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ON POINTWISE APPROXIMATION OF FUNCTIONS BY SOME MATRIX MEANS OF CONJUGATE FOURIER SERIES

Abstract. The results corresponding to some theorems of S. Lal [Tamkang J. Math., 31(4)(2000), 279-288] and the results of the authors [Banach Center Publ. 92(2011), 237-247] are shown. The same degrees of pointwise approximation as in mentioned papers by significantly weaker assumptions on considered functions are obtained. From presented pointwise results the estimation on norm approximation with essentialy better degrees are derived. Some special cases as corollaries for iteration of the Nörlund or the Riesz method with the Euler one are also formulated.

Key words: conjugate function, Fourier series, degree of approximation.

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1. Introduction

Let L^p $(1 \leq p < \infty)$ [respectively L^∞] be the class of all 2π–periodic real–valued functions integrable in the Lebesgue sense with p–th power [essentially bounded] over $Q = [-\pi, \pi]$ with the norm

$$\|f\| := \|f(\cdot)\|_{L^p} = \begin{cases} \left(\int_Q \mid f(t) \mid^p dt \right)^{1/p}, & \text{when} \quad 1 \leq p < \infty, \\ ess \sup\limits_{t \in Q} \mid f(t) \mid, & \text{when} \quad p = \infty \end{cases}$$

and consider the conjugate trigonometric Fourier series

$$\widetilde{S}f(x) := \sum_{\nu=1}^{\infty} (a_\nu(f) \sin \nu x - b_\nu(f) \cos \nu x)$$

with the partial sums $\widetilde{S}_k f$. We know that if $f \in L$ then

$$\widetilde{f}(x) := -\frac{1}{\pi} \int_0^\pi \psi_x(t) \frac{1}{2} \cot \frac{t}{2} dt = \lim_{\epsilon \to 0^+} \widetilde{f}(x, \epsilon),$$

where

$$\widetilde{f}(x,\epsilon) := -\frac{1}{\pi}\int_{\epsilon}^{\pi} \psi_x(t)\,\frac{1}{2}\cot\frac{t}{2}dt$$

with

$$\psi_x(t) := f(x+t) - f(x-t),$$

exists for almost all x [11, Th.(3.1)IV].

Let $A := (a_{n,k})$ and $B := (b_{n,k})$ be infinite lower triangular matrices of real numbers such that

$$\begin{aligned} a_{n,k} &\geq 0 \text{ and } b_{n,k} \geq 0, \quad \text{when } k = 0,1,2,...n, \\ a_{n,k} &= 0 \text{ and } b_{n,k} = 0, \quad \text{when } k > n, \end{aligned}$$

$$\sum_{k=0}^{n} a_{n,k} = 1 \text{ and } \sum_{k=0}^{n} b_{n,k} = 1, \text{ where } n = 0,1,2,\ldots,$$

and let, for $m = 0,1,2,..,n$,

$$A_{n,m} = \sum_{k=0}^{m} a_{n,k} \quad \text{and} \quad \overline{A}_{n,m} = \sum_{k=m}^{n} a_{n,k}.$$

Let the $AB-$transformation of $\widetilde{S}_k f$ be given by

$$\widetilde{T}_{n,A,B} f(x) := \sum_{r=0}^{n}\sum_{k=0}^{r} a_{n,r} b_{r,k} \widetilde{S}_k f(x) \quad (n = 0,1,2,\ldots).$$

We define two classes of sequences (see [3]).

A sequence $c := (c_k)$ of nonnegative numbers tending to zero is called the Rest Bounded Variation Sequence, or briefly $c \in RBVS$, if it has the property

$$\sum_{k=m}^{\infty} |c_k - c_{k+1}| \leq K(c)\, c_m,$$

for all positive integer m, where $K(c)$ is a constant depending only on c.

A sequence $c := (c_k)$ of nonnegative numbers will be called the Head Bounded Variation Sequence, or briefly $c \in HBVS$, if it has the property

$$\sum_{k=0}^{m-1} |c_k - c_{k+1}| \leq K(c)\, c_m,$$

for all positive integer m, or only for all $m \leq n$ if the sequence c has only finite nonzero terms and the last nonzero term is c_n.

Now, we define the another classes of sequences.

Following by L. Leindler [4], a sequence $c := (c_r)$ of nonnegative numbers tending to zero is called the Mean Rest Bounded Variation Sequence, or briefly $c \in MRBVS$, if it has the property

$$\sum_{r=m}^{\infty} |c_r - c_{r+1}| \leq K(c) \frac{1}{m+1} \sum_{r \geq m/2}^{m} c_r,$$

for all positive integer m.

Analogously as in [6], a sequence $c := (c_r)$ of nonnegative numbers will be called the Mean Head Bounded Variation Sequence, or briefly $c \in MHBVS$, if it has the property

$$\sum_{r=0}^{n-m-1} |c_r - c_{r+1}| \leq K(c) \frac{1}{m+1} \sum_{r=n-m}^{n} c_r,$$

for all positive integers $m < n$, where the sequence c has only finite nonzero terms and the last nonzero term is c_n. Consequently, we assume that the sequence $(K(\alpha_n))_{n=0}^{\infty}$ is bounded, that is, that there exists a constant K such that

$$0 \leq K(\alpha_n) \leq K$$

holds for all n, where $K(\alpha_n)$ denote the constants appearing in the before inequalities for the sequences $\alpha_n = (a_{n,r})_{r=0}^{n}$, $n = 0, 1, 2, \ldots$.

Now we can give the conditions to be used later on. We assume that for all n and $0 \leq m < n$

$$\sum_{k=m}^{n-1} |a_{n,r} - a_{n,r+1}| \leq K \frac{1}{m+1} \sum_{r \geq m/2}^{m} a_{n,r}$$

and

$$\sum_{r=0}^{n-m-1} |a_{n,r} - a_{n,r+1}| \leq K \frac{1}{m+1} \sum_{r=n-m}^{n} a_{n,r}$$

hold if $(a_{n,r})_{r=0}^{n}$ belongs to $MRBVS$ and $MHBVS$, for $n = 1, 2, \ldots$, respectively.

As a measure of approximation of $\widetilde{f}$ by $\widetilde{T}_{n,A,B}f$ we use the pointwise moduli of continuity of f in the space L^p defined by the formulas

$$\widetilde{w}_x^p f(\delta)_\beta = \begin{cases} \left\{ \frac{1}{\delta} \int_0^\delta \left| \psi_x(u) \sin^\beta \frac{u}{2} \right|^p du \right\}^{\frac{1}{p}}, & \text{when } 1 \leq p < \infty, \\ ess \sup\limits_{0<u\leq\delta} \mid \psi_x(u) \sin^\beta \frac{u}{2} \mid, & \text{when } p = \infty, \end{cases}$$

$$\widetilde{w}_x^p f(\delta)_\beta = \begin{cases} \sup_{0<t\leq\delta} \left\{ \frac{1}{t} \int_0^t \left| \psi_x(u) \sin^\beta \frac{u}{2} \right|^p du \right\}^{\frac{1}{p}} & \text{when } 1 \leq p < \infty, \\ ess \sup\limits_{0<u\leq\delta} \left| \psi_x(u) \sin^\beta \frac{u}{2} \right| & \text{when } p = \infty, \end{cases}$$

and the classical ones

$$\widetilde{\omega}_\beta f(\delta)_{L^p} = \sup_{0<t\leq\delta} \| \sin^\beta \frac{t}{2} \psi_\cdot(t) \|_{L^p}.$$

The deviation $\widetilde{T}_{n,A} f - \widetilde{f} = \widetilde{T}_{n,A,B} f - \widetilde{f}$, with $b_{r,r} = 1$ and 0 otherwise, was estimated at the point as well as in the norm of L^p by K. Qureshi [7] and S. Lal, H. Nigam [1]. These results were generalized by K. Qureshi [8]. The next generalization was obtained by S. Lal [2]. In the case

$$a_{n,r} = \frac{1}{n+1} \quad \text{when} \quad r = 0, 1, 2, ...n \quad \text{and} \quad a_{n,r} = 0 \quad \text{when} \quad r > n$$

and

$$b_{r,k} = \frac{\binom{r}{k}\gamma^k}{(1+\gamma)^r} \quad \text{when} \quad k = 0, 1, 2, ...r \quad \text{and}$$
$$b_{n,r} = 0 \quad \text{when} \quad k > r \quad \text{with } \gamma > 0,$$

the deviation $\widetilde{T}_{n,A,B} f - \widetilde{f}$, was estimated by S. Sonker and U. Singh [9] as follows:

Theorem. *Let $f(x)$ be a 2π-periodic, Lebesgue integrable function and belongs to the $Lip(\alpha, r)$-class with $r \geq 1$ and $\alpha r \geq 1$. Then the degree of approximation of $\widetilde{f}(x)$, the conjugate of $f(x)$ by $(C,1)(E,\gamma)$ means of its conjugate Fourier series is given by*

$$\left\| \frac{1}{n+1} \sum_{r=0}^{n} \frac{1}{(1+\gamma)^r} \sum_{k=0}^{r} \binom{r}{k} \gamma^k \widetilde{S}_k f(\cdot) - \widetilde{f}(\cdot) \right\|_{L^r} = O_x\left((n+1)^{-\alpha+1/r} \right)$$

provided

$$\left\{ \int_0^{\pi/(n+1)} \left(\frac{|\psi_x(t)|}{t^\alpha} \right)^r dt \right\}^{1/r} = O\left((n+1)^{-1} \right)$$

and

$$\left\{ \int_{\pi/(n+1)}^{\pi} \left(\frac{t^{-\delta} |\psi_x(t)|}{t^\alpha} \right)^p dt \right\}^{1/p} = O\left((n+1)^\delta \right),$$

where δ is an arbitrary positive number with $(\alpha+\delta)s + 1 < 0$, and $r^{-1} + s^{-1} = 1$, $r > 1$.

In this paper we shall consider the deviations $\widetilde{T}_{n,A,B} f(\cdot) - \widetilde{f}(\cdot)$ and $\widetilde{T}_{n,A,B} f(\cdot) - \widetilde{f}(\cdot, \frac{\pi}{n+1})$ in general form. In the theorems we formulate the general conditions for the functions and the modulus of continuity obtaining the same and sometimes essentially better degrees of approximation than the above one. Finally, we also give some results on norm approximation with essentially better degrees of approximation. The obtained results generalize the results from [1] and [6].

We shall write $I_1 \ll I_2$ if there exists a positive constant K, sometimes depending on some parameters, such that $I_1 \leq K I_2$.

2. Statement of the results

Let

$$L^p\left(\widetilde{w}_x\right)_\beta = \left\{ f \in L^p : \widetilde{w}_x^p f\left(\delta\right)_\beta \leq \widetilde{w}_x\left(\delta\right) \right\},$$

where $\widetilde{w}_x$ is positive, with $\widetilde{w}_x(0) = 0$, and almost nondecreasing continuous function. We can now formulate our main results.

At the beginning, we formulate the results on the degrees of pointwise summability of conjugate series.

Theorem 1. *Let $f \in L^1$ and $\beta \in (0,1]$. If $(a_{n,k})_{k=0}^n \in MRBVS$ and*

$$\left| \sum_{r=\mu}^{\nu} \sum_{k=0}^{r} b_{r,k} \cos \frac{(2k+1)\,t}{2} \right| \ll \tau \tag{1}$$

for $0 \leq \mu \leq \nu$, then

$$\left| \widetilde{T}_{n,A,B} f\left(x\right) - \widetilde{f}\left(x, \frac{\pi}{n+1}\right) \right| = O_x\left((n+1)^\beta \sum_{k=0}^{n} a_{n,k} \widetilde{\widetilde{w}}_x^1 f\left(\frac{\pi}{k+1}\right)_\beta \right),$$

for almost all considered x.

Theorem 2. *Let $f \in L^1$ and $\beta \in (0,1]$. If $(a_{n,k})_{k=0}^n \in MHBVS$ and the entries of matrix B satisfy the condition (1) for $0 \leq \mu \leq \nu$, then*

$$\left| \widetilde{T}_{n,A,B} f\left(x\right) - \widetilde{f}\left(x, \frac{\pi}{n+1}\right) \right| = O_x\left((n+1)^\beta \sum_{k=0}^{n} a_{n,n-k} \widetilde{\widetilde{w}}_x^1 f\left(\frac{\pi}{k+1}\right)_\beta \right),$$

for almost all considered x.

Theorem 3. *Let $f \in L^p\left(\widetilde{w}_x\right)_\beta$ with $1 < p < \infty$, and let $\widetilde{w}_x$ satisfy*

$$\left\{ \int_0^{\frac{\pi}{n+1}} \left(\frac{\widetilde{w}_x\left(t\right)}{t \sin^\beta \frac{t}{2}} \right)^{p/(p-1)} dt \right\}^{(p-1)/p} = O\left((n+1)^{\beta+1/p}\, \widetilde{w}_x\left(\frac{\pi}{n+1}\right) \right) \tag{2}$$

and

$$\left\{ \int_0^{\frac{\pi}{n+1}} \left(\frac{|\psi_x(t)|}{\widetilde{w}_x(t)} \right)^p \sin^{\beta p} \frac{t}{2} dt \right\}^{1/p} = O_x\left((n+1)^{-\frac{1}{p}} \right) \tag{3}$$

with some $\beta \geq 0$. *If* $(a_{n,k})_{k=0}^n \in MRBVS$ *and the entries of matrix* B *satisfy the condition (1) for* $0 \leq \mu \leq \nu$, *then*

$$\left| \widetilde{T}_{n,A,B} f(x) - \widetilde{f}(x) \right| = O_x \left((n+1)^\beta \sum_{k=0}^n a_{n,k} \widetilde{w}_x \left(\frac{\pi}{k+1} \right) \right), \tag{4}$$

for almost all considered x *such that* $\widetilde{f}(x)$ *exists.*

Theorem 4. *Let* $f \in L^p(\widetilde{w}_x)_\beta$, *with* $1 < p < \infty$, *and and let* $\widetilde{w}_x$ *satisfy (2) and (3) with some* $\beta \geq 0$. *If* $(a_{n,k})_{k=0}^n \in MHBVS$ *and the entries of matrix* B *satisfy the condition (1) for* $0 \leq \mu \leq \nu$, *then*

$$\left| \widetilde{T}_{n,A,B} f(x) - \widetilde{f}(x) \right| = O_x \left((n+1)^\beta \sum_{k=0}^n a_{n,n-k} \widetilde{w}_x \left(\frac{\pi}{k+1} \right) \right),$$

for almost all considered x *such that* $\widetilde{f}(x)$ *exists.*

Next, we formulate the results on estimates of L^p norm of the deviation $\widetilde{T}^f_{n,A,B}(\cdot) - \widetilde{f}(\cdot, \frac{\pi}{n+1})$.

Theorem 5. *Let* $f \in L^p$ *with* $1 < p < \infty$. *If* $(a_{n,k})_{k=0}^n \in MRBVS$ *and the entries of matrix* B *satisfy the condition (1) for* $0 \leq \mu \leq \nu$, *then*

$$\left\| \widetilde{T}_{n,A,B} f(\cdot) - \widetilde{f}\left(\cdot, \frac{\pi}{n+1} \right) \right\|_{L^p} = O \left((n+1)^\beta \sum_{k=0}^n a_{n,k} \widetilde{\omega}_\beta f \left(\frac{\pi}{k+1} \right)_{L^p} \right)$$

where $0 < \beta \leq 1$.

Theorem 6. *Let* $f \in L^p$ *with* $1 < p < \infty$. *If* $(a_{n,k})_{k=0}^n \in MHBVS$ *and the entries of matrix* B *satisfy the condition (1) for* $0 \leq \mu \leq \nu$, *then*

$$\left\| \widetilde{T}_{n,A,B} f(\cdot) - \widetilde{f}\left(\cdot, \frac{\pi}{n+1} \right) \right\|_{L^p} = O \left((n+1)^\beta \sum_{k=0}^n a_{n,n-k} \widetilde{\omega}_\beta f \left(\frac{\pi}{k+1} \right)_{L^p} \right)$$

where $0 < \beta \leq 1$.

In case of the deviation $\widetilde{T}_{n,A,B} f(\cdot) - \widetilde{f}(\cdot)$ let

$$L^p(\widetilde{\omega})_\beta = \left\{ f \in L^p : \widetilde{\omega}_\beta f(\delta)_{L^p} \leq \widetilde{\omega}(\delta) \right\},$$

where $\widetilde{\omega}$ is positive, with $\widetilde{\omega}(0) = 0$, and almost nondecreasing continuous function.

Theorem 7. *Let* $f \in L^p(\widetilde{\omega})_\beta$ *with* $1 < p < \infty$ *and* $0 < \beta < 1 - \frac{1}{p}$, *where* $\widetilde{\omega}$ *instead of* $\widetilde{w}_x$ *satisfy (2) and (3) with some* $\beta \geq 0$. *If* $(a_{n,k})_{k=0}^{n} \in MRBVS$ *and the entries of matrix* B *satisfy the condition (1) for* $0 \leq \mu \leq \nu$, *then*

$$\left\| \widetilde{T}_{n,A,B} f(\cdot) - \widetilde{f}(\cdot) \right\|_{L^p} = O\left((n+1)^\beta \sum_{k=0}^{n} a_{n,k} \widetilde{\omega}\left(\frac{\pi}{k+1}\right)\right).$$

Theorem 8. *Let* $f \in L^p(\widetilde{\omega})_\beta$ *with* $1 < p < \infty$ *and* $0 < \beta < 1 - \frac{1}{p}$, *where* $\widetilde{\omega}$ *instead of* $\widetilde{w}_x$ *satisfy (2) and (3) with some* $\beta \geq 0$. *If* $(a_{n,k})_{k=0}^{n} \in MHBVS$ *and the entries of matrix* B *satisfy the condition (1) for* $0 \leq \mu \leq \nu$, *then*

$$\left\| \widetilde{T}_{n,A,B} f(\cdot) - \widetilde{f}(\cdot) \right\|_{L^p} = O\left((n+1)^\beta \sum_{k=0}^{n} a_{n,n-k} \widetilde{\omega}\left(\frac{\pi}{k+1}\right)\right).$$

Remark 1. Under the above remarks we can observe that in the special case $\beta = 0$, when our sequences (a_{nk}) are monotonic with respect to k we also have the corrected form of the result of S. Lal and H. K. Nigam [1].

Finally, we give applications of our results as corollary and some remarks

Taking $a_{n,r} = p_{n-r} / \sum_{\nu=0}^{n} p_\nu$ when $r = 0, 1, 2, \ldots, n$ and $a_{n,r} = 0$ when $r > n$ with $p_\nu > 0$, $p_\nu \leq p_{\nu+1}$ and $b_{r,k} = \frac{\binom{r}{k}\gamma^k}{(1+\gamma)^r}$ when $k = 0, 1, 2, \ldots, r$ and $b_{n,r} = 0$ when $k > r$ with $\gamma > 0$, Theorem 1 and Theorem 2 imply:

Corollary 1. *If* $f \in L^1$ *and* $0 < \beta \leq 1$, *then*

$$\left| \frac{1}{\sum_{\nu=0}^{n} p_\nu} \sum_{r=0}^{n} \frac{p_r}{(1+\gamma)^r} \sum_{k=0}^{r} \binom{r}{k} \gamma^k \widetilde{S}_k f(x) - \widetilde{f}\left(x, \frac{\pi}{n+1}\right) \right|$$

$$= O_x\left(\frac{(n+1)^\beta}{\sum_{\nu=0}^{n} p_\nu} \sum_{k=0}^{n} p_k \widetilde{\overline{w}}_x^1 f\left(\frac{\pi}{k+1}\right)_\beta \right),$$

and

$$\left| \frac{1}{\sum_{\nu=0}^{n} p_\nu} \sum_{r=0}^{n} \frac{p_{n-r}}{(1+\gamma)^r} \sum_{k=0}^{r} \binom{r}{k} \gamma^k \widetilde{S}_k f(x) - \widetilde{f}\left(x, \frac{\pi}{n+1}\right) \right|$$

$$= O_x\left(\frac{(n+1)^\beta}{\sum_{\nu=0}^{n} p_\nu} \sum_{k=0}^{n} p_{n-k} \widetilde{\overline{w}}_x^1 f\left(\frac{\pi}{k+1}\right)_\beta \right),$$

for almost all considered x, where $p_\nu > 0$, $p_\nu \leq p_{\nu+1}$ and $\gamma > 0$. In special case $p_r = p_{n-r} = 1$ we have estimate

$$\left| \frac{1}{n+1} \sum_{r=0}^{n} \frac{1}{(1+\gamma)^r} \sum_{k=0}^{r} \binom{r}{k} \gamma^k \widetilde{S}_k f(x) - \widetilde{f}\left(x, \frac{\pi}{n+1}\right) \right|$$
$$= O_x \left((n+1)^{\beta-1} \sum_{k=0}^{n} \left[\widetilde{\widetilde{w}}_x^1 f\left(\frac{\pi}{k+1}\right)_\beta \right] \right).$$

Analogical corollary we can derive from Theorem 3 and Theorem 4.

Remark 2. If we take $\beta = 0$ in the above considerations then we have to estimate the quantities $\left|\widetilde{I}_2\right|, \left|\widetilde{I}_2^\circ\right|$ and $\left\|\widetilde{I}_2\right\|_{L^p}, \left\|\widetilde{I}_2^\circ\right\|_{L^p}$ using the Hölder inequality analogously as in estimate of $\left|\widetilde{I}_1\right|$. Thus we obtain in the all above estimates $(n+1)^{1/p}$ instead of $(n+1)^\beta$.

Remark 3. Analyzing the proofs of Theorem 1 - 4 we can deduce that taking the assumption $(a_{n,k})_{k=0}^n \in RBVS$ or $(a_{n,k})_{k=0}^n \in HBVS$ instead of $(a_{n,k})_{k=0}^n \in MRBVS$ or $(a_{n,k})_{k=0}^n \in MHBVS$, respectively, we obtain the results from the paper [5].

3. Auxiliary results

We begin this section by some notations following A. Zygmund [11, Section 5 of Chapter II].

It is clear that

$$\widetilde{S}_k f(x) = -\frac{1}{\pi} \int_{-\pi}^{\pi} f(x+t) \widetilde{D}_k(t)\, dt,$$

and

$$\widetilde{T}_{n,A,B} f(x) = -\frac{1}{\pi} \int_{-\pi}^{\pi} f(x+t) \sum_{r=0}^{n} \sum_{k=0}^{r} a_{n,r} b_{r,k} \widetilde{D_k}(t)\, dt,$$

where

$$\widetilde{D_k}(t) = \sum_{\nu=0}^{k} \sin \nu t = \frac{\cos\frac{t}{2} - \cos\frac{(2k+1)t}{2}}{2\sin\frac{t}{2}}.$$

Hence

$$\widetilde{T}_{n,A,B} f(x) - \widetilde{f}\left(x, \frac{\pi}{n+1}\right) = -\frac{1}{\pi} \int_0^{\frac{\pi}{n+1}} \psi_x(t) \sum_{r=0}^{n} \sum_{k=0}^{r} a_{n,r} b_{r,k} \widetilde{D_k}(t)\, dt$$
$$+ \frac{1}{\pi} \int_{\frac{\pi}{n+1}}^{\pi} \psi_x(t) \sum_{r=0}^{n} \sum_{k=0}^{r} a_{n,r} b_{r,k} \widetilde{D_k^\circ}(t)\, dt$$

and

$$\widetilde{T}_{n,A,B}f(x)-\widetilde{f}(x)=\frac{1}{\pi}\int_0^{\pi}\psi_x(t)\sum_{r=0}^{n}\sum_{k=0}^{r}a_{n,r}b_{r,k}\widetilde{D_k^{\circ}}(t)\,dt,$$

where

$$\widetilde{D_k^{\circ}}(t)=\frac{1}{2}\cot\frac{t}{2}-\sum_{\nu=0}^{k}\sin\nu t=\frac{\cos\frac{(2k+1)t}{2}}{2\sin\frac{t}{2}}.$$

Now, we formulate some estimates for the conjugate Dirichlet kernels.

Lemma 1 (see [11]). *If* $0<|t|\le\pi/2$, *then*

$$\left|\widetilde{D_k^{\circ}}(t)\right|\le\frac{\pi}{2|t|}$$

and for any real t we have

$$\left|\widetilde{D_k}(t)\right|\le\frac{1}{2}k(k+1)|t|\quad\text{and}\quad\left|\widetilde{D_k}(t)\right|\le k+1.$$

Lemma 2. *Let* $(b_{r,k})_{k=0}^{r}$ *be such that the condition (1) holds for* $0\le\mu\le\nu$. *If* $(a_{n,k})_{k=0}^{n}\in MRBVS$, *then*

$$\left|\sum_{r=0}^{n}\sum_{k=0}^{r}a_{n,r}b_{r,k}\widetilde{D_k^{\circ}}(t)\right|\ll\tau A_{n,\tau}$$

and if $(a_{n,k})_{k=0}^{n}\in MHBVS$, *then*

$$\left|\sum_{r=0}^{n}\sum_{k=0}^{r}a_{n,r}b_{r,k}\widetilde{D_k^{\circ}}(t)\right|\ll\tau\overline{A}_{n,n-\tau}$$

with $\tau=[\pi/t]$ *and* $t\in\left[\frac{\pi}{n+1},\pi\right]$, *for* $n=0,1,2,\ldots$

Proof. Let

$$K_n(t):=\sum_{r=0}^{n}\sum_{k=0}^{r}a_{n,r}b_{r,k}\cos\frac{(2k+1)t}{2}.$$

The relation $(a_{n,k})_{k=0}^{n}\in MRBVS$ implies

$$\begin{aligned}a_{n,s}-a_{n,m}\ &\le\ |a_{n,m}-a_{n,s}|\le\sum_{k=m}^{s-1}|a_{n,k}-a_{n,k+1}|\\ &\ll\sum_{k=r}^{n-1}|a_{n,k}-a_{n,k+1}|\ll\frac{1}{r+1}\sum_{k\ge r/2}a_{n,k}\quad(0\le r\le m<s\le n)\end{aligned}$$

whence

$$a_{n,s} \ll a_{n,m} + \frac{1}{r+1} \sum_{k \geq r/2}^{r} a_{n,k} \quad (0 \leq r \leq m < s \leq n)$$

and thus, by our assumption

$$\left| \sum_{l=\tau+1}^{r} \sum_{k=0}^{l} b_{l,k} \cos \frac{(2k+1)\,t}{2} \right| \ll \tau,$$

we obtain

$$\begin{aligned}
|K_n(t)| &= \left| \left(\sum_{r=0}^{\tau} + \sum_{r=\tau+1}^{n} \right) a_{n,r} \sum_{k=0}^{r} b_{r,k} \cos \frac{(2k+1)\,t}{2} \right| \\
&\leq \sum_{r=0}^{\tau} a_{n,r} \sum_{k=0}^{r} b_{r,k} \\
&\quad + \sum_{r=\tau+1}^{n-1} |a_{n,r} - a_{n,r+1}| \left| \sum_{l=\tau+1}^{r} \sum_{k=0}^{l} b_{l,k} \cos \frac{(2k+1)\,t}{2} \right| \\
&\quad + a_{n,n} \left| \sum_{l=\tau+1}^{n} \sum_{k=0}^{l} b_{l,k} \cos \frac{(2k+1)\,t}{2} \right|
\end{aligned}$$

$$\begin{aligned}
&\ll A_{n.\tau} + \left(\frac{1}{\tau+1} \sum_{k \geq \tau/2}^{\tau} a_{n,k} + \frac{1}{\tau+1} \sum_{k \geq \tau/2}^{\tau} a_{n,n} \right) \tau \\
&\ll A_{n,\tau} + \left[\frac{1}{\tau+1} \sum_{k \geq \tau/2}^{\tau} a_{n,k} + \frac{1}{\tau+1} \sum_{k \geq \tau/2}^{\tau} \left(a_{n,k} + \frac{1}{k+1} \sum_{l \geq k/2}^{k} a_{n,l} \right) \right] \tau \\
&\leq A_{n,\tau} + \left[\frac{1}{\tau+1} \sum_{k \geq \tau/2}^{\tau} a_{n,k} + \frac{1}{\tau+1} \sum_{k \geq \tau/2}^{\tau} \left(a_{n,k} + \frac{1}{\tau/2+1} \sum_{l \geq \tau/4}^{\tau} a_{n,l} \right) \right] \tau \\
&\leq A_{n,\tau} + \left[\frac{1}{\tau+1} \sum_{k \geq \tau/2}^{\tau} a_{n,k} + \frac{1}{\tau+1} \sum_{k \geq \tau/2}^{\tau} a_{n,k} + \frac{1}{\tau+1} \sum_{l=0}^{\tau} a_{n,l} \right] \tau \\
&\leq A_{n,\tau} + \left[\frac{1}{\tau+1} \sum_{k \geq \tau/2}^{\tau} a_{n,k} + \frac{1}{\tau+1} \sum_{k \geq \tau/2}^{\tau} a_{n,k} + \frac{1}{\tau+1} \sum_{l=0}^{\tau} a_{n,l} \right] \tau \\
&\leq A_{n,\tau} + 3 \sum_{l=0}^{\tau} a_{n,l} = 4A_{n,\tau},
\end{aligned}$$

but the relation $(a_{n,k})_{k=0}^{n} \in MHBVS$ implies

$$a_{n,m} - a_{n,s} \le |a_{n,m} - a_{n,s}| \le \sum_{k=m}^{s-1} |a_{n,k} - a_{n,k+1}|$$
$$\ll \sum_{k=0}^{r-1} |a_{n,k} - a_{n,k+1}| \ll \frac{1}{n-r+1} \sum_{k=r}^{n} a_{n,k} \;\; (0 \le m < s \le r \le n)$$

whence

$$a_{n,m} \ll a_{n,s} + \frac{1}{n-r+1} \sum_{k=r}^{n} a_{n,k} \quad (0 \le m < s \le r \le n)$$

and thus, by our assumption

$$\left| \sum_{l=0}^{r} \sum_{k=0}^{l} b_{l,k} \cos \frac{(2k+1)\,t}{2} \right| \ll \tau,$$

we obtain

$$|K_n\,(t)| = \left| \left(\sum_{r=n-\tau}^{n} + \sum_{r=0}^{n-\tau} \right) a_{n,r} \sum_{k=0}^{r} b_{r,k} \cos \frac{(2k+1)\,t}{2} \right|$$
$$= \overline{A}_{n.n-\tau} + \sum_{r=0}^{n-\tau-1} |a_{n,r} - a_{n,r+1}| \left| \sum_{l=0}^{r} \sum_{k=0}^{l} b_{l,k} \cos \frac{(2k+1)\,t}{2} \right|$$
$$+ a_{n,n-\tau-1} \left| \sum_{l=0}^{n-\tau} \sum_{k=0}^{l} b_{l,k} \cos \frac{(2k+1)\,t}{2} \right|$$
$$\ll \overline{A}_{n.n-\tau} + \left[\frac{1}{\tau+1} \sum_{k=n-\tau}^{n} a_{n,k} + a_{n,n-\tau-1} \right] \tau$$
$$\ll \overline{A}_{n.n-\tau} + \left[\frac{1}{\tau+1} \sum_{k=n-\tau}^{n} a_{n,k} + \frac{1}{\tau+1} \sum_{k=n-\tau}^{n-\tau/2} a_{n,n-\tau-1} \right] \tau$$
$$\ll \overline{A}_{n.n-\tau} + \left[\frac{1}{\tau+1} \sum_{k=n-\tau}^{n} a_{n,k} + \frac{1}{\tau+1} \sum_{k=n-\tau}^{n-\tau/2} \left(a_{n,k} + \frac{1}{\tau+1} \sum_{\nu=n-\tau/2}^{n} a_{n,\nu} \right) \right] \tau$$
$$\le \overline{A}_{n.n-\tau} + 2 \sum_{k=n-\tau}^{n} a_{n,k} + \sum_{k=n-\tau}^{n} \frac{1}{\tau+1} \sum_{\nu=n-\tau}^{n} a_{n,\nu} \ll \overline{A}_{n.n-\tau}.$$

Now, our proof is complete. ■

4. Proofs of the results

4.1. Proof of Theorem 1

We start with the obvious relations

$$\begin{aligned}\widetilde{T}_{n,A,B}f\left(x\right)-\widetilde{f}\left(x,\frac{\pi}{n+1}\right) &= -\frac{1}{\pi}\int_{0}^{\frac{\pi}{n+1}}\psi_{x}\left(t\right)\sum_{r=0}^{n}\sum_{k=0}^{r}a_{n,r}b_{r,k}\widetilde{D_{k}}\left(t\right)dt \\ &\quad+\frac{1}{\pi}\int_{\frac{\pi}{n+1}}^{\pi}\psi_{x}\left(t\right)\sum_{r=0}^{n}\sum_{k=0}^{r}a_{n,r}b_{r,k}\widetilde{D_{k}^{\circ}}\left(t\right) \\ &= \widetilde{I}_{1}+\widetilde{I}_{2}^{\circ}\end{aligned}$$

and

$$\left|\widetilde{T}_{n,A}f\left(x\right)-\widetilde{f}\left(x,\frac{\pi}{n+1}\right)\right|\leq\left|\widetilde{I}_{1}\right|+\left|\widetilde{I}_{2}^{\circ}\right|.$$

By Lemma 1, we have

$$\begin{aligned}\left|\widetilde{I}_{1}\right| &\ll \left(n+1\right)^{2}\int_{0}^{\frac{\pi}{n+1}}t\left|\psi_{x}\left(t\right)\right|dt\leq\left(n+1\right)^{2}\int_{0}^{\frac{\pi}{n+1}}t\left|\psi_{x}\left(t\right)\right|\sin^{\beta}\frac{t}{2}\sin^{-\beta}\frac{t}{2}dt \\ &\ll \left(n+1\right)^{2}\int_{0}^{\frac{\pi}{n+1}}\left|\psi_{x}\left(t\right)\right|\sin^{\beta}\frac{t}{2}dt\left(\frac{\pi}{n+1}\right)^{1-\beta}\ll\left(n+1\right)^{\beta}\widetilde{w}_{x}^{1}f\left(\frac{\pi}{n+1}\right)_{\beta} \\ &= \left(n+1\right)^{\beta}\widetilde{w}_{x}^{1}f\left(\frac{\pi}{n+1}\right)_{\beta}\sum_{k=1}^{n}a_{n,k}\leq\left(n+1\right)^{\beta}\sum_{k=0}^{n}a_{n,k}\widetilde{\widetilde{w}}_{x}^{1}f\left(\frac{\pi}{k+1}\right)_{\beta},\end{aligned}$$

for $\beta\in[0,1]$.

Using Lemma 2 we obtain

$$\begin{aligned}\left|\widetilde{I}_{2}^{\circ}\right| &\ll \int_{\frac{\pi}{n+1}}^{\pi}\frac{\left|\psi_{x}\left(t\right)\right|}{t}\sum_{k=0}^{\tau}a_{n,k}dt=\sum_{m=1}^{n}\int_{\frac{\pi}{m+1}}^{\frac{\pi}{m}}\frac{\left|\psi_{x}\left(t\right)\right|}{t}\sum_{k=0}^{\tau}a_{n,k}dt \\ &\leq \sum_{m=1}^{n}\sum_{k=0}^{m+1}a_{n,k}\int_{\frac{\pi}{m+1}}^{\frac{\pi}{m}}\frac{\left|\psi_{x}\left(t\right)\right|}{t}dt\end{aligned}$$

$$\begin{aligned}&= \sum_{m=1}^{n}\sum_{k=1}^{m}a_{n,k}\int_{\frac{\pi}{m+1}}^{\frac{\pi}{m}}\frac{\left|\psi_{x}\left(t\right)\right|}{t}dt+\sum_{m=1}^{n}\left(a_{n,0}+a_{n,m+1}\right)\int_{\frac{\pi}{m+1}}^{\frac{\pi}{m}}\frac{\left|\psi_{x}\left(t\right)\right|}{t}dt \\ &= \left(\sum_{k=1}^{n}a_{n,k}\sum_{m=k}^{n}+a_{n,0}\sum_{m=1}^{n}+\sum_{m=1}^{n}a_{n,m+1}\right)\int_{\frac{\pi}{m+1}}^{\frac{\pi}{m}}\frac{\left|\psi_{x}\left(t\right)\right|}{t}dt \\ &\leq \left(\sum_{k=1}^{n}a_{n,k}\sum_{m=k}^{n}+a_{n,0}\sum_{m=1}^{n}+\sum_{m=1}^{n}a_{n,m+1}\right)\left\{\int_{\frac{\pi}{m+1}}^{\frac{\pi}{m}}\left[\frac{\left|\psi_{x}\left(t\right)\right|\sin^{\beta}\frac{t}{2}}{t^{1+\beta}}\right]dt\right\}\end{aligned}$$

$$\ll \left(\sum_{k=1}^{n}a_{n,k}\sum_{m=k}^{n}+a_{n,0}\sum_{m=1}^{n}\right)\left\{\int_{\frac{\pi}{m+1}}^{\frac{\pi}{m}}\left[\frac{|\psi_x(t)|\sin^{\beta}\frac{t}{2}}{t^{1+\beta}}\right]dt\right\}$$

$$+\sum_{m=1}^{n}a_{n,m+1}(m+1)^{\beta}\left\{\frac{m+1}{\pi}\int_{\frac{\pi}{m+1}}^{\frac{\pi}{m}}\left[|\psi_x(t)|\sin^{\beta}\frac{t}{2}\right]dt\right\}$$

$$\ll \sum_{k=0}^{n}a_{n,k}\left\{\int_{\frac{\pi}{n+1}}^{\frac{\pi}{k+1}}t^{-1-\beta}\frac{d}{dt}\left[\int_0^t|\psi_x(u)|\sin^{\beta}\frac{u}{2}du\right]dt\right\}$$

$$+\ (n+1)^{\beta}\sum_{m=1}^{n}a_{n,m+1}\widetilde{w}_x^1f\left(\frac{\pi}{m+1}\right)_{\beta}$$

$$\ll \sum_{k=0}^{n}a_{n,k}\left\{\left[\frac{1}{t^{1+\beta}}\int_0^t|\psi_x(u)|\sin^{\beta}\frac{u}{2}du\right]_{\frac{\pi}{n+1}}^{\frac{\pi}{k+1}}\right.$$

$$\left.+\ (1+\beta)\int_{\frac{\pi}{n+1}}^{\frac{\pi}{k+1}}\left[\frac{1}{t^{2+\beta}}\int_0^t|\psi_x(u)|\sin^{\beta}\frac{u}{2}du\right]dt\right\}$$

$$+\ (n+1)^{\beta}\sum_{m=1}^{n}a_{n,m+1}\widetilde{w}_x^1f\left(\frac{\pi}{m+1}\right)_{\beta}$$

$$\ll \sum_{k=0}^{n}a_{n,k}\left\{\left[(k+1)^{1+\beta}\int_0^{\frac{\pi}{k+1}}|\psi_x(u)|\sin^{\beta}\frac{u}{2}du\right]\right.$$

$$\left.+\ (1+\beta)\int_{\frac{\pi}{n+1}}^{\frac{\pi}{k+1}}\left[\frac{1}{t^{1+\beta}}\widetilde{w}_x^1f(t)_{\beta}\right]dt\right\}$$

$$+\ (n+1)^{\beta}\sum_{m=1}^{n}a_{n,m+1}\widetilde{w}_x^1f\left(\frac{\pi}{m+1}\right)_{\beta}$$

$$\ll \sum_{k=0}^{n}a_{n,k}\left\{\left[(k+1)^{\beta}\,\widetilde{w}_x^1f\left(\frac{\pi}{k+1}\right)_{\beta}\right]\right.$$

$$\left.+\ (1+\beta)\int_{\frac{\pi}{n+1}}^{\frac{\pi}{k+1}}\left[\frac{1}{t^{1+\beta}}\widetilde{w}_x^1f(t)_{\beta}\right]dt\right\}$$

$$+\ (n+1)^{\beta}\sum_{m=1}^{n}a_{n,m+1}\widetilde{w}_x^1f\left(\frac{\pi}{m+1}\right)_{\beta}$$

$$\ll \sum_{k=0}^{n}a_{n,k}\left\{\int_{k+1}^{n+1}\left[\frac{1}{t^{1-\beta}}\widetilde{w}_x^1f\left(\frac{\pi}{t}\right)_{\beta}\right]dt\right\}$$

$$+\ (n+1)^{\beta}\sum_{m=1}^{n}a_{n,m+1}\widetilde{w}_x^1f\left(\frac{\pi}{m+1}\right)_{\beta}$$

Since $\widetilde{\widetilde{w}}_x^p f(\delta)_\beta$ is nondecreasing majorant of $\widetilde{w}_x^p f(\delta)_\beta$ we have with $\beta > 0$

$$\ll \sum_{k=0}^{n} a_{n,k} \widetilde{\widetilde{w}}_x^1 f\left(\frac{\pi}{k+1}\right)_\beta \int_{k+1}^{n+1} \frac{1}{t^{1-\beta}} dt$$

$$+ \ (n+1)^\beta \sum_{m=1}^{n} a_{n,m+1} \widetilde{\widetilde{w}}_x^1 f\left(\frac{\pi}{m+1}\right)_\beta$$

$$\ll (n+1)^\beta \sum_{k=0}^{n} a_{n,k} \widetilde{\widetilde{w}}_x^1 f\left(\frac{\pi}{k+1}\right)_\beta .$$

Collecting these estimates we obtain the desired result. ■

4.2. Proof of Theorem 2

Let as usual

$$\widetilde{T}_{n,A,B} f(x) - \widetilde{f}\left(x, \frac{\pi}{n+1}\right) = \widetilde{I}_1 + \widetilde{I}_2^\circ$$

and

$$\left|\widetilde{T}_{n,A,B} f(x) - \widetilde{f}\left(x, \frac{\pi}{n+1}\right)\right| \le \left|\widetilde{I}_1\right| + \left|\widetilde{I}_2^\circ\right| .$$

The term $|\widetilde{I_1}|$ we can estimate by the same way as in the proof of Theorem 1. Therefore

$$\left|\widetilde{I}_1\right| \ll (n+1)^\beta \, \widetilde{w}_x^1 f\left(\frac{\pi}{n+1}\right)_\beta = (n+1)^\beta \, \widetilde{w}_x^1 f\left(\frac{\pi}{n+1}\right)_\beta \sum_{k=0}^{n} a_{n,n-k}$$

$$\le \ (n+1)^\beta \sum_{k=0}^{n} a_{n,n-k} \widetilde{\widetilde{w}}_x^1 f\left(\frac{\pi}{k+1}\right)_\beta .$$

Analogously to the above, by Lemma 2

$$\left|\widetilde{I}_2^\circ\right| \ll \int_{\frac{\pi}{n+1}}^{\pi} \frac{|\psi_x(t)|}{t} \sum_{k=n-\tau}^{n} a_{n,k} dt = \sum_{m=1}^{n} \int_{\frac{\pi}{m+1}}^{\frac{\pi}{m}} \frac{|\psi_x(t)|}{t} \sum_{k=0}^{\tau} a_{n,n-k} dt$$

$$\le \sum_{m=1}^{n} \sum_{k=0}^{m+1} a_{n,n-k} \int_{\frac{\pi}{m+1}}^{\frac{\pi}{m}} \frac{|\psi_x(t)|}{t} dt$$

$$\ll (n+1)^\beta \sum_{k=0}^{n} a_{n,n-k} \widetilde{\widetilde{w}}_x^1 f\left(\frac{\pi}{k+1}\right)_\beta .$$

Collecting these estimates we obtain the desired result. ■

4.3. Proof of Theorem 3

We start with the obvious relations

$$\begin{aligned}\widetilde{T}_{n,A}f(x)-\widetilde{f}(x) &= \frac{1}{\pi}\int_0^{\frac{\pi}{n+1}}\psi_x(t)\sum_{r=0}^{n}\sum_{k=0}^{r}a_{n,r}b_{r,k}\widetilde{D_k^{\circ}}(t)\,dt\\ &+\frac{1}{\pi}\int_{\frac{\pi}{n+1}}^{\pi}\psi_x(t)\sum_{r=0}^{n}\sum_{k=0}^{r}a_{n,r}b_{r,k}\widetilde{D_k^{\circ}}(t)\,dt\\ &= \widetilde{I_1^{\circ}}+\widetilde{I_2^{\circ}}\end{aligned}$$

and

$$\left|\widetilde{T}_{n,A}f(x)-\widetilde{f}(x)\right|\leq\left|\widetilde{I_1^{\circ}}\right|+\left|\widetilde{I_2^{\circ}}\right|.$$

By the Hölder inequality $\left(\frac{1}{p}+\frac{1}{q}=1\right)$, Lemma 1, (3) and (2), for $\beta<1-\frac{1}{p}$

$$\begin{aligned}\left|\widetilde{I_1^{\circ}}\right| &\ll \int_0^{\frac{\pi}{n+1}}\frac{|\psi_x(t)|}{t}dt\\ &\leq \left\{\int_0^{\frac{\pi}{n+1}}\left[\frac{|\psi_x(t)|}{\widetilde{w}_x(t)}\sin^{\beta}\frac{t}{2}\right]^p dt\right\}^{\frac{1}{p}}\left\{\int_0^{\frac{\pi}{n+1}}\left[\frac{\widetilde{w}_x(t)}{t\sin^{\beta}\frac{t}{2}}\right]^q dt\right\}^{\frac{1}{q}}\\ &\ll (n+1)^{\beta}\,\widetilde{w}_x\left(\frac{\pi}{n+1}\right)=(n+1)^{\beta}\,\widetilde{w}_x\left(\frac{\pi}{n+1}\right)\sum_{k=0}^{n}a_{n,k}\\ &\leq (n+1)^{\beta}\sum_{k=0}^{n}a_{n,k}\widetilde{w}_x\left(\frac{\pi}{k+1}\right).\end{aligned}$$

The term $\left|\widetilde{I_2^{\circ}}\right|$ we can estimate by the same way like in the proof of Theorem 1.

So

$$\left|\widetilde{I_2^{\circ}}\right|\ll(n+1)^{\beta}\sum_{k=0}^{n}a_{n,k}\widetilde{w}_x\left(\frac{\pi}{k+1}\right).$$

Collecting these estimates we obtain the desired result. ■

4.4. Proof of Theorem 4

Let as usual

$$\widetilde{T}_{n,A}f(x)-\widetilde{f}(x)=\widetilde{I_1^{\circ}}+\widetilde{I_2^{\circ}}$$

and

$$\left|\widetilde{T}_{n,A}f(x)-\widetilde{f}(x)\right|\leq\left|\widetilde{I_1^{\circ}}\right|+\left|\widetilde{I_2^{\circ}}\right|.$$

For the first term, by the proof of Theorem 3, we have

$$\left|\widetilde{I_1^{\circ}}\right| \ll (n+1)^{\beta}\,\widetilde{w}_x\left(\frac{\pi}{n+1}\right) \leq (n+1)^{\beta}\sum_{k=0}^{n} a_{n,n-k}\widetilde{w}_x\left(\frac{\pi}{k+1}\right),$$

where $\beta < 1-\frac{1}{p}$, and for the second one, by the proof of Theorem 2, we can write

$$\left|\widetilde{I_2^{\circ}}\right| \ll (n+1)^{\beta}\sum_{k=0}^{n} a_{n,n-k}\widetilde{w}_x\left(\frac{\pi}{k+1}\right).$$

Collecting these estimates we obtain the desired result. ■

4.5. Proofs of Theorems 5-8

The proofs are similar to these above and follows from the evident inequality

$$\left\|\widetilde{w}_{\cdot}^{p} f\left(\delta\right)_{\beta}\right\|_{L^p} \ll \widetilde{\omega}_{\beta} f\left(\delta\right)_{L^p}$$

immediately. ■

4.6. Proof of corollary

We have to verify the assumptions of Lemma 2 only.

For the first one we note that any non decreasing sequence belongs to the class $MRBVS$ and any non increasing sequence belongs to the class $MHBVS$. The second one follows from the following calculations

$$\left|\sum_{r=\mu}^{\nu}(1+\gamma)^{-r}\sum_{k=0}^{r}\binom{r}{k}\gamma^k\cos\frac{(2k+1)\,t}{2}\right|$$
$$=\left|\mathrm{Re}\sum_{r=\mu}^{\nu}(1+\gamma)^{-r}\sum_{k=0}^{r}\binom{r}{k}\gamma^k\exp\frac{(2k+1)\,it}{2}\right|$$
$$=\left|\mathrm{Re}\sum_{r=\mu}^{\nu}(1+\gamma)^{-r}\exp\frac{it}{2}\sum_{k=0}^{r}\binom{r}{k}\gamma^k\exp ikt\right|$$
$$=\left|\mathrm{Re}\sum_{r=\mu}^{\nu}(1+\gamma)^{-r}\exp\frac{it}{2}\left(1+\gamma\exp it\right)^r\right|$$
$$=\left|\mathrm{Re}\left[\exp\frac{it}{2}\sum_{r=\mu}^{\nu}\left(\frac{1+\gamma\exp it}{1+\gamma}\right)^r\right]\right|$$

$$= \left| \operatorname{Re}\left[\exp\frac{it}{2}\left(\frac{1+\gamma\exp it}{1+\gamma}\right)^{\mu}\frac{1-\left(\frac{1+\gamma\exp it}{1+\gamma}\right)^{\nu-\mu+1}}{1-\frac{1+\gamma\exp it}{1+\gamma}} \right] \right|$$

$$\leq \left| \frac{1-\left(\frac{1+\gamma\exp it}{1+\gamma}\right)^{\nu-\mu+1}}{1-\frac{1+\gamma\exp it}{1+\gamma}} \right|$$

$$\leq \frac{2}{\left|1-\frac{1+\gamma\exp it}{1+\gamma}\right|} = \frac{2(1+\gamma)}{|1+\gamma-1-\gamma\exp it|}$$

$$= \frac{2(1+\gamma)}{\gamma|1-\exp it|} = \frac{2(1+\gamma)}{\gamma\sqrt{(1-\cos t)^2+\sin^2 t}}$$

$$= \frac{2(1+\gamma)}{\gamma\sqrt{2-2\cos t}} = \frac{2(1+\gamma)}{\gamma\sqrt{2}\sqrt{2\sin^2\frac{t}{2}}} = \frac{1+\gamma}{\gamma\left|\sin\frac{t}{2}\right|} \leq \frac{1+\gamma}{\gamma}\frac{\pi}{t} \ll \tau.$$

Thus proof is complete. ■

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