$, and $e_{ki} = 0$ otherwise.

Theorem 3. *Let T and Φ be the same as in Theorem 2. If the solid DS space Λ is topologized by an absolutely monotone seminorm g, then*

$$h(\mathbf{x}) = \inf\left\{\rho > 0 : g\left(\Phi\left(\rho^{-1}T\mathbf{x}\right)\right) \leq 1\right\}$$

is a seminorm on $\Lambda^{\exists}(\Phi, T, \mathbf{X})$. The seminorm h is absolutely monotone if (3) *holds. In particuler, if*

$$T\mathbf{x} = 0 \implies \mathbf{x} = 0, \tag{6}$$

then h is a norm on $\Lambda^{\exists}(\Phi, T, \mathbf{X})$.

Proof. For any $\mathbf{x}$ from $\Lambda^{\exists}(\Phi, T, \mathbf{X})$ there exists a number $\sigma > 0$ such that $g(\Phi(\sigma^{-1}T\mathbf{x})) < \infty$. Defining $d = \max\{1, g(\Phi(\sigma^{-1}T\mathbf{x}))\}$ and $\rho = \sigma d$, by $0 < 1/d \leq 1$ we have

$$g(\Phi(\rho^{-1}T\mathbf{x})) \leq \frac{1}{d}g(\Phi(\sigma^{-1}T\mathbf{x})) \leq 1.$$

This shows that the functional h is determined on $\Lambda^{\exists}(\Phi, T, \mathbf{X})$. It is obvious that h satisfies (N1).

To prove (N2), we fix $\mathbf{x}, \mathbf{y} \in \Lambda^{\exists}(\Phi, T, \mathbf{X})$ and $\varepsilon > 0$. Then, as in the proof of Theorem 2, denoting $s = h(\mathbf{x}) + \varepsilon$, $t = h(\mathbf{y}) + \varepsilon$, by

$$g(\Phi(s^{-1}T\mathbf{x})) \leq 1, \qquad g(\Phi(t^{-1}T\mathbf{y})) \leq 1$$

we get

$$g\left(\Phi\left(\frac{T(\mathbf{x}+\mathbf{y})}{s+t}\right)\right) \leq \frac{s}{s+t}g\left(\Phi\left(\frac{T\mathbf{x}}{s}\right)\right) + \frac{t}{s+t}g\left(\Phi\left(\frac{T\mathbf{y}}{t}\right)\right) \leq 1$$

which gives

$$h(\mathbf{x}+\mathbf{y}) \le h(\mathbf{x}) + h(\mathbf{y}).$$

Now, if we take $\alpha \neq 0$, then

$$\begin{aligned} h(\alpha\mathbf{x}) &= \inf\left\{\rho > 0 : g\left(\Phi\left(\frac{T(\alpha\mathbf{x})}{\rho}\right)\right) \le 1\right\} \\ &= |\alpha| \inf\left\{\frac{\rho}{|\alpha|} > 0 : g\left(\Phi\left(\frac{T\mathbf{x}}{\rho/|\alpha|}\right)\right) \le 1\right\} = |\alpha|\, h(\mathbf{x}). \end{aligned}$$

Hence h satisfies also (N8), i.e., h is a seminorm on $\Lambda^{\exists}(\Phi, T, \mathbf{X})$.

It remains to prove that h is a norm on $\Lambda^{\exists}(\Phi, T, \mathbf{X})$ if (6) holds. To prove the axiom (N5) for h, let $h(\mathbf{x}) = 0$. Then

$$g\left(\Phi\left(\rho^{-1}T\mathbf{x}\right)\right) \le 1 \quad (\rho > 0). \tag{7}$$

If we suppose $\mathbf{x} \neq 0$, then also $T\mathbf{x} \neq 0$ by (6), and there exist indices m, n with $T_{mn}(\mathbf{x}) \neq 0$. But then also $c_\rho = \phi_m\left(\rho^{-1}\dot{|}T_{mn}(\mathbf{x})\dot{|}_m\right) \neq 0$. Since the elements of double sequences $\mathbf{u} = \Phi\left(\rho^{-1}T\mathbf{x}\right)$ and $\mathbf{v} = c_\rho \mathbf{e}_{mn}$ are connected with $|v_{ki}| \le |u_{ki}|$, the sequence $c_\rho \mathbf{e}_{mn}$ is in Λ by the solidity of Λ. Moreover, since g is absolutely monotone, we have $g(\mathbf{u}) \ge g(\mathbf{v})$ which gives

$$g\left(\Phi\left(\rho^{-1}T\mathbf{x}\right)\right) \ge \phi_m\left(\rho^{-1}\dot{|}T_{mn}(\mathbf{x})\dot{|}_m\right) g(\mathbf{e}_{mn}).$$

Therefore, using also the fact that the Orlicz function ϕ_m is unbounded, for sufficiently small ρ we get

$$g(\Phi(\rho^{-1}T\mathbf{x})) > 1$$

contrary to (7). Consequently, it must be $\mathbf{x} = 0$. ■

3. Applications related to summability

In the following we apply Theorems 1−3 in the special case, where Λ is the strong summability domain of a non-negative infinity matrix, and the operator T is determined by means of a sequence of summability matrices.

The most common summability method is the matrix method defined by an infinite scalar matrix $A = (a_{nk})$. If for a sequence $\mathbf{x} \in s(X)$ the series $A_n\mathbf{x} = \sum_k a_{nk}x_k$ converge and the limit $\lim_n A_n\mathbf{x} = l$ exists in X, then we say that $\mathbf{x}$ *is A-summable* to l and write A-$\lim x_k = l$. A summability method (or a matrix) A is called *regular in X* if for all convergent in X sequences $\mathbf{x} = (x_k)$ we have

$$\lim_k x_k = l \implies \lim_n A_n\mathbf{x} = l.$$

A well-known example of a regular matrix method is the Cesàro method C_1 defined by the matrix $C_1 = (c_{nk})$, where, for any $n \in \mathbb{N}$, $c_{nk} = n^{-1}$ if $k \leq n$ and $c_{nk} = 0$ otherwise. A (trivial) regular method is defined by the *unit matrix* $I = (i_{nk})$, where $i_{nn} = 1$ and $i_{nk} = 0$ for $n \neq k$. Recall also that a matrix $A = (a_{nk})$ is called *normal* if, for any $n \in \mathbb{N}$, $a_{nn} \neq 0$ and $a_{nk} = 0$ if $k > n$. For example, Cesàro matrix C_1 is normal. Every scalar sequence (c_k) defines a *diagonal matrix* $D(c_k) = (d_{ni})$ by the equalities $d_{nn} = c_n$ and $d_{ni} = 0$ if $n \neq i$. Clearly, a diagonal matrix $D(c_k)$ is regular if and only if $\lim_k c_k = 1$, and it is normal if $c_k \neq 0$ for all $k \in \mathbb{N}$. More information about the matrix summability may be found, for example, in [2].

An another class of summability methods is determined by sequences $\mathcal{B} = (B^i)$ of infinite scalar matrices $B^i = (b^i_{nk})$. Recall that a sequence $\mathbf{x} = (x_k) \in s(X)$ is called $\mathcal{B}$*-summable* to the point $l \in X$ if $B^i\text{-}\lim x_k = l$ uniformly in i, i.e., the series $B^i_n\mathbf{x} = \sum_k b^i_{nk}x_k$ converge in X and

$$\lim_n \left| B^i_n\mathbf{x} - l \right| = 0 \text{ uniformly in } i.$$

The summability method $\mathcal{B}$ is also known as the sequential matrix method (SM method) of summability (see [2], p. 19). In the special case

$$b^i_{nk} = \begin{cases} \frac{1}{n}, & \text{if } \ i \leq k \leq n+i-1, \\ 0, & \text{otherwise} \end{cases}$$

the $\mathcal{B}$-summability reduces to so-called *almost convergence*. Almost convergence is a non-matrix method of summability. Any matrix method B we can consider as the SM method $\mathcal{B}$ with $B^i = B$. By the unit SM method $\mathcal{I}$ we mean the SM method $\mathcal{B}$ with $B^i = I$.

Now, let $A = (a_{nk})$ be a non-negative matrix, i.e., $a_{nk} \geq 0$. We say that A is *column-positive* if for any $k \in \mathbb{N}$ there exists an index n_k such that $a_{n_k,k} > 0$. A sequence $\mathbf{u} = (u_k) \in s$ is called *strongly* A*-summable* to l if $\lim_n \sum_k a_{nk}|u_k - l| = 0$, and *strongly* A*-bounded* if $\sup_n \sum_k a_{nk}|u_k| < \infty$. It is clear that the set $c_0[A]$ of all strongly A-summable to zero sequences and the set $\ell_\infty[A]$ of all strongly A-bounded sequences are linear spaces. Moreover, the functional

$$g_A(\mathbf{u}) = \sup_n \sum_k a_{nk}|u_k|$$

is a seminorm on $\ell_\infty[A]$ and $c_0[A]$, it is a norm if A is column-positive.

More generally, if $\mathcal{M}$ is a solid DS space, then the set

$$\mathcal{M}[A] = \left\{ \mathbf{u}^2 = (u_{ki}) \in s^2 : A|\mathbf{u}^2| \in \mathcal{M} \right\},$$

where

$$A|\mathbf{u}^2| = \left(A_n|\mathbf{u}^2|\right) = \left(\sum_k a_{nk}|u_{ki}|\right),$$

is also a solid DS space. Moreover, if $\mathcal{M}$ is topologized by an absolutely monotone F-seminorm (seminorm) $g_{\mathcal{M}}$, then on $\mathcal{M}[A]$ we may define an absolutely monotone F-seminorm (seminorm) by the equality (cf. [13, pp. 188–189])

$$g_{\mathcal{M},A}(\mathbf{u}^2) = g_{\mathcal{M}}\left(A|\mathbf{u}^2|\right).$$

At it, $g_{\mathcal{M},A}$ is an F-norm (norm) on $\mathcal{M}[A]$ if $g_{\mathcal{M}}$ is an F-norm (norm) on $\mathcal{M}$ and A is column-positive.

In the following let $\Psi = (\psi_k)$ be a sequence of modulus functions and $\Phi = (\phi_k)$ be a sequence of Orlicz functions. As a GS space of Orlicz type, connected with summability, we consider the set

$$\mathcal{M}^{\exists}[A, \Psi\Phi, \mathcal{B}, X] = \{\mathbf{x} \in s(X) : (\exists\rho > 0)\ A|\Psi\Phi(\rho^{-1}\mathcal{B}\mathbf{x})| \in \mathcal{M}\},$$

where $\mathcal{B}\mathbf{x} = (B_n^i\mathbf{x})$. Let us denote by

$$s_{\mathcal{B}}(X) = \left\{\mathbf{x} \in s(X) : \sum_k b_{nk}^i x_k \quad (n, i \in \mathbb{N}) \text{ converge in } X\right\}$$

the *application domain* of the SM method $\mathcal{B}$. Then the operator $\mathcal{B} : s_{\mathcal{B}}(X) \to s^2(X)$, $\mathcal{B}\mathbf{x} = (B_n^i\mathbf{x})$, is linear. Consequently, since

$$\mathcal{M}^{\exists}[A, \Psi\Phi, \mathcal{B}, X] = \mathcal{M}[A]^{\exists}\left(\Psi\Phi, \mathcal{B}, X\right), \tag{8}$$

and $\Psi\Phi = (\psi_k\phi_k)$ is a sequence of φ-functions, from Theorem 1 (with $\Lambda = \mathcal{M}(A)$ and $T = \mathcal{B}$) we immediately get the following proposition.

Proposition 1. *If $\mathcal{M} \subset s^2$ is a solid DS space, A is a non-negative matrix, and $\mathcal{B}$ is a SM method of summability, then $\mathcal{M}^{\exists}[A, \Psi\Phi, \mathcal{B}, X]$ is a GS space.*

For the topologization of GS spaces $\mathcal{M}^{\exists}[A, \Psi\Phi, \mathcal{B}, X]$ we replace (8) with the representation

$$\mathcal{M}^{\exists}[A, \Psi\Phi, \mathcal{B}, X] = \mathcal{M}[A, \Psi]^{\exists}\left(\Phi, \mathcal{B}, X\right), \tag{9}$$

where

$$\mathcal{M}[A, \Psi] = \left\{\mathbf{u}^2 = (u_{ki}) \in s^2 : A|\Psi\left(\mathbf{u}^2\right)| \in \mathcal{M}\right\}.$$

Proposition 2. *Let $\mathcal{M}$ be a solid DS space which is topologized by an absolutely monotone F-seminorm $g_{\mathcal{M}}$.*

(*i*) *If the sequence of moduli $\Psi = (\psi_k)$ satisfies one of* (*equivalent*) *conditions*

(M1) There exist a function ν and a number $\delta > 0$ such that $\psi_k(ut) \le \nu(u)\psi_k(t)$ $(0 \le u < \delta,\ t \ge 0)$ and $\lim_{u\to 0+} \nu(u) = 0$,

(M2) $\lim\limits_{u\to 0+} \sup\limits_{t>0} \sup\limits_{k} \dfrac{\psi_k(ut)}{\psi_k(t)} = 0$,

then the GS space $\mathcal{M}^{\exists}[A, \Psi\Phi, \mathcal{B}, X]$ may be topologized by the F-seminorm

$$\widehat{h}_{\mathcal{M},A,\mathcal{B}}(\mathbf{x}) = \inf\left\{\rho > 0 : g_{\mathcal{M}}\left(A\left|\Psi\Phi\left(\rho^{-1}\mathcal{B}\mathbf{x}\right)\right|\right) \le \rho\right\}.$$

The F-seminorm $\widehat{h}_{\mathcal{M},A,\mathcal{B}}$ is absolutely monotone if the operator $\mathcal{B}$ satisfies (3)*.*

(ii) If $(\mathcal{M}[A], g_{\mathcal{M},A})$ is double AK-space, then $\widehat{h}_{\mathcal{M},A,\mathcal{B}}$ is a F-seminorm on GS space $\mathcal{M}^{\exists}[A, \Psi\Phi, \mathcal{B}, X] \cap Us(\Phi, \mathcal{B}, X)$ for an arbitrary sequence of moduli Ψ.

(iii) If $g_{\mathcal{M}}$ is a seminorm on $\mathcal{M}$, then the GS space $\mathcal{M}^{\exists}[A, \Phi, \mathcal{B}, X]$ may be topologized by the seminorm

$$h_{\mathcal{M},A,\mathcal{B}}(\mathbf{x}) = \inf\left\{\rho > 0 : g_{\mathcal{M}}\left(A\left|\Phi\left(\rho^{-1}\mathcal{B}\mathbf{x}\right)\right|\right) \le 1\right\}.$$

The seminorm $h_{\mathcal{M},A,\mathcal{B}}$ is total (i.e., a norm) if $\mathcal{B}$ satisfies (6)*.*

Proof. (i) The DS space $\mathcal{M}[A, \Psi]$ from (9) may be written in the form

$$\mathcal{M}[A](\Psi) = \{\mathbf{u}^2 \in s^2 : \Psi(\mathbf{u}^2) \in \mathcal{M}[A]\}.$$

Applying [14, Theorem 1 and Remark 1] with $\Lambda = \mathcal{M}[A]$, $\mathbf{\Phi} = \Psi^{(2)}$, $T = \imath$ and $\mathbf{X}^2 = \mathbf{Y}^2 = \mathbb{K}^2$, we get that $\mathcal{M}[A, \Psi]$ is topologized by the absolutely monotone F-seminorm

$$g_{\mathcal{M},A,\Psi}(\mathbf{u}^2) = g_{\mathcal{M},A}(\Psi(\mathbf{u}^2)) = g_{\mathcal{M}}(A|\Psi(\mathbf{u}^2)|)$$

whenever Ψ satisfies one of conditions (M1) and (M2). Therefore, (i) follows by Theorem 2 because of the representation (9).

(ii) If $\mathbf{x} \in \mathcal{M}^{\exists}[A, \Psi\Phi, \mathcal{B}, X] \cap Us(\Phi, \mathcal{B}, X)$, then by the solidity of $\mathcal{M}$ and Us there exists a number $\rho \ge 1$ such that $\Phi(\rho^{-1}\mathcal{B}\mathbf{x}) \in \mathcal{M}[A, \Psi] \cap Us$. This shows that

$$\mathcal{M}^{\exists}[A, \Psi\Phi, \mathcal{B}, X] \cap Us(\Phi, \mathcal{B}, X) \subset \left(\mathcal{M}[A, \Psi] \cap Us\right)^{\exists}(\Phi, \mathcal{B}, X)$$

and since $\mathcal{M}[A, \Psi] \cap Us$ may be topologized by the absolutely monotone F-seminorm $g_{\mathcal{M},A,\Psi}$ in view of [14, Theorem 2], our statement follows again by Theorem 2.

(iii) We know that if $g_{\mathcal{M}}$ is a seminorm on $\mathcal{M}$, then $\mathcal{M}[A]$ is topologized by the absolutely monotone seminorm $g_{\mathcal{M},A}$. Therefore, using (9) (with $\psi_k = \imath$), the required statements follow immediately from Theorem 3. ∎

Recall that an infinite matrix $A = (a_{nk})$ is said to be *row-finite* if for any $n \in \mathbb{N}$ there exists an index k_n with $a_{nk} = 0$ for all $k > k_n$. The following modification of Proposition 2 is related to the case $\mathcal{M} = U\lambda$.

Proposition 3. *Let λ be a solid sequence space which is topologized by an absolutely monotone F-seminorm g_λ.*

(i) Suppose that the matrix A is row-finite and column-positive, and the moduli ψ_k are unbounded. If (λ, g_λ) is AK-space and

$$(a_{nk})_{n\in\mathbb{N}} \in \lambda \quad (k \in \mathbb{N}), \tag{10}$$

then the GS space $U\lambda^{\exists}[A, \Psi\Phi, \mathcal{B}, X]$ may be topologized by the F-seminorm

$$\widehat{h}_{\lambda,\widetilde{A},\mathcal{B}}(\mathbf{x}) = \inf\left\{\rho > 0 : g_\lambda\left(\widetilde{A}\left|\Psi\Phi\left(\rho^{-1}\mathcal{B}\mathbf{x}\right)\right|\right) \le \rho\right\},$$

where

$$\widetilde{A}\left|\mathbf{u}^2\right| = \left(\sup_i \sum_k a_{nk}|u_{ki}|\right).$$

The F-seminorm $\widehat{h}_{\lambda,\widetilde{A},\mathcal{B}}$ is absolutely monotone if the operator $\mathcal{B}$ satisfies (3)*.*

(ii) If g_λ is a seminorm on λ, then the GS space $U\lambda^{\exists}[A, \Phi, \mathcal{B}, X]$ may be topologized by the seminorm

$$h_{\lambda,\widetilde{A},\mathcal{B}}(\mathbf{x}) = \inf\left\{\rho > 0 : g_\lambda\left(\widetilde{A}\left|\Phi\left(\rho^{-1}\mathcal{B}\mathbf{x}\right)\right|\right) \le 1\right\}.$$

The seminorm $h_{\lambda,\widetilde{A},\mathcal{B}}$ is total (*i.e., a norm*) *if $\mathcal{B}$ satisfies* (6)*.*

Proof. (*i*) It follows from [13, Proposition 2 *b*)] that, by our suppositions, the DS space $U\lambda[A, \Psi]$ may be topologized by the absolutely monotone F-seminorm

$$g_{\lambda,\widetilde{A},\Psi}(\mathbf{u}^2) = g_\lambda\left(\widetilde{A}\left|\Psi\left(\mathbf{u}^2\right)\right|\right).$$

In view of (9) (with $\mathcal{M} = U\lambda$) it remains to apply Theorem 2.

(*ii*) is a special case of Proposition 2 (*iii*), because the equality

$$g_{U\lambda}(\mathbf{u}^2) = g_\lambda(\widetilde{\mathbf{u}})$$

defines an absolutely monotone seminorm on $U\lambda$. ∎

4. Some special cases

Let $\Phi = (\phi_k)$ be a sequence of Orlicz functions, $\mathbf{p} = (p_k)$ be a sequence of positive numbers, and let $\imath^{\mathbf{p}}$ be the sequence of φ-functions $\imath^{p_k}(t) = t^{p_k}$. The authors of [1, 5, 6, 7, 8, 9, 17, 19, 22] consider, for various concrete non-negative summability methods A and SM methods $\mathcal{B}$, the GS spaces

$$\Lambda^{\exists}\left[A, \imath^{\mathbf{p}}\Phi, \mathcal{B}, X\right] = \left\{\mathbf{x} \in s(X) : (\exists\rho > 0)\, A|\imath^{\mathbf{p}}\Phi(\rho^{-1}\mathcal{B}\mathbf{x})| \in \Lambda\right\},$$

where $\Lambda \in \{uc_0, U\ell_\infty\}$ and $A|\imath^{\mathbf{p}}\Phi(\mathcal{B}\mathbf{x})| = \left(A_n^i|\imath^{\mathbf{p}}\Phi(\mathcal{B}\mathbf{x})|\right)$ with

$$A_n^i|\imath^{\mathbf{p}}\Phi(\mathcal{B}\mathbf{x})| = \sum_k a_{nk}\left(\phi_k\left(\Big|\sum_j b_{kj}^i x_k\Big|\right)\right)^{p_k}.$$

For bounded sequences $\mathbf{p}$, they determine on these spaces the paranorms of type

$$h_0(\mathbf{x}) = \inf_{\rho>0,\, m\in\mathbb{N}}\left\{\rho^{p_m/r} : \sup_{n,\,i}\left(A_n^i|\imath^{\mathbf{p}}\Phi(\rho^{-1}\mathcal{B}\mathbf{x})|\right)^{1/r} \le 1\right\}, \tag{11}$$

where $r = \max\{1, \sup_k p_k\}$.

Since $t^{1/r} \le 1$ if and only if $t \le 1$, the functional h_0 my be replaced with

$$h_0^*(\mathbf{x}) = \inf_{\rho>0,\, m\in\mathbb{N}}\left\{\rho^{p_m/r} : \sup_{n,\,i} A_n^i|\imath^{\mathbf{p}}\Phi(\rho^{-1}\mathcal{B}\mathbf{x})| \le 1\right\}. \tag{12}$$

We remark that the paranorm of type h_0^* is used in [10].

It is clear, by (11) and (12), that the functionals h_0 and h_0^* are determined on $uc_0{}^{\exists}(A, \imath^{\mathbf{p}}\Phi, \mathcal{B}, X)$ only if

$$uc_0{}^{\exists}\left[A, \imath^{\mathbf{p}}\Phi, \mathcal{B}, X\right] \subset U\ell_\infty{}^{\exists}\left[A, \imath^{\mathbf{p}}\Phi, \mathcal{B}, X\right] \tag{13}$$

or, equivalently,

$$uc_0{}^{\exists}\left[A, \imath^{\mathbf{p}}\Phi, \mathcal{B}, X\right] = Uc_0{}^{\exists}\left[A, \imath^{\mathbf{p}}\Phi, \mathcal{B}, X\right]. \tag{14}$$

The following example shows that (13) is not automatically fulfilled.

Example 1. Let $A = I$, $\phi_k = \imath$, $p_k = 1$, and let $\mathcal{B}$ be the sequence of matrices $B^i = (b_{nk}^i)$ with the elements

$$b_{nk}^i = \begin{cases} i, & \text{if } n = 1,\ k = i, \\ 1/n, & \text{if } n \ge 2,\ k = i, \\ 0, & \text{otherwise.} \end{cases}$$

Since

$$B_n^i\mathbf{u} = \sum_k b_{nk}^i u_k = \begin{cases} iu_i, & \text{if } n = 1, \\ n^{-1}u_i, & \text{if } n \ge 2, \end{cases}$$

for any $\rho > 0$ and every sequence $\mathbf{u} \in s$ with $0 < \inf_k |u_k| \le \sup_k |u_k| < \infty$ we have that $\lim_n |\rho^{-1}B_n^i\mathbf{u}| = 0$ uniformly in i, but $\sup_i |\rho^{-1}B_n^i\mathbf{u}| = \sup_i \rho^{-1}i|u_i| = \infty$. Thus, the inclusion (13) and also the equality (14) hold not in our case.

It should be remarked that the validity of (13) is unnoticed in [1, 6, 7, 8, 9, 17, 22] by the definition of paranorms h_0 and h_0^* on some spaces of type ${uc_0}^{\exists}[A, \imath^{\mathbf{p}}\Phi, \mathcal{B}, X]$. Moreover, the proofs of (13) presented in [3, 5, 10, 19, 20, 21] for various special cases are, unfortunately, in like manner inconclusive ($\lim_k u_{ki} = 0$ uniformly in i implies not $\sup_{k,i} |u_{ki}| < \infty$ in general). This leads us to the following question.

Problem 1. *On what conditions the inclusion* (13) *or, equivalently, the equality* (14) *hold?*

Our Propositions 2 and 3 allow to determine F-seminorm topologies for the GS spaces $\Lambda^{\exists}[A, \imath^{\mathbf{p}}\Phi, \mathcal{B}, X]$ and $U\lambda^{\exists}[A, \imath^{\mathbf{p}}\Phi, \mathcal{B}, X]$. Supposing that the sequence $\mathbf{p} = (p_k)$ is bounded and $r = \max\{1, \sup_k p_k\}$, we may write, similarly to (9),

$$\Lambda^{\exists}[A, \imath^{\mathbf{p}}\Phi, \mathcal{B}, X] = \Lambda\left[A, \imath^{\mathbf{p}/r}\right]^{\exists}(\Phi^r, \mathcal{B}, X),$$

where $\imath^{\mathbf{p}/r}$ is the sequence of moduli $\imath^{p_k/r}(t) = t^{p_k/r}$ and Φ^r is the sequence of Orlicz functions $\phi_k^r(t) = (\phi_k(t))^r$. Since $\imath^{\mathbf{p}/r}$ satisfies (M2) if and only if $\inf_k p_k > 0$, and $U\ell_\infty$ may be topologized by the absolutely monotone norm $\|\mathbf{u}^2\|_\infty^2 = \sup_{ki} |u_{ki}|$, from Proposition 2 we immediately get the following corollary.

Corollary 1. *Let Λ be a solid DS space which is topologized by an absolutely monotone F-seminorm g_{Λ} and let $\mathbf{p} = (p_k)$ be a bounded sequence of positive numbers.*

(a) If $\inf_k p_k > 0$, then the GS space $\Lambda^{\exists}[A, \imath^{\mathbf{p}}\Phi, \mathcal{B}, X]$ may be topologized by the F-seminorm

$$\widehat{h}^{\mathbf{p}}_{\Lambda,A,\mathcal{B}}(\mathbf{x}) = \inf\left\{\rho > 0 : g_{\Lambda}\left(A\left|\imath^{\mathbf{p}}\Phi\left(\rho^{-1}\mathcal{B}\mathbf{x}\right)\right|\right) \le \rho\right\}.$$

In particular, on the GS space ${U\ell_\infty}^{\exists}[A, \imath^{\mathbf{p}}\Phi, \mathcal{B}, X]$ we may define the F-seminorm

$$\begin{aligned}\widehat{h}^{\mathbf{p}}_{\infty,A,\mathcal{B}}(\mathbf{x}) &= \inf\left\{\rho > 0 : \left\|A\left|\imath^{\mathbf{p}}\Phi\left(\rho^{-1}\mathcal{B}\mathbf{x}\right)\right|\right\|_\infty^2 \le \rho\right\} \\ &= \inf\left\{\rho > 0 : \sup_{n,i}\sum_k a_{nk}\left(\phi_k\left(\left|\rho^{-1}\sum_j b^i_{kj}x_k\right|\right)\right)^{p_k} \le \rho\right\}.\end{aligned}$$

(b) If $\left(\Lambda[A], g_{\Lambda,A}\right)$ is a double AK-space, then $\widehat{h}^{\mathbf{p}}_{\Lambda,A,\mathcal{B}}$ is a F-seminorm on the GS space $\Lambda^{\exists}[A, \imath^{\mathbf{p}}\Phi, \mathcal{B}, X] \cap Us(\Phi, \mathcal{B}, X)$.

It is known that $(c_0, \|\cdot\|_\infty)$ and $(\ell_q, \|\cdot\|_q)$ $(1 \le q < \infty)$ are normed AK-spaces with the norms $\|\mathbf{u}\|_\infty = \sup_k |u_k|$ and $\|\mathbf{u}\|_q = \left(\sum_k |u_k|^q\right)^{1/q}$, respectively. Therefore, by Proposition 3 we can formulate our next corollary.

Corollary 2. *Let λ be a solid sequence space which is topologized by an absolutely monotone F-seminorm g_λ. Suppose that the matrix A is row-finite and column-positive. If (λ, g_λ) is the AK-space and (11) holds, then on the GS space $U\lambda^{\exists}[A, \imath^{\mathbf{p}}\Phi, \mathcal{B}, X]$ we may define the F-seminorm*

$$\widehat{h}^{\mathbf{p}}_{\lambda,\widetilde{A},\mathcal{B}}(\mathbf{x}) = \inf\left\{\rho > 0 : g_\lambda\left(\widetilde{A}\left|\imath^{\mathbf{p}}\Phi\left(\rho^{-1}\mathcal{B}\mathbf{x}\right)\right|\right) \le \rho\right\}.$$

In particular, $Uc_0^{\exists}[A, \imath^{\mathbf{p}}\Phi, \mathcal{B}, X]$ may be topologized by the F-seminorm $\widehat{h}^{\mathbf{p}}_{\infty,A,\mathcal{B}}$ whenever $\lim_n a_{nk} = 0$ $(k \in \mathbb{N})$, and on $U\ell_q^{\exists}[A, \imath^{\mathbf{p}}\Phi, \mathcal{B}, X]$ $(1 \le q < \infty)$ we may determine the F-seminorm

$$\widehat{h}^{\mathbf{p}}_{q,\widetilde{A},\mathcal{B}}(\mathbf{x}) = \inf\left\{\rho > 0 : \left\|\widetilde{A}\left|\imath^{\mathbf{p}}\Phi\left(\rho^{-1}\mathcal{B}\mathbf{x}\right)\right|\right\|_q \le \rho\right\}$$

$$= \inf\left\{\rho > 0 : \left(\sum_n \left|\sup_i \sum_k a_{nk}\left(\phi_k\left(\left|\rho^{-1}\sum_j b^i_{kj}x_k\right|\right)\right)^{p_k}\right|^q\right)^{1/q} \le \rho\right\}$$

provided that $\left(\sum_n |a_{nk}|^q\right)^{1/q} < \infty$ for any $k \in \mathbb{N}$.

Remark 1. Since any F-seminorm is a paranorm (see [12, Remark 1]), all F-seminorms defined above are also paranorms satisfying the axiom (N3).

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