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**ON THE APPROXIMATION OF FUNCTIONS
FROM L^p BY SOME SPECIAL MATRIX
MEANS OF FOURIER SERIES**

ABSTRACT. The results corresponding to some theorems of S. Lal [Appl. Math. and Comput. 209 (2009), 346-350] and the results of W. Lenski and B. Szal [Banach Center Publ., 95, (2011), 339-351] are shown. The better degrees of pointwise approximation than these in mentioned papers by another assumptions on summability methods for considered functions are obtained. From presented pointwise results the estimation on norm approximation are derived. Some special cases as corollaries are also formulated.

KEY WORDS: degree of approximation, Fourier series, matrix means.

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1. Introduction

Let L^p ($1 \leq p < \infty$) [resp. L^∞] be the class of all 2π -periodic real-valued functions integrable in the Lebesgue sense with p -th power [essentially bounded] over $Q = [-\pi, \pi]$ with the norm

$$\|f\| := \|f(\cdot)\|_{L^p} = \begin{cases} \left(\frac{1}{2\pi} \int_Q |f(t)|^p dt \right)^{1/p}, & \text{when } 1 \leq p < \infty, \\ \operatorname{ess\,sup}_{t \in Q} |f(t)|, & \text{when } p = \infty \end{cases}$$

and consider the trigonometric Fourier series

$$Sf(x) := \frac{a_0(f)}{2} + \sum_{\nu=1}^{\infty} (a_\nu(f) \cos \nu x + b_\nu(f) \sin \nu x)$$

with the partial sums $S_k f$ [4, Th. (3.1)IV].

Let $A := (a_{n,k})$ and $B := (b_{n,k})$ be infinite lower triangular matrices of real numbers such that

$$(1) \quad \begin{aligned} a_{n,k} &\geq 0 \quad \text{and} \quad b_{n,k} \geq 0 \quad \text{when} \quad k = 0, 1, 2, \dots, n, \\ a_{n,k} &= 0 \quad \text{and} \quad b_{n,k} = 0 \quad \text{when} \quad k > n, \end{aligned}$$

$$(2) \quad \sum_{k=0}^n a_{n,k} = 1 \quad \text{and} \quad \sum_{k=0}^n b_{n,k} = 1, \quad \text{where} \quad n = 0, 1, 2, \dots$$

Let the AB -transformation of $(S_k f)$ be given by

$$T_{n,A,B}f(x) := \sum_{r=0}^n \sum_{k=0}^r a_{n,n-r} b_{r,k} S_k f(x) \quad (n = 0, 1, 2, \dots).$$

As a measure of approximation of function f by $T_{n,A,B}f$ in the space L^p we will use the pointwise moduli of continuity of f defined, for $\beta \geq 0$, by the formulas

$$\begin{aligned} \omega_x f(\delta)_{p,\beta} &= \begin{cases} \left\{ \frac{1}{\delta} \int_0^\delta |\varphi_x(u) \sin^\beta \frac{u}{2}|^p du \right\}^{\frac{1}{p}}, & \text{when } 1 \leq p < \infty, \\ \operatorname{ess\,sup}_{0 < u \leq \delta} |\varphi_x(u) \sin^\beta \frac{u}{2}|, & \text{when } p = \infty, \end{cases} \\ \omega_{L^p} f(\delta)_\beta &= \sup_{0 < t \leq \delta} \|\varphi_x(t) \sin^\beta \frac{t}{2}\|_{L^p}, \end{aligned}$$

where

$$\varphi_x(t) := f(x+t) + f(x-t) - 2f(x).$$

The deviation $T_{n,A,B}f - f$ with the lower triangular infinite matrix A , defined by $a_{n,r} = \frac{1}{n+1}$ when $r = 0, 1, 2, \dots, n$ and $a_{n,r} = 0$ when $r > n$, and with the lower triangular infinite matrix B , defined by $b_{r,k} = p_{r-k} / \sum_{\nu=0}^r p_\nu$ when $k = 0, 1, 2, \dots, r$ and $b_{r,k} = 0$ when $k > r$, was estimated by S. Lal [1, Theorem 2]. The similar deviation $T_{n,A,B}f - f$, but in general form, was estimated at the point as well as in the norm of L^p in [3]. The pointwise estimation from this paper is following

Theorem. *Let $f \in L^p$ ($1 < p \leq \infty$) and let ω satisfy*

$$\begin{aligned} \left\{ \int_{\frac{\pi}{n+1}}^{\frac{\pi}{n}} \left(\frac{|\varphi_x(t)|}{\omega(t)} \right)^p \sin^{\beta p} \frac{t}{2} dt \right\}^{\frac{1}{p}} &= O_x \left((n+1)^{-\frac{2}{p}} \right), \quad \text{when } 1 < p < \infty, \\ \operatorname{ess\,sup}_{t \in [\frac{\pi}{n+1}, \frac{\pi}{n}]} \left| \frac{|\varphi_x(t)|}{\omega(t)} \sin^\beta \frac{t}{2} \right| &= O_x(1), \quad \text{when } p = \infty, \end{aligned}$$

and

$$\left\{ \int_0^{\frac{\pi}{n+1}} \left(\frac{|\varphi_x(t)|}{\omega(t)} \right)^p \sin^{\beta p} \frac{t}{2} dt \right\}^{\frac{1}{p}} = O_x \left((n+1)^{-\frac{1}{p}} \right), \text{ when } 1 < p < \infty,$$

$$\text{ess sup}_{t \in [0, \frac{\pi}{n+1}]} \left| \frac{|\varphi_x(t)|}{\omega(t)} \sin^{\beta} \frac{t}{2} \right| = O_x(1), \text{ when } p = \infty,$$

with $0 \leq \beta < 1 - \frac{1}{p}$.

If the entries of matrices $(a_{n,r})$ and $(b_{r,k})$ satisfy conditions

$$a_{n,n} \ll \frac{1}{n+1}$$

and

$$|a_{n,r}b_{r,r-l} - a_{n,r+1}b_{r+1,r+1-l}| \ll \frac{a_{n,r}}{(r+1)^2} \text{ for } 0 \leq l \leq r \leq n-1,$$

then

$$\begin{aligned} |T_{n,A,B}f(x) - f(x)| &= O_x \left(\sum_{r=0}^n a_{n,r} \frac{1}{r+1} \sum_{s=0}^r (s+1)^{\beta} \omega \left(\frac{\pi}{s+1} \right) \right. \\ &\quad \left. + \frac{1}{n+1} \sum_{s=0}^n (s+1)^{\beta} \omega \left(\frac{\pi}{s+1} \right) \right), \end{aligned}$$

and, in the case $0 < \beta < 1 - \frac{1}{p}$

$$\begin{aligned} &|T_{n,A,B}f(x) - f(x)| \\ &= O_x \left((n+1)^{\beta} \omega \left(\frac{\pi}{n+1} \right) \left[(n+1)^{1-\beta} \sum_{s=0}^n a_{n,s} (s+1)^{\beta-1} \right] \right), \end{aligned}$$

for considered x .

In this paper we shall consider the pointwise deviation $T_{n,A,B}f - f$ under another assumptions on summability method obtaining better degree of approximation than the above. Finally, we also give a result on the norm approximation and some corollaries.

We shall write $K_1 \ll K_2$ if there exists a positive constant C , sometimes depending on some parameters, such that $K_1 \leq CK_2$.

2. Statement of the results

Now we formulate our main results.

Theorem 1. Let $f \in L^p$ ($1 \leq p \leq \infty$). If the entries of matrices $(a_{n,r})_{r=0}^n$ and $(b_{r,k})_{k=0}^n$ satisfy the conditions (1), (2) and

$$(3) \quad \sum_{r=0}^n a_{n,r} \sum_{k=0}^{n-r} |b_{n-r,k} - b_{n-r,k+1}| \ll \frac{1}{n+1},$$

then

$$\begin{aligned} & |T_{n,A,B}f(x) - f(x)| \\ &= O_x(n+1)^\beta \left\{ w_x f\left(\frac{\pi}{n+1}\right)_{p,\beta} + \frac{1}{n+1} \sum_{k=0}^n w_x f\left(\frac{\pi}{k+1}\right)_{1,\beta} \right\}, \end{aligned}$$

for almost all considered x and $0 \leq \beta < 1 - \frac{1}{p}$, when $p > 1$, and $\beta = 0$, when $p = 1$.

The result of estimate of L^p norm of the above deviation is following

Theorem 2. Let $f \in L^p$ ($1 \leq p \leq \infty$). If the entries of matrices $(a_{n,r})_{r=0}^n$ and $(b_{r,k})_{k=0}^n$ satisfy the conditions (1), (2) and (3), then

$$\|T_{n,A,B}f(\cdot) - f(\cdot)\|_{L^p} \leq (n+1)^{\beta - \frac{1}{p}} \sum_{k=0}^n \omega_{L^p} f\left(\frac{\pi}{k+1}\right)_\beta,$$

for $0 \leq \beta < 1 - \frac{1}{p}$, when $p > 1$, and $\beta = 0$, when $p = 1$.

3. Corollaries

The following corollaries can be derived from Theorem 1

Corollary 1. By the assumptions of Theorem 1, we have

$$|T_{n,A,B}f(x) - f(x)| = O_x(n+1)^{\beta - \frac{1}{p}} \left\{ \sum_{k=0}^n \left[w_x f\left(\frac{\pi}{k+1}\right)_{p,\beta} \right]^p \right\}^{\frac{1}{p}}.$$

Proof. Using Lemma 3 we obtain

$$\begin{aligned} & |T_{n,A,B}f(x) - f(x)| \\ & \ll (n+1)^\beta \left\{ \left(\frac{1}{n+1} \sum_{k=0}^n \left[w_x f\left(\frac{\pi}{k+1}\right)_{p,\beta} \right]^p \right)^{\frac{1}{p}} \right. \\ & \quad \left. + \frac{1}{n+1} \sum_{k=0}^n w_x f\left(\frac{\pi}{k+1}\right)_{p,\beta} \right\} \\ & \leq (n+1)^{\beta - \frac{1}{p}} \left(\sum_{k=0}^n \left[w_x f\left(\frac{\pi}{k+1}\right)_{p,\beta} \right]^p \right)^{\frac{1}{p}}. \end{aligned}$$

This completes the proof of Corollary 1. ■

If matrix A is defined by $a_{n,r} = \frac{1}{(r+1)\ln(n+1)}$ when $r = 0, 1, 2, \dots, n$ and $a_{n,r} = 0$ when $r > n$, and matrix B is defined by $b_{r,k} = \frac{1}{r+1}$ when $k = 0, 1, 2, \dots, r$ and $b_{r,k} = 0$ when $k > r$, then from Theorem 1 we obtain

Corollary 2. *Let $f \in L^p$ ($1 \leq p \leq \infty$), then*

$$\begin{aligned} & |T_{n,A,B}f(x) - f(x)| \\ &= \left| \frac{1}{\ln(n+1)} \sum_{r=0}^n \frac{1}{(r+1)^2} \sum_{k=0}^r S_k f(x) - f(x) \right| \\ &= O_x(n+1)^\beta \left\{ w_x f \left(\frac{\pi}{n+1} \right)_{p,\beta} \right. \\ &\quad \left. + \frac{1}{n+1} \sum_{k=0}^n w_x f \left(\frac{\pi}{k+1} \right)_{1,\beta} \right\}, \end{aligned}$$

for almost all considered x and $0 \leq \beta < 1 - \frac{1}{p}$, when $p > 1$, and $\beta = 0$, when $p = 1$.

Proof. For the proof we will show that $(a_{n,r})$ and $(b_{r,k})$ satisfy the assumptions of Lemma 2. The conditions (1) and (2) are satisfied evidently. Since

$$\begin{aligned} & \sum_{r=0}^n \frac{1}{(r+1)\ln(n+1)} \left(\sum_{k=0}^{n-r-1} \left| \frac{1}{n-r+1} - \frac{1}{n-r+1} \right| + \frac{1}{n-r+1} \right) \\ &= \frac{1}{\ln(n+1)} \sum_{r=0}^n \frac{1}{r+1} \frac{1}{n-r+1} \\ &= \frac{1}{\ln(n+1)} \frac{1}{n+2} \sum_{r=0}^n \left(\frac{1}{r+1} + \frac{1}{n-r+1} \right) \\ &\leq \frac{1}{\ln(n+1)} \frac{1}{n+2} 2 \ln(n+1) \ll \frac{1}{n+1} \end{aligned}$$

and the proof of Corollary 2 is complete. ■

4. Auxiliary results

We begin this section by some notations following A. Zygmund [4, Section 5 of Chapter II]. It is clear that

$$S_k f(x) = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x+t) D_k(t) dt$$

and

$$T_{n,A,B}f(x) = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x+t) \sum_{r=0}^n \sum_{k=0}^r a_{n,n-r} b_{r,k} D_k(t) dt,$$

where

$$(4) \quad D_k(t) = \frac{1}{2} + \sum_{\nu=1}^k \cos \nu t = \frac{\sin \frac{(2k+1)t}{2}}{2 \sin \frac{t}{2}}.$$

Hence

$$T_{n,A,B}f(x) - f(x) = \frac{1}{\pi} \int_0^{\pi} \varphi_x(t) \sum_{r=0}^n \sum_{k=0}^r a_{n,n-r} b_{r,k} D_k(t) dt.$$

Now, we formulate some estimates for the Dirichlet kernels.

Lemma 1. (see [4]) *If $0 < |t| \leq \pi/2$, then*

$$|D_k(t)| \leq \frac{\pi}{2|t|},$$

and for any real t we have

$$(5) \quad |D_k(t)| \leq k + 1.$$

Lemma 2. *If the entries of matrices $(a_{n,r})_{r=0}^n$ and $(b_{r,k})_{k=0}^n$ satisfy the conditions (1), (2) and (3) then*

$$\left| \sum_{r=0}^n a_{n,n-r} \sum_{k=0}^r b_{r,k} D_k(t) \right| \ll \frac{\tau^2}{n+1}$$

for $n = 0, 1, 2, 3, \dots$, where $\tau = [\pi/|t|] \leq n/2$.

Proof. Since

$$K_n(t) := \sum_{r=0}^n a_{n,n-r} \sum_{k=0}^r b_{r,k} D_k(t) = \sum_{r=0}^n a_{n,r} \sum_{k=0}^{n-r} b_{n-r,k} D_k(t)$$

and using Abel's transformation and from (4) we get

$$\begin{aligned} K_n(t) &= \sum_{r=0}^n \frac{a_{n,r}}{2 \sin \frac{t}{2}} \left[\sum_{k=0}^{n-r-1} (b_{n-r,k} - b_{n-r,k+1}) \sum_{l=0}^k \sin \frac{(2l+1)t}{2} \right. \\ &\quad \left. + b_{n-r,n-r} \sum_{l=0}^{n-r} \sin \frac{(2l+1)t}{2} \right] \\ &= \frac{1}{2 \sin \frac{t}{2}} \sum_{r=0}^n a_{n,r} \sum_{k=0}^{n-r} (b_{n-r,k} - b_{n-r,k+1}) \sum_{l=0}^k \sin \frac{(2l+1)t}{2}, \end{aligned}$$

then

$$|K_n(t)| \leq \frac{1}{\sin \frac{t}{2}} \sum_{r=0}^n a_{n,r} \sum_{k=0}^{n-r} |b_{n-r,k} - b_{n-r,k+1}| \left| \sum_{l=0}^k \sin \frac{(2l+1)t}{2} \right|.$$

A simple calculation for the last sum give

$$\left| \sum_{l=0}^k \sin \frac{(2l+1)t}{2} \right| \ll \frac{1}{\sin \frac{t}{2}}.$$

So

$$|K_n(t)| \ll \frac{\pi}{2t^2} \sum_{r=0}^n a_{n,r} \sum_{k=0}^{n-r} |b_{n-r,k} - b_{n-r,k+1}|$$

and from (3) we obtain

$$|K_n(t)| \ll \frac{\tau^2}{n+1}.$$

The desired estimate is now evident. ■

Lemma 3. *Let $f \in L^p$ ($1 \leq p \leq \infty$) and $\beta \geq 0$, then*

$$w_x f \left(\frac{\pi}{n+1} \right)_{p,\beta} \ll \left\{ \frac{1}{n+1} \sum_{k=0}^n w_x^p f \left(\frac{\pi}{k+1} \right)_{p,\beta} \right\}^{\frac{1}{p}}$$

holds for every natural n and all real x .

Proof. By our assumption we get

$$\begin{aligned} w_x f \left(\frac{\pi}{n+1} \right)_{p,\beta} &= \left\{ \frac{n+1}{\pi} \int_0^{\frac{\pi}{n+1}} |\varphi_x(u) \sin^\beta \frac{u}{2}|^p du \sum_{k=0}^n \frac{2(k+1)}{(n+1)(n+2)} \right\}^{\frac{1}{p}} \\ &= \left\{ \frac{2}{n+1} \sum_{k=0}^n \frac{(k+1)(n+1)}{\pi(n+2)} \int_0^{\frac{\pi}{n+1}} |\varphi_x(u) \sin^\beta \frac{u}{2}|^p du \right\}^{\frac{1}{p}} \\ &\ll \left\{ \frac{1}{n+1} \sum_{k=0}^n \frac{(k+1)}{\pi} \int_0^{\frac{\pi}{k+1}} |\varphi_x(u) \sin^\beta \frac{u}{2}|^p du \right\}^{\frac{1}{p}} \\ &= \left\{ \frac{1}{n+1} \sum_{k=0}^n \left[w_x f \left(\frac{\pi}{k+1} \right)_{p,\beta} \right]^p \right\}^{\frac{1}{p}}, \end{aligned}$$

and this completes the proof. ■

Proof of Theorem 1. Let as usually

$$\begin{aligned}
 T_{n,A,B}f(x) - f(x) &= \frac{1}{\pi} \int_0^\pi \varphi_x(t) \sum_{r=0}^n \sum_{k=0}^r a_{n,n-r} b_{r,k} D_k(t) dt \\
 &= \frac{1}{\pi} \int_0^{\frac{\pi}{n+1}} \varphi_x(t) \sum_{r=0}^n \sum_{k=0}^r a_{n,n-r} b_{r,k} D_k(t) dt \\
 &\quad + \frac{1}{\pi} \int_{\frac{\pi}{n+1}}^\pi \varphi_x(t) \sum_{r=0}^n \sum_{k=0}^r a_{n,n-r} b_{r,k} D_k(t) dt \\
 &= I_1(x) + I_2(x)
 \end{aligned}$$

and

$$|T_{n,A,B}f(x) - f(x)| \leq |I_1(x)| + |I_2(x)|.$$

In case $p = 1$ and $\beta = 0$, by Lemma 1 and (5), we have

$$\begin{aligned}
 |I_1(x)| &\leq \frac{1}{\pi} \sum_{r=0}^n \sum_{k=0}^r a_{n,n-r} b_{r,k} (k+1) \int_0^{\frac{\pi}{n+1}} |\varphi_x(t)| dt \\
 &\leq \frac{1}{\pi} (n+1) \sum_{r=0}^n \sum_{k=0}^r a_{n,n-r} b_{r,k} \int_0^{\frac{\pi}{n+1}} |\varphi_x(t)| dt \\
 &= w_x f\left(\frac{\pi}{n+1}\right)_{1,0}
 \end{aligned}$$

In case $p > 1$ and $0 < \beta < 1 - \frac{1}{p}$, from the Hölder inequality for integrals, with $\frac{1}{p} + \frac{1}{q} = 1$, and by Lemma 1 and (5), we obtain

$$\begin{aligned}
 |I_1(x)| &\leq \frac{1}{\pi} (n+1) \int_0^{\frac{\pi}{n+1}} |\varphi_x(t)| dt \\
 &\leq (n+1)^{\frac{1}{q}} \left\{ \frac{n+1}{\pi} \int_0^{\frac{\pi}{n+1}} \left[|\varphi_x(t)| \sin^\beta \frac{t}{2} \right]^p dt \right\}^{\frac{1}{p}} \\
 &\quad \times \left\{ \int_0^{\frac{\pi}{n+1}} \left[\frac{1}{\sin^\beta \frac{t}{2}} \right]^q dt \right\}^{\frac{1}{q}} \\
 &\ll (n+1)^{\frac{1}{q}} w_x f\left(\frac{\pi}{n+1}\right)_{p,\beta} \left\{ \int_0^{\frac{\pi}{n+1}} t^{-\beta q} dt \right\}^{\frac{1}{q}} \\
 &\ll (n+1)^{\frac{1}{q}} (n+1)^{\beta - \frac{1}{q}} w_x f\left(\frac{\pi}{n+1}\right)_{p,\beta} \\
 &= (n+1)^\beta w_x f\left(\frac{\pi}{n+1}\right)_{p,\beta}.
 \end{aligned}$$

Now we shall estimate the term I_2 . Using Lemma 2 and the inequality $\sin \frac{t}{2} \geq \frac{t}{\pi}$ ($0 \leq t \leq \pi$), we obtain

$$\begin{aligned} |I_2(x)| &\ll \frac{\pi}{n+1} \int_{\frac{\pi}{n+1}}^{\pi} \frac{|\varphi_x(t)|}{t^2} dt \ll \frac{1}{n+1} \int_{\frac{\pi}{n+1}}^{\pi} \frac{|\varphi_x(t)| \sin^{\beta} \frac{t}{2}}{t^{2+\beta}} dt \\ &\ll (n+1)^{\beta} \frac{1}{n+1} \int_{\frac{\pi}{n+1}}^{\pi} \frac{|\varphi_x(t)| \sin^{\beta} \frac{t}{2}}{t^2} dt, \end{aligned}$$

for $\beta \geq 0$. Integrating by parts we get

$$\begin{aligned} |I_2(x)| &\ll (n+1)^{\beta-1} \left\{ \left[\frac{1}{t^2} \int_0^t |\varphi_x(s)| \sin^{\beta} \frac{s}{2} ds \right]_{\frac{\pi}{n+1}}^{\pi} \right. \\ &\quad \left. + 2 \int_{\frac{\pi}{n+1}}^{\pi} \frac{1}{t^3} \left[\int_0^t |\varphi_x(s)| \sin^{\beta} \frac{s}{2} ds \right] dt \right\} \\ &\leq (n+1)^{\beta-1} \left\{ \frac{1}{\pi^2} \int_0^{\pi} |\varphi_x(s)| \sin^{\beta} \frac{s}{2} ds + 2 \int_{\frac{\pi}{n+1}}^{\pi} \frac{1}{t^2} w_x f(t)_{1,\beta} dt \right\} \\ &\ll (n+1)^{\beta-1} \left\{ w_x f(\pi)_{1,\beta} + \int_1^{n+1} w_x \left(\frac{\pi}{u} \right)_{1,\beta} du \right\} \\ &\leq (n+1)^{\beta-1} \left\{ w_x f(\pi)_{1,\beta} + \sum_{m=1}^n \int_m^{m+1} w_x f \left(\frac{\pi}{u} \right)_{1,\beta} du \right\} \\ &\ll (n+1)^{\beta-1} \left\{ w_x f(\pi)_{1,\beta} + \sum_{m=0}^n w_x f \left(\frac{\pi}{m+1} \right)_{1,\beta} \right\} \\ &\leq (n+1)^{\beta-1} \sum_{m=0}^n w_x f \left(\frac{\pi}{m+1} \right)_{1,\beta}. \end{aligned}$$

Collecting these estimates we obtain the desired result. ■

Proof of Theorem 2. Let $\delta := \frac{\pi}{k+1}$. If $f \in L^p$, then by monotonicity of the norm as a functional

$$\begin{aligned} \|T_{n,A,B}f(\cdot) - f(\cdot)\|_{L^p} &\leq \left\| (n+1)^{\beta-\frac{1}{p}} \sum_{k=0}^n w.f(\delta)_{p,\beta} \right\|_{L^p} \\ &\leq (n+1)^{\beta-\frac{1}{p}} \sum_{k=0}^n \left\| w.f(\delta)_{p,\beta} \right\|_{L^p} \end{aligned}$$

and by the general Minkowski inequality we obtain

$$\begin{aligned} \left\| w.f(\delta)_{p,\beta} \right\|_{L^p} &= \left\{ \frac{1}{2\pi} \int_{-\pi}^{\pi} \left[\frac{1}{\delta} \int_0^{\delta} \left| \varphi_x(u) \sin^{\beta} \frac{u}{2} \right|^p du \right] dx \right\}^{\frac{1}{p}} \\ &\leq \left\{ \frac{1}{\delta} \int_0^{\delta} \left[\frac{1}{2\pi} \int_{-\pi}^{\pi} \left| \varphi_x(u) \sin^{\beta} \frac{u}{2} \right|^p dx \right] du \right\}^{\frac{1}{p}} \\ &= \left\{ \frac{1}{\delta} \int_0^{\delta} \left\| \varphi_x(u) \sin^{\beta} \frac{u}{2} \right\|_{L^p}^p du \right\}^{\frac{1}{p}} \\ &\leq \sup_{0 < u \leq \delta} \left\| \varphi_x(u) \sin^{\beta} \frac{u}{2} \right\|_{L^p} = \omega_{L^p} f(\delta)_{\beta}. \end{aligned}$$

Thus, the desired result follows. ■

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