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SOLVABILITY OF SEQUENCE SPACES EQUATIONS OF THE FORM $(E_a)_\Delta + F_x = F_b$

Abstract. Given any sequence $a = (a_n)_{n\geq 1}$ of positive real numbers and any set E of complex sequences, we write E_a for the set of all sequences $y = (y_n)_{n\geq 1}$ such that $y/a = (y_n/a_n)_{n\geq 1} \in E$; in particular, $s_a^{(c)}$ denotes the set of all sequences y such that y/a converges. For any linear space F of sequences, we have $F_x = F_b$ if and only if x/b and $b/x \in M(F, F)$. The question is: what happens when we consider the perturbed equation $\mathcal{E} + F_x = F_b$ where $\mathcal{E}$ is a special linear space of sequences? In this paper we deal with the perturbed sequence spaces equations (SSE), defined by $(E_a)_\Delta + s_x^{(c)} = s_b^{(c)}$ where $E = c_0$, or ℓ_p, $(p > 1)$ and Δ is the operator of the first difference defined by $\Delta_n y = y_n - y_{n-1}$ for all $n \geq 1$ with the convention $y_0 = 0$. For $E = c_0$ the previous perturbed equation consists in determining the set of all positive sequences $x = (x_n)_n$ that satisfy the next statement. The condition $y_n/b_n \to L_1$ holds if and only if there are two sequences u, v with $y = u + v$ such that $\Delta_n u/a_n \to 0$ and $v_n/x_n \to L_2$ $(n \to \infty)$ for all y and for some scalars L_1 and L_2. Then we deal with the resolution of the equation $(E_a)_\Delta + s_x^0 = s_b^0$ for $E = c$, or s_1, and give applications to particular classes of (SSE).

Key words: BK space, spaces of strongly bounded sequences, sequence spaces equations, sequence spaces equations with operator.

AMS Mathematics Subject Classification: 40C05, 46A45.

1. Introduction

We write ω for the set of all complex sequences $y = (y_n)_{n\geq 1}$, ℓ_∞, c and c_0 for the sets of all bounded, convergent and null sequences, respectively, also $\ell^p = \{y \in \omega : \sum_{k=1}^\infty |y_k|^p < \infty\}$ for $1 \leq p < \infty$. If y, $z \in \omega$, then we write $yz = (y_n z_n)_{n\geq 1}$. Let $U = \{y \in \omega : y_n \neq 0\}$ and $U^+ = \{y \in \omega : y_n > 0\}$. We write $z/u = (z_n/u_n)_{n\geq 1}$ for all $z \in \omega$ and all $u \in U$, in particular

$1/u = e/u$, where $e = \mathbf{1}$ is the sequence with $e_n = 1$ for all n. Finally, if $a \in U^+$ and E is any subset of ω, then we put $E_a = (1/a)^{-1} * E = \{y \in \omega : y/a \in E\}$. Let E and F be subsets of ω. Then the set $M(E, F) = \{y \in \omega : yz \in F \text{ for all } z \in E\}$ is called the *multiplier space of E and F*. In [1], the sets s_a, s_a^0 and $s_a^{(c)}$ were defined for positive sequences a by $(1/a)^{-1} * E$ and $E = \ell_\infty, c_0, c$, respectively. In [3] the sum $E_a + F_b$ and the product $E_a * F_b$ were defined where E, F are any of the symbols s, s^0, or $s^{(c)}$. Then in [6] the solvability was determined of sequences spaces inclusion equations $G_b \subset E_a + F_b$ where $E, F, G \in \{s^0, s^{(c)}, s\}$ and some applications were given to sequence spaces inclusions with operators.

In this paper we deal with the solvability of perturbed equations defined as follows. Let F be any linear space of sequences, and b be a positive sequence. It is known that the solutions of the equation $F_x = F_b$ where x is the unknown, are determined by $x \in cl^{M(F,F)}(b)$. Then we consider the perturbed equation $\mathcal{E} + F_x = F_b$, where $\mathcal{E}$ is a particular linear space of sequences. For example, the solutions of the equation $c_x = c$ are determined by $\lim_{n\to\infty} x_n = L > 0$. Then the perturbed equation defined by $c_a + c_x = c$, has the same solutions if and only if $a_n \to 0$ as n tends to infinity; then if $a_n \to l > 0$ as n tends to infinity the set of all its solutions is equal to c; finally, if $a \notin c$ the perturbed equation has no solutions, (cf. [7]). Here we extend some results given in [12], [6], [4], [5], [11], [7], [9], [10]. In [11] for given sequences a and b was determined the set of all positive sequences x for which $y_n/b_n \to l$ if and only if there are sequences u and v for which $y = u+v$ and $u_n/a_n \to 0$, $v_n/x_n \to l'$ $(n \to \infty)$ for all y and for some scalars l and l'. This statement is equivalent to the sequence spaces equation $s_a^0 + s_x^{(c)} = s_b^{(c)}$. In [7] we determined the set of all $x \in U^+$ such that for every sequence y, we have $y_n/b_n \to l$ if and only if there are sequences u and v with $y = u+v$ and $|u_n/a_n|^{1/n} \to 0$ and $v_n/x_n \to l'$ $(n \to \infty)$ for some scalars l and l'. This statement means $\mathbf{\Gamma}_a + s_x^{(c)} = s_b^{(c)}$, where $\mathbf{\Gamma}$ is the set of all entire sequences. So we are led to deal with special*sequence spaces equations (SSE), (resp. sequence spaces inclusion equations (SSIE))*, which are determined by an identity, (resp. inclusion), for which each term is a *sum* or*a sum of products of sets of the form* $(E_a)_T$*and* $\left(E_{f(x)}\right)_T$ where f maps U^+ to itself, E is a linear space of sequences, x is the unknown and T is a triangle. It can be found in [5] a solvability of the (SSE) $E_a + \left(s_x^{(c)}\right)_{B(r,s)} = s_x^{(c)}$ where $E = s$, s^0, or $s^{(c)}$ and x is the unknown. In [11] we determined the sets of all positive sequences x that satisfy each of the systems $s_a^0 + (s_x)_\Delta = s_b$, $s_x \supset s_b$ and $s_a + (c_x)_\Delta = c_b$, $c_x \supset c_b$. Then a resolution can be found of the (SSE) with operators defined by $(E_a)_{C(\lambda)D_\tau} + (c_x)_{C(\mu)D_\tau} = c_b$ with $E = c_0$, or ℓ_∞. Recently in [8] a study can be found on the (SSE) with operator

$(E_a)_{C(\lambda)C(\mu)} + (E_x)_{C(\lambda\sigma)C(\mu)} = E_b$, where $b \in \widehat{C}_1$ and E is any of the sets ℓ_∞, or c_0. For $E = c_0$ the resolution of this equation consists in determining the set of all $x \in U^+$ such that for every sequence y the condition $y_n/b_n \to 0$ $(n \to \infty)$ holds if and only if there are u, $v \in \omega$ such that $y = u + v$ and

$$(1) \qquad \frac{1}{\lambda_n a_n} \sum_{k=1}^{n} \left(\frac{1}{\mu_k} \sum_{i=1}^{k} u_i \right) \to 0 \text{ and}$$
$$\frac{1}{\lambda_n \sigma_n x_n} \sum_{k=1}^{n} \left(\frac{1}{\mu_k} \sum_{i=1}^{k} v_i \right) \to 0 \ (n \to \infty).$$

There is also a resolution of the (SSE) $(s_a)_{(C(\lambda)D_\tau)} + \left(s_x^0\right)_{(C(\mu)D_\tau)} = s_b^0$.

In this paper we deal with some classes of (SSE) with the operators of the form $(E_a)_\Delta + s_x^{(c)} = s_b^{(c)}$ where $E = c_0$, or ℓ^p, $(p > 1)$ and of the form $(E_a)_\Delta + s_x^0 = s_b^0$.

This paper is organized as follows. In Section 2 we recall some definitions and results on sequence spaces and matrix transformations. In Section 3 we recall general results on the multiplier $M(E, F)$ of some sequence spaces. In Section 4 we recall some results on the solvability of the equation $E + F_x = F_b$ in the general case. In Section 5 we determine the sets $M((E_a)_\Delta, F)$ and we deal with the (SSIE) $F_b \subset (E_a)_\Delta + F_x$. In Section 6 we apply the previous results to solve the (SSE) $(E_a)_\Delta + c_x = c_b$ where $E = c_0$, or ℓ^p, $(p > 1)$. In Section 7 we apply the results of Section 6 to solve special (SSE) of the form $(E_a)_\Delta + c_x = c_b$. Finally in Section 8 we deal with the (SSE) $(E_a)_\Delta + s_x^0 = s_b^0$ for $E = c$, or s_1.

2. Preliminaries and notations

An FK space is a *complete metric space*, for which convergence implies *coordinatewise convergence*. A *BK space* is a Banach space of sequences that is an *FK space*. A BK space E *is said to have AK if for every sequence* $y = (y_n)_n \in E$, *then* $y = \lim_{p\to\infty} \sum_{k=1}^{p} y_k e^{(k)}$, *where* $e^{(k)} = (0, ..., 1, ...)$, 1 *being in the* $k-th$ *position.*

For any given infinite matrix $A = (\mathbf{a}_{nk})_{n,k}$ we define the operators $A_n = (\mathbf{a}_{nk})_{k\geq 1}$ for any integer $n \geq 1$, by $A_n y = \sum_{k=1}^{\infty} \mathbf{a}_{nk} y_k$, where $y = (y_k)_{k\geq 1}$, and the series are assumed convergent for all n. So we are led to the study of the operator A defined by $Ay = (A_n y)_n$ mapping between sequence spaces. When A *maps* E *into* F, *where* E *and* F *are subsets of* ω, *we write* $A \in (E, F)$, (cf. [13]). It is well known that if E has AK, then the set $\mathcal{B}(E)$ of all *bounded linear operators* L mapping in E, with norm $\|L\| = \sup_{y\neq 0} (\|L(y)\|_E / \|y\|_E)$ satisfies the identity $\mathcal{B}(E) = (E, E)$. We will use the operator Δ of the first difference defined by $\Delta_n y = y_n - y_{n-1}$

for $n \geq 1$ with the convention $y_0 = 0$. It is well known that the operator Σ defined by $\Sigma_n y = \sum_{k=1}^{n} y_k$ for all n, is the inverse of Δ, that is, $\Delta(\Sigma y) = \Sigma(\Delta y) = y$ for all y. Let $U^+ \subset \omega$ be the set of all sequences $\mathbf{u} = (u_n)_n$ with $u_n > 0$ for all n. Then for any given sequence $u = (u_n)_n \in \omega$ we define the infinite diagonal matrix D_u by $[D_u]_{nn} = u_n$ for all n. It is interesting to rewrite the set E_u using a diagonal matrix. Let E be any subset of ω and $u \in U^+$ we have

$$E_u = D_u E = \{y = (y_n)_{n\geq 1} \in \omega \ : y/u \in E\}.$$

We will use the sets s_a^0, $s_a^{(c)}$, s_a and ℓ_a^p defined as follows (cf. [1]). For given $a \in U^+$ and $p \geq 1$ we put $D_a c_0 = s_a^0$, $D_a c = s_a^{(c)}$, $D_a \ell_\infty = s_a$, and $D_a \ell_p = \ell_a^p$. Each of the spaces $D_a E$, where $E \in \{c_0, c, \ell_\infty, \ell^p\}$ with $p > 1$, is a *BK space, and we have* $\|y\|_{s_a} = \sup_{n\geq 1}(|y_n|/a_n)$, and $\|y\|_{\ell_a^p} = \left(\sum_{k=1}^{\infty}(|y_k|/a_k)^p\right)^{1/p}$. Then and s_a^0 and ℓ_a^p have *AK*. If $a = (r^n)_n$ with $r > 0$, we write s_r, s_r^0 and $s_r^{(c)}$ for the sets s_a, s_a^0 and $s_a^{(c)}$ respectively. When $r = 1$, we obtain $s_1 = \ell_\infty$, $s_1^0 = c_0$ and $s_1^{(c)} = c$. Recall that $S_1 = (s_1, s_1)$ is a Banach algebra and $(c_0, s_1) = (c, \ell_\infty) = (s_1, s_1) = S_1$. We have $A \in S_1$ if and only if

$$\sup_n \left(\sum_{k=1}^{\infty} |\mathbf{a}_{nk}| \right) < \infty. \tag{2}$$

We will also use the well-known characterizations of (c_0, c_0) and (c_0, c). We have $A \in (c_0, c_0)$ if and only if (2) holds and $\lim_{n\to\infty} \mathbf{a}_{nk} = 0$ for all k; and we have $A \in (c_0, c)$ if and only if (2) holds and $\lim_{n\to\infty} \mathbf{a}_{nk} = l_k$ for some scalar l_k and for all k. For any subset F of ω, we write $F_A = \{y \in \omega \ : \ Ay \in F\}$. Let $cs = c_\Sigma$ denote the set of all convergent series. For any subset E of ω we write $AE = \{y \in \omega : \ y = Ax \text{ for some } x \in \omega\}$. We will use the well-known property, stated as follows. For any given triangles T and T', we have $T' \in (E_T, F)$ if and only if $T'T^{-1} \in (E, F)$ for any subsets E, $F \subset \omega$. It is also well known that $A \in (E, F_T)$ if and only if $TA \in (E, F)$.

3. The multipliers of some sets and matrix transformations

3.1. The multipliers of classical sets

First we need to recall some well known results. Let y and z be sequences and let E and F be two subsets of ω, we then write $yz = (y_n z_n)_{n\geq 1}$ and

$$M(E, F) = \{y \in \omega : \ yz \in F \text{ for all } z \in E\},$$

$M(E, F)$ is called the *multiplier space of* E *and* F. In the following we will use the next elementary results.

Lemma 1. *Let E, $\widetilde{E}$, F and $\widetilde{F}$ be arbitrary subsets of ω. Then*
(i) $M(E,F) \subset M\left(\widetilde{E},F\right)$ *for all* $\widetilde{E} \subset E$,
(ii) $M(E,F) \subset M\left(E,\widetilde{F}\right)$ *for all* $F \subset \widetilde{F}$.

Lemma 2. *Let a, $b \in U^+$ and let E and F be two subsets of ω. Then $D_aE \subset D_bF$ if and only if $a/b \in M(E,F)$.*

Lemma 3. *Let a, $b \in U^+$ and let E, $F \subset \omega$. Then $A \in (D_aE, D_bF)$ if and only if $D_{1/b}AD_a \in (E,F)$.*

By [14, Lemma 3.1, p. 648] and [16, Example 1.28, p. 157], we obtain the next result.

Lemma 4. *We have*
(i) $M(c,c_0) = M(\ell_\infty,c) = M(\ell_\infty,c_0) = c_0$ *and* $M(c,c) = c$;
(ii) $M(\chi,\ell_\infty) = M(c_0,\chi') = \ell_\infty$ *for* χ, $\chi' = c_0$, c, *or* ℓ_∞ .

4. On the (SSE) $E + F_x = F_b$

In this section we apply the previous results to the solvability of the (SSE) $E + F_x = F_b$ with $\mathbf{1} \in F$.

4.1. Regular sequence spaces equations

For $b \in U^+$ and for any subset F of ω, we denote by $cl^F(b)$ the equivalent class for the equivalence relation R_F defined by

$$xR_Fy \text{ if } D_xF = D_yF \text{ for } x,\ y \in U^+.$$

It can easily be seen that $cl^F(b)$ is the set of all $x \in U^+$ such that $x/b \in M(F,F)$ and $b/x \in M(F,F)$, (cf. [11]). We then have $cl^F(b) = cl^{M(F,F)}(b)$. For instance $cl^c(b)$ is the set of all $x \in U^+$ such that $D_xc = D_bc$, that is, $s_x^{(c)} = s_b^{(c)}$. This is the set of all sequences $x \in U^+$ such that $x_n \sim Cb_n$ $(n \to \infty)$ for some $C > 0$. In [11] we denote by $cl^\infty(b)$ the class $cl^{\ell_\infty}(b)$. Recall that $cl^\infty(b)$ is the set of all $x \in U^+$ such that $K_1 \le x_n/b_n \le K_2$ for all n and for some K_1, $K_2 > 0$.

Let X and Y be two linear spaces of sequences. Then the sum of X and Y defined by $Z = X + Y = \{x + y : x \in X \text{ and } y \in Y\}$, is a linear space of sequences. Let b be a positive sequence and F be a linear subspace of ω. As we have seen above the solutions of the equation $F_x = F_b$ are defined by $x \in cl^F(b)$. Then the question is: what are the solutions of the perturbed equation $E + F_x = F_b$, where E is a linear space of sequences? In this way we are led to consider the set $\mathcal{S}(E,F) = \{x \in U^+ : E + F_x = F_b\}$, where $b \in U^+$, and E is a linear subspace of ω.

Definition 1. *We say that $\mathcal{S}(E,F)$, (or the equation $E+F_x=F_b$), is regular if*

$$\mathcal{S}(E,F)=\begin{cases} cl^{M(F,F)}(b), & \text{if } 1/b\in M(E,F),\\ \varnothing, & \text{if } 1/b\notin M(E,F).\end{cases}$$

Note that $E+F_x=F_b$ is not regular in general. Indeed for $E=F=\ell_\infty$ we have $M(\ell_\infty,\ell_\infty)=\ell_\infty$ and if $1/b\in\ell_\infty\diagdown c_0$ and $s_a=s_1$ we have $\mathcal{S}(\ell_\infty,\ell_\infty)=s_b\cap U^+\neq cl^\infty(b)$, (cf. [12, Theorem 11, pp. 916-917]). In particular the solutions of the (SSE) $\ell_\infty+s_x=\ell_\infty$ are determined by $0<x_n\le M$ for all n and for some $M>0$. It is interesting to notice that by [7, Theorem 5.2, p. 108], the (SSE) $c+c_x=c_b$ is not regular, since $1/b\in c\diagdown c_0$ implies $\mathcal{S}(c,c)=c_b$.

In the following we will use the condition

$$\chi\subset\chi(D_\alpha)\quad\text{for all}\quad \alpha\in c(1), \tag{3}$$

where $\chi\subset\omega$ is any linear space, and $c(1)$ is the set of all sequences that tend to 1. It can easily seen that this condition is true for any of the spaces $F=c$, s_1. To state the next results we also need the next conditions:

$$\mathbf{1}\in F, \tag{4}$$

$$F\subset M(F,F). \tag{5}$$

We then recall the next result which is a direct consequence of [7, Theorem 5.1, pp. 106-107].

Lemma 5. *Let $b\in U^+$ and let E, F be two linear subspaces of ω. We assume F satisfies the conditions in (3), (4), (5), and that $M(E,F)\subset M(E,c_0)$. Then the set $S(E,F)$ is regular.*

5. Some results on the mutiplier $M((E_a)_\Delta,F)$ and on the (SSIE) $F_b\subset(E_a)_\Delta+F_x$

In this section we explicitly calculate the multiplier $M((E_a)_\Delta,F)$ where $E=c_0$, or ℓ_p and $F=c_0$, c, or s_1. Then we deal with the (SSIE) defined by $F_b\subset(E_a)_\Delta+F_x$ where E and F are linear spaces of sequences, and $E\subset s_1$ and $c_0\subset F\subset s_1$.

In the following we will use the factorable matrix $D_\alpha\Sigma D_\beta$, with α and $\beta\in\omega$ defined by $(D_\alpha\Sigma D_\beta)_{nk}=\alpha_n\beta_k$ for $k\le n$ for all n, the other entries being equal to zero.

5.1. The mutipliers $M\left(\left(E_a\right)_\Delta, F\right)$ where $E = c_0$, or ℓ_p and $F = c_0$, c, or s_1

Lemma 6. *Let $a \in U^+$ and let $p > 1$. Then*
(i) the condition $a \notin cs$ implies

$$M\left(\left(s_a^0\right)_\Delta, F\right) = s_{\left(\frac{1}{\sum_{k=1}^n a_k}\right)_n} \text{ for } F = c_0,\ c,\ \text{or } s_1. \tag{6}$$

(ii) The condition $a^q \notin cs$ where $q = p/(p-1)$ implies

$$M\left(\left(\ell_a^p\right)_\Delta, F\right) = s_{\left(\left(\sum_{k=1}^n a_k^q\right)^{-1/q}\right)_n} \text{ for } F = c_0,\ c,\ \text{or } s_1. \tag{7}$$

Proof. (i) We have $\alpha \in M\left(\left(s_a^0\right)_\Delta, c_0\right)$ if and only if $D_\alpha \Sigma D_a \in (c_0, c_0)$. By the characterization of (c_0, c_0) we have

$$|\alpha_n| \sum_{k=1}^n a_k \le K \text{ for all } n \text{ and for some } K > 0 \tag{8}$$

and

$$\alpha \in c_0. \tag{9}$$

But since $a \notin cs$ the condition in (8) implies (9) and we have $\alpha \in M\left(\left(s_a^0\right)_\Delta, c_0\right)$ if and only if (8) holds. This shows the identity in (6) for $F = c_0$. In a similar way the identity (6) for $F = s_1$ can easily be shown. From the inclusions $M\left(\left(s_a^0\right)_\Delta, c_0\right) \subset M\left(\left(s_a^0\right)_\Delta, c\right) \subset M\left(\left(s_a^0\right)_\Delta, s_1\right)$, we conclude that the identity in (6) holds for $F = c$.

(ii) We have $\alpha \in M\left(\left(\ell_a^p\right)_\Delta, c_0\right)$ if and only if $D_\alpha \Sigma D_a \in (\ell^p, c_0)$. By the characterization of (ℓ^p, c_0), (see for instance [16, Theorem 1.37, pp. 160-161]), we have

$$|\alpha_n|^q \sum_{k=1}^n a_k^q \le K \text{ for all } n \text{ and for some } K > 0 \tag{10}$$

and (9) holds. But since $a^q \notin cs$ the condition in (10) implies (9) and we have $\alpha \in M\left(\left(\ell_a^p\right)_\Delta, c_0\right)$ if and only if (10) holds. So we have shown that the identity in (7) holds for $F = c_0$. In a similar way the identity in (7) with $F = s_1$ can easily be shown. We conclude the proof using the inclusions $M\left(\left(\ell_a^p\right)_\Delta, c_0\right) \subset M\left(\left(\ell_a^p\right)_\Delta, c\right) \subset M\left(\left(\ell_a^p\right)_\Delta, s_1\right)$. ■

5.2. Some properties of the (SSIE) $F_b \subset \left(E_a\right)_\Delta + F_x$

Let E and F be two linear subspaces of ω. We define by $\mathcal{I}\left(\left(E_a\right)_\Delta, F\right)$ the set of all $x \in U^+$ such that $F_b \subset \left(E_a\right)_\Delta + F_x$. It can easily be seen that the sets $(E_a)_\Delta$ and F_x are linear spaces of sequences, and we have $z \in (E_a)_\Delta + F_x$ if and only if there are $\xi \in E$ and $f \in F$ such that $z_n = \sum_{k=1}^{n} a_k \xi_k + f_n x_n$. To simplify we will denote by $\mathcal{I}_E^F$ the set $\mathcal{I}\left((E_a)_\Delta, F\right)$.

In the following we will use the sequence $\sigma = (\sigma_n)_n$, defined for a, $b \in U^+$ by

$$\sigma_n = \frac{1}{b_n} \sum_{k=1}^{n} a_k.$$

For any given $b \in U^+$ we write $s_b^\bullet$ for the set of all sequences x such that $x_n \geq K b_n$ for all n and for some $K > 0$. Notice that we have $s_b \cap s_b^\bullet = cl^\infty(b)$. First we state the next lemma.

Lemma 7. *Let a, $b \in U^+$, and let E and F be two linear subspaces of s, that satisfy E, $F \subset s_1$ and $F \supset c_0$. Then we have*

(i) Assume $\sigma \in c_0$. Then

a) $\mathcal{I}_E^F \subset \mathcal{I}_{s_1}^{s_1}$, b) $\mathcal{I}_E^F \subset s_b^\bullet$.

(ii) Assume $a \in c_0$. Then we have $\mathcal{I}_E^F \subset s_1^\bullet$ for $b = e$.

Proof. (i) $a)$ Let $x \in \mathcal{I}_E^F$. Then we have $F_b \subset (E_a)_\Delta + F_x$ and since E, $F \subset s_1$ we obtain

$$(E_a)_\Delta + F_x = \Sigma D_a E + D_x F \subset (\Sigma D_a + D_x)\, s_1,$$

where $\Sigma D_a + D_x$ is a triangle, and

$$F_b \subset (s_1)_T\,, \tag{11}$$

with $T = (\Sigma D_a + D_x)^{-1}$. Now the condition in (11) implies $T \in (F_b, s_1)$, but we have, since $F \supset c_0$

$$(F_b, s_1) \subset \left(s_b^0, s_1\right) = (s_b, s_1)$$

and then $T \in (s_b, s_1)$. Finally we obtain $s_b \subset (s_a)_\Delta + s_x$. This shows the inclusion $\mathcal{I}_E^F \subset \mathcal{I}_{s_1}^{s_1}$.

(i) $b)$ As we have just seen $x \in \mathcal{I}_E^F$ implies $s_b \subset (s_a)_\Delta + s_x$ and there are $u \in (s_a)_\Delta$ and $v \in s_x$ such that $b = u + v$. Since $(s_a)_\Delta = (\Sigma D_a)\, s_1$ and $b \in s_b$, there are two sequences h, $k \in s_1$ such that $b_n = \sum_{k=1}^{n} a_k h_k + x_n k_n$ and

$$\frac{b_n}{x_n}\left(1 - \frac{1}{b_n}\sum_{k=1}^{n} a_k h_k\right) = k_n \text{ for all } n.$$

Then we have

$$\frac{1}{b_n}\left|\sum_{k=1}^{n} a_k h_k\right| \leq K\sigma_n \text{ for all } n \text{ and for some } K > 0,$$

and since $\sigma \in c_0$, we conclude $b/x \in s_1$, that is, $x \in s_b^{\bullet}$.

(ii) Let $x \in \mathcal{I}_E^F$. Then from (i) we obtain $s_1 \subset (s_a)_\Delta + s_x$. So the sequence $\xi = ((-1)^n)_n \in s_1$ can be written as $\xi = u+v$, where $u \in (s_a)_\Delta$ and $v \in s_x$. There are K_1 and $K_2 > 0$ such that $|\Delta_n u| = |u_n - u_{n-1}| \leq K_1 a_n$, $|\Delta_n v| = |v_n - v_{n-1}| \leq K_2 (x_n + x_{n-1})$ and

$$|(\Delta\xi)_n| = 2 = |\Delta_n u + \Delta_n v| \leq K_1 a_n + K_2 (x_n + x_{n-1}) \quad \text{for all} \quad n \geq 2.$$

Then we have

$$x_n + x_{n-1} \geq \frac{1}{K_2} (2 - K_1 a_n),$$

and since $a \in c_0$, there is $K_3 > 0$ such that $x_n + x_{n-1} \geq K_3$ for all sufficiently large n, and it can easily be shown $x \in s_1^{\bullet}$. We conclude $\mathcal{I}_E^F \subset s_1^{\bullet}$. This completes the proof. ∎

6. Solvability of sequence spaces equations of the form $(E_a)_\Delta + s_x^{(c)} = s_b^{(c)}$

In this section we solve the (SSE) $(E_a)_\Delta + s_x^{(c)} = s_b^{(c)}$ where $E = c_0$, or ℓ^p with $p > 1$. For instance, the (SSE) defined by $(s_a^0)_\Delta + s_x^{(c)} = s_b^{(c)}$ is equivalent to the statement: $y_n/b_n \to l_1$ $(n \to \infty)$ if and only if there are two sequences u, v with $y = u + v$ such that $(\Delta_n u)/a_n \to 0$ and $v_n/x_n \to l_2$ $(n \to \infty)$ for all y and for some scalars l_1 and l_2.

6.1. Solvability of the (SSE) $(s_a^0)_\Delta + s_x^{(c)} = s_b^{(c)}$ and $(\ell_a^p)_\Delta + s_x^{(c)} = s_b^{(c)}$ in the general case

For any given a, $b \in U^+$ we denote by $S((E_a)_\Delta, F)$ the set of all the solutions of the (SSE) defined by $(E_a)_\Delta + F_x = F_b$ where E and F are linear spaces.

Theorem 1. *Let a, $b \in U^+$. Then we have:*

(i) *The set $S_0^c = S((s_a^0)_\Delta, c)$ of all the solutions of the (SSE) $(s_a^0)_\Delta + s_x^{(c)} = s_b^{(c)}$ is determined in the following way.*

a) If $a \notin cs$, (that is, $\sum_k a_k = \infty$), then we have

$$S_0^c = \begin{cases} cl^c(b), & \text{if } \sigma \in s_1, \\ \varnothing, & \text{if } \sigma \notin s_1. \end{cases}$$

b) If $a \in cs$, then we have

$$S_0^c = \begin{cases} cl^c(b), & \text{if } \frac{1}{b} \in c_0, \\ cl^c(e), & \text{if } \frac{1}{b} \in c \setminus c_0, \\ \varnothing, & \text{if } \frac{1}{b} \notin c. \end{cases} \tag{12}$$

(ii) The set $S_p^c = S\left(\left(\ell_a^p\right)_\Delta, c\right)$ *with* $p > 1$*, of all the solutions of the (SSE)* $\left(\ell_a^p\right)_\Delta + s_x^{(c)} = s_b^{(c)}$ *is determined in the following way.*

a) If $a^q \notin cs$*, then*

$$S_p^c = \begin{cases} cl^c(b), & \text{if } \left(\frac{a_1^q + \ldots + a_n^q}{b_n^q}\right)_n \in s_1, \\ \varnothing, & \text{if } \left(\frac{a_1^q + \ldots + a_n^q}{b_n^q}\right)_n \notin s_1. \end{cases}$$

b) If $a^q \in cs$*, then* $S_p^c = S_0^c$ *defined by (12).*

Proof. (i) $a)$ First consider the case $a \notin cs$. By Lemma 6 we have $M\left(\left(s_a^0\right)_\Delta, c\right) = M\left(\left(s_a^0\right)_\Delta, c_0\right)$ and we can apply Lemma 5 where $1/b \in M\left(\left(s_a^0\right)_\Delta, c\right)$ if and only if $\sigma \in s_1$.

(i) $b)$ Case $a \in cs$. We deal with the 3 cases $\alpha)$ $1/b \notin c$, $\beta)$ $1/b \in c_0$ and $\gamma)$ $1/b \in c \setminus c_0$.

Case $\alpha)$. We have $S_0^c = \varnothing$. Indeed, assume there is $x \in S_0^c$, then we have $\left(s_a^0\right)_\Delta \subset s_b^{(c)}$ and $D_{1/b}\Sigma D_a \in (c_0, c)$. From the characterization of (c_0, c) we deduce $1/b \in c$, which is a contradiction. We conclude $S_0^c = \varnothing$.

Case $\beta)$. Let $1/b \in c_0$. Then $x \in S_0^c$ implies

$$x \in s_b^{(c)} \tag{13}$$

and $s_b^{(c)} \subset \left(s_a^0\right)_\Delta + s_x^{(c)}$. Using similar arguments as those in Lemma 7, we easily see that since $b \in s_b^{(c)}$ there are $\varepsilon \in c_0$ and $\varphi \in c$ such that

$$\frac{b_n}{x_n}\left(1 - \frac{1}{b_n}\sum_{k=1}^{n} a_k \varepsilon_k\right) = \varphi_n \text{ for all } n.$$

We deduce $b/x \in c$ since $\sigma \in c_0$. Using the condition in (13) we conclude that $x \in S_0^c$ implies $s_x^{(c)} = s_b^{(c)}$. Conversely, assume $s_x^{(c)} = s_b^{(c)}$. Then we have

$$\left(s_a^0\right)_\Delta + s_x^{(c)} = \left(s_a^0\right)_\Delta + s_b^{(c)} = s_b^{(c)}$$

since we have $\sigma \in s_1$ and $1/b \in c$. We conclude $S_0^c = cl^c(b)$.

Case $\gamma)$. Here we have $\lim_{n\to\infty} b_n = L > 0$ and $s_b^{(c)} = c$ and we are led to study the (SSE)

$$\left(s_a^0\right)_\Delta + s_x^{(c)} = c. \tag{14}$$

We have $x \in S_0^c$ implies $s_x^{(c)} \subset c$, that is, $x_n \to l$ $(n \to \infty)$. Then by Lemma 7 (ii) with $E = s_1$ and $F = c$, the condition $x \in S_0^c$ implies $x \in s_1^{\bullet}$. This means $l > 0$ and $S_0^c = cl^c(e)$. This completes the proof.

(ii) a) Case $a^q \notin cs$. By Lemma 6 we have $M\left((\ell_a^p)_\Delta, c\right) = M\left((\ell_a^p)_\Delta, c_0\right)$. Let $\alpha \in M\left((\ell_a^p)_\Delta, c\right)$. Then we can apply Lemma 5 where $1/b \in M\left((\ell_a^p)_\Delta, c\right)$ if and only if $\left(\left(\sum_{k=1}^n a_k^q\right)/b_n^q\right)_n \in s_1$.

(ii) b) Case $a^q \in cs$. As above we deal with the 3 cases α) $1/b \notin c$, β) $1/b \in c_0$ and γ) $1/b \in c \setminus c_0$. Case α). We have $x \in S_p^c$ implies $(\ell_a^p)_\Delta \subset s_b^{(c)}$ and $D_{1/b}\Sigma D_a \in (\ell^p, c)$. From the characterization of (ℓ^p, c) we deduce $1/b \in c$. We conclude that if $1/b \notin c$, then $S_p^c = \varnothing$. Case β). We have $x \in S_p^c$ implies

$$x \in s_b^{(c)} \tag{15}$$

and

$$s_b^{(c)} \subset (\ell_a^p)_\Delta + s_x^{(c)}. \tag{16}$$

Again using similar arguments that as those in Lemma 7, we easily see that since $b \in s_b^{(c)}$ there are $\lambda \in \ell^p$ and $\varphi \in c$ such that

$$\frac{b_n}{x_n}\left(1 - \frac{1}{b_n}\sum_{k=1}^n a_k \lambda_k\right) = \varphi_n \text{ for all } n.$$

From the characterization of (ℓ^p, c_0), (cf. [16, Theorem 1.37, pp. 160-161]) we have $D_{1/b}\Sigma D_a \in (\ell^p, c_0)$ since $1/b \in c_0$ and $a^q \in cs$ together imply $\left(b_n^{-q}\left(a_1^q + \ldots + a_n^q\right)\right)_n \in s_1$. We deduce

$$\left(D_{1/b}\Sigma D_a\right)_n \lambda = \frac{1}{b_n}\sum_{k=1}^n a_k \lambda_k \to 0 \quad (n \to \infty),$$

and $b/x \in c$. Using the condition in (15) we conclude $x \in S_p^c$ implies $s_x^{(c)} = s_b^{(c)}$. Conversely, assume $s_x^{(c)} = s_b^{(c)}$. Since $1/b \in c_0$ and $a^q \in cs$ together imply $D_{1/b}\Sigma D_a \in (\ell^p, c)$, (cf. [16, Theorem 1.37]), we successively obtain $(\ell_a^p)_\Delta \subset s_b^{(c)}$, $(\ell_a^p)_\Delta + s_x^{(c)} = (\ell_a^p)_\Delta + s_b^{(c)} = s_b^{(c)}$ and $x \in S_p^c$. We conclude $S_p^c = cl^c(b)$.

Case γ). Here we have $s_b^{(c)} = c$ and we are led to study the (SSE)

$$(\ell_a^p)_\Delta + s_x^{(c)} = c. \tag{17}$$

We have $x \in S_p^c$ implies $s_x^{(c)} \subset c$, that is, $x_n \to l\ (n \to \infty)$. Then by Lemma 7 *(ii)* with $E = \ell^p$ and $F = c$, the condition $x \in S_p^c$ implies $x \in \mathcal{I}_{\ell^p}^c$ and $x \in s_1^\bullet$. This means $l > 0$ and $S_p^c = cl^c(e)$. This completes the proof. ■

From Theorem 1 we immediately obtain the following.

Corollary 1. *Let $b \in U^{+}$ and let $\mathcal{S}$ be the set of all positive sequences x that satify the (SSE) $(c_0)_{\Delta} + s_x^{(c)} = s_b^{(c)}$. Then the next statements are equivalent, where,*

(i) $\mathcal{S} \neq \varnothing$,

(ii) $\mathcal{S} = cl^{c}(b)$,

(iii) $1/b \in s_{(1/n)_n}$.

We also obtain the following corollary, where $bv_p = \ell_{\Delta}^{p}$ is the set of all sequences of p-bounded variation.

Corollary 2. *Let $b \in U^{+}$ and $p > 1$, and denote by $\mathcal{S}_p$ the set of all positive sequences x that satify the (SSE) $bv_p + s_x^{(c)} = s_b^{(c)}$. Then the next statements are equivalent, where,*

(i) $\mathcal{S}_p \neq \varnothing$,

(ii) $\mathcal{S}_p = cl^{c}(b)$,

(iii) $1/b \in s_{(1/n^q)_n}$ *with* $q = p/(p-1)$.

6.2. The equation $s_x^{(c)} = s_b^{(c)}$ and the perturbed equation $\left(s_a^0\right)_{\Delta} + s_x^{(c)} = s_b^{(c)}$

In view of perturbed equations we can state the following. Let b be a positive sequence. The equation

$$s_x^{(c)} = s_b^{(c)} \tag{18}$$

is equivalent to $x_n/b_n \to l \ (n \to \infty)$ for some $l > 0$. Then the (SSE)

$$\left(s_a^0\right)_{\Delta} + s_x^{(c)} = s_b^{(c)} \tag{19}$$

can be considered as a perturbed equation of (18), and the question is: what are the conditions on a for which the perturbed equation and the (SSE) defined by (18) have the same solutions. As a direct consequence of Theorem 1 we obtain the next corollary, where $\overline{cs}$ is the complement of cs.

Corollary 3. *Let a, $b \in U^{+}$. Then we have*

(i) if $1/b \in c$, then the equations in (18) and (19) are equivalent if and only if $a \in cs \cup (\overline{cs} \cap (s_b)_{\Sigma})$.

(ii) If $1/b \notin c$, the perturbed equation in (19) has no solutions.

Proof. *(i)* is an immediate consequence of Theorem 1. *(ii)* Let $a \notin cs$. The condition $\sigma \in s_1$ should imply $1/b_n \leq K \left(\sum_{k=1}^{n} a_k\right)^{-1}$ for all n and for some $K > 0$ and $1/b \in c_0$, which is contradictory. So the perturbed equation in (19) has no solutions. The case $a \in cs$ is a direct consequence of Theorem 1 *(i)* *b*). ∎

Remark 1. We may state a similar result for the perturbed equation $(\ell_a^p)_\Delta + s_x^{(c)} = s_b^{(c)}$.

6.3. Cases when b, or b^q is in $\widehat{C_1}$.

Now we state the next elementary results, where $\widehat{C_1}$ is the set of all positive sequences x that satisfy $\left(x_n^{-1} \sum_{k=1}^n x_k\right)_n \in \ell_\infty$, (cf. [1]).

Corollary 4. *Let a, $b \in U^+$. Then we have*

(*i*) *Let $b \in \widehat{C_1}$. Then the set S_0^c of all positive $x \in U^+$ such that $\left(s_a^0\right)_\Delta + s_x^{(c)} = s_b^{(c)}$ is determined in the following way.*

a) Let $a \notin cs$. Then we have

$$S_0^c = \begin{cases} cl^c(b), & \text{if } a/b \in s_1, \\ \varnothing, & \text{if } a/b \notin s_1. \end{cases} \tag{20}$$

b) Let $a \in cs$. Then we have $S_0^c = cl^c(b)$.

(*ii*) *Let $p > 1$ and let $b^q \in \widehat{C_1}$ with $q = p/(p-1)$. Then the set S_p^c of all $x \in U^+$ such that $(\ell_a^p)_\Delta + s_x^{(c)} = s_b^{(c)}$ is determined in the following way.*

a) Let $a^q \notin cs$. Then $S_p^c = S_0^c$ defined by (20).

b) Let $a^q \in cs$. Then $S_p^c = S_0^c = cl^c(b)$.

Proof. (*i*) *a*) We have $\sigma \in s_1$ if and only if $a \in s_b(\Sigma)$. But by [1, Theorem 2.6, p. 1789] we have $b \in \widehat{C_1}$ if and only if $s_b(\Delta) = s_b$. This implies that $\Delta \in (s_b, s_b)$ is bijective and so is for $\Sigma = \Delta^{-1}$. So we have $s_b(\Sigma) = s_b$. We have $\sigma \in s_1$ if and only if $a/b \in s_1$, and we conclude by Theorem 1. This completes the proof of (*i*) *a*).

(*i*) *b*) comes from that fact that $b \in \widehat{C_1}$ implies $1/b \in c_0$, (see [1, Proposition 2.1, p. 1786]).

(*ii*) *a*) Here we have

$$\left(\frac{a_1^q + \ldots + a_n^q}{b_n^q}\right)_n \in s_1 \text{ if and only if } a^q \in s_{b^q}(\Sigma),$$

and as we have just seen we have $s_{b^q}(\Sigma) = s_{b^q}$ since $b^q \in \widehat{C_1}$. So we obtain (*ii*) *a*). (*ii*) *b*) we have $b^q \in \widehat{C_1}$ implies that there are $C > 0$ and $\gamma > 1$ such that $b_n^q \geq C\gamma^n$, for all n, (cf. [1, Proposition 2.1, p. 1786]). So we have $b_n \geq C^{1/q}\gamma^{n/q}$ for all n, and $1/b \in c_0$. We conclude by Theorem 1. This completes the proof. ∎

Remark 2. Notice that for $b \in \widehat{C_1}$ we have $S_0^c \neq \varnothing$ if and only if $a \in (cs \cup (\overline{cs} \cap s_b)) \cap U^+$.

Example 1. Consider the (SSE) with operator defined by

$$\left(s^0_{(n^{-\alpha})}\right)_\Delta + s^{(c)}_x = s^{(c)}_b, \tag{21}$$

with $0 < \alpha \leq 1$ and $b \in \widehat{C_1}$. We have $a/b = (n^{-\alpha}/b_n)_n$. By [1, Proposition 2.1, p 1786], $b \in \widehat{C_1}$ implies that there are $K > 0$ and $\gamma > 1$ such that $b_n \geq K\gamma^n$ for all n. This implies $a/b \in c_0$. We may apply Corollary 4 and conclude that the solutions of the (SSE) in (21) satisfy the condition $x_n \sim Cb_n$ $(n \to \infty)$ for some $C > 0$.

Example 2. Let $b^q \in \widehat{C_1}$. It can easily be shown that the solutions of the (SSE) $\left(\ell^p_{(n^\alpha)}\right)_\Delta + s^{(c)}_x = s^{(c)}_b$ are defined by $x_n \sim Cb_n$ $(n \to \infty)$ for some $C > 0$ and for all reals α.

Remark 3. Notice that if $a \in \widehat{C_1}$, the set $S^c_0 = S\left(s^0_a, c\right)$ is determined by Corollary 4 (i). Indeed, $a \in \widehat{C_1}$ implies $\left(s^0_a\right)_\Delta = s^0_a$, (cf. [1, Theorem 2.6, p. 1789]) and we conclude from the solvability of the (SSE) $s^0_a + s^{(c)}_x = s^{(c)}_b$, (cf. [11, Theorem 4.4, p. 7]).

Remark 4. If $\overline{\lim}_{n\to\infty}(a_{n-1}/a_n) < 1$, then $(\ell^p_a)_\Delta = \ell^p_a$, (cf. [2, Theorem 6.5 p. 3200]). So we have $S^c_p = cl^c(b)$ if $a/b \in s_1$, and $S^c_p = \varnothing$ if $a/b \notin s_1$.

7. Applications to particular (SSE) where a and b are classical sequences

7.1. On the (SSE) $\left(s^0_R\right)_\Delta + s^{(c)}_x = s^{(c)}_{\overline{R}}$

We obtain the next corollary whose the proof is elementary and is left to the reader.

Corollary 5. *Let R, $\overline{R} > 0$, and denote by $\mathrm{S}_{R,\overline{R}}$ the set of all positive sequences x that satify the (SSE) $\left(s^0_R\right)_\Delta + s^{(c)}_x = s^{(c)}_{\overline{R}}$. Then we obtain*

(i) Case $R < 1$. We have

$$\mathrm{S}_{R,\overline{R}} = \begin{cases} cl^c\left(\overline{R}\right), & \text{if } \overline{R} \geq 1, \\ \varnothing, & \text{if } \overline{R} < 1. \end{cases}$$

(ii) Case $R = 1$. We have

$$\mathrm{S}_{R,\overline{R}} = \begin{cases} cl^c\left(\overline{R}\right), & \text{if } \overline{R} > 1, \\ \varnothing, & \text{if } \overline{R} \leq 1. \end{cases}$$

(iii) Case $R > 1$. *We have*

$$\mathrm{S}_{R,\overline{R}} = \begin{cases} cl^c\left(\overline{R}\right), & \text{if } R \le \overline{R}, \\ \varnothing, & \text{if } R > \overline{R}. \end{cases}$$

As a direct consequence of the preceding we can state the next remark.

Remark 5. Let R, $\overline{R} > 0$. We have $\mathrm{S}_{R,\overline{R}} \neq \varnothing$ if and only if $R = 1 < \overline{R}$, or $1 < R \le \overline{R}$, or $R < 1 \le \overline{R}$. For instance the set of all positive sequences that satisfy the (SSE) $\left(s_R^0\right)_\Delta + s_x^{(c)} = s_2^{(c)}$ is non empty if and only if $R \le 2$.

7.2. On the (SSE) $\left(s_{1/r}^0\right)_\Delta + s_{1/x}^{(c)} = s_{(1/n^\alpha)_n}^{(c)}$ and $\left(s_{(n^{-\alpha})_n}^0\right)_\Delta + s_{1/x}^{(c)} = s_{1/r}^{(c)}$

7.2.1. The (SSE) $\left(s_{1/r}^0\right)_\Delta + s_{1/x}^{(c)} = s_{(1/n^\alpha)_n}^{(c)}$

Now we consider the next statement: the condition $n^\alpha y_n \to l_1$ $(n \to \infty)$ holds if and only if there are two sequences u, v, with $y = u + v$ such that

$$r^n\left(u_n - u_{n-1}\right) \to 0 \text{ and } x_n v_n \to l_2 \ \ (n \to \infty)$$

for some scalars l_1, l_2 and for all $y \in \omega$. The set of all x that satisfy the previous statement is equivalent to the (SSE)

$$\left(s_{1/r}^0\right)_\Delta + s_{1/x}^{(c)} = s_{(1/n^\alpha)_n}^{(c)}. \tag{22}$$

We obtain the following.

Corollary 6. *Let* $r > 0$ *and* α *be a real and let* $\overline{S}_{r,\alpha}$ *be the set of all positive sequences* x *that satisfy the (SSE) defined by (22). Then we obtain*

(i) if $r < 1$, *then* $\overline{S}_{r,\alpha} = \varnothing$.

(ii) If $r = 1$, *then we have*

$$\overline{S}_{r,\alpha} = \begin{cases} cl^c\left(\left(n^\alpha\right)_n\right), & \text{if } \alpha \le -1, \\ \varnothing, & \text{if } \alpha > -1. \end{cases}$$

(iii) If $r > 1$, *then we have*

$$\overline{S}_{r,\alpha} = \begin{cases} cl^c\left(\left(n^\alpha\right)_n\right), & \text{if } \alpha \le 0, \\ \varnothing, & \text{if } \alpha > 0. \end{cases}$$

Proof. Notice that $r<1$ implies $a=(r^{-n})_n \notin cs$. So the statement in (i) comes from the equivalence $\sigma_n \sim (1-r)^{-1} n^{\alpha} r^{-n}$ $(n\to\infty)$ and $\sigma \notin s_1$, for $r<1$. Let $r=1$. Then we have $\sigma_n \sim n^{\alpha+1}$ $(n\to\infty)$ and $\sigma \in s_1$ if and only if $\alpha \leq -1$, and we conclude by conclude by Theorem 1. This shows ii). Finally for $r>1$, we have $a \in cs$ and $1/b=(n^{\alpha})_n \in c$ if and only if $\alpha \leq 0$, and we conclude by Theorem 1. This completes the proof. ∎

We immediately deduce the next remark.

Remark 6. We have $\overline{S}_{r,\alpha} \neq \varnothing$ if and only if $r=1\leq\alpha$, or $r>1$ and $\alpha\leq 0$. We also have $\overline{S}_{r,0} \neq \varnothing$ if and only if $r>1$.

Example 3. Consider the statement: $y_n/n \to l_1$ $(n\to\infty)$ holds if and only if there are two sequences u, v, with $y=u+v$ such that $u_n-u_{n-1}\to 0$ and $x_n v_n \to l_2$ $(n\to\infty)$ for some scalars l_1, l_2 and for all $y\in\omega$. This statement holds if and only if $x\in\overline{S}_{1,-1}$, that is, $x_n/n\to L$ $(n\to\infty)$ with $L>0$.

7.2.2. On the (SSE) $\left(s^0_{(1/n^{\alpha})}\right)_{\Delta} + s^{(c)}_{1/x} = s^{(c)}_{1/r}$

As an application of Theorem 1 the following can easily be shown.

Corollary 7. *Let $r>0$, α be a real and $\overline{\overline{S}}_{\alpha,r}$ be the set of all $x\in U^+$ such that $\left(s^0_{(1/n^{\alpha})}\right)_{\Delta} + s^{(c)}_{1/x} = s^{(c)}_{1/r}$. The next statements are equivalent.*

(i) $\overline{\overline{S}}_{\alpha,r}\neq\varnothing$,

(ii) $\overline{\overline{S}}_{\alpha,r} = cl^c\left((r^n)_n\right)$,

(iii) $r\leq 1<\alpha$, *or* $\alpha\leq 1$ *and* $r<1$.

Proof. This result is a direct consequence of the equivalences

$$\int_n = \sum_{k=1}^{n} k^{-\alpha} \sim \frac{n^{1-\alpha}}{1-\alpha}\ (n\to\infty) \text{ if } \alpha\neq 1; \text{ and}$$
$$\int_n \sim \ln n\ (n\to\infty) \text{ if } \alpha = 1.$$

Then if $\alpha\neq 1$, we have $\left(r^n n^{1-\alpha}\right)_n \in \ell_\infty$ if and only if $r\leq 1<\alpha$, or α and $r<1$; and if $\alpha=1$ we have $(r^n \ln n)_n \in \ell_\infty$ if and only if $r<1$. This concludes the proof. ∎

Example 4. For $r=1/2$ we have $\overline{\overline{S}}_{\alpha,1/2} = cl^c\left((2^{-n})_n\right)$ for all reals α.

7.3. On the (SSE) $\left(s^0_{(1/n^{\alpha})_n}\right)_{\Delta} + s^{(c)}_{1/x} = s^{(c)}_{(1/n^{\beta})_n}$

Now let $\mathcal{S}_{\alpha,\beta}$ for all reals α and β, be the set of all positive sequences $x=(x_n)_n$ that satisfy the following statement. For every y the condition

$n^{\beta}y_n \to l_1$ $(n \to \infty)$ holds if and only if there are two sequences u, v, with $y = u + v$ such that $n^{\alpha}(u_n - u_{n-1}) \to 0$ and $x_n v_n \to l_2$ $(n \to \infty)$ for some scalars l_1, l_2. This statement leads to the solvability of the (SSE) defined by $\left(s^0_{(1/n^{\alpha})_n}\right)_{\Delta} + s^{(c)}_{1/x} = s^{(c)}_{\left(1/n^{\beta}\right)_n}$. We obtain the next result which can be obtained by similar arguments as those used above.

Corollary 8. *Let α, β be reals. Then*
(i) if $\alpha < 1$, then we have

$$\mathcal{S}_{\alpha,\beta} = \begin{cases} cl^c\left(\left(n^{\beta}\right)_n\right), & \text{if } \beta \le \alpha - 1, \\ \varnothing, & \text{if } \beta > \alpha - 1. \end{cases}$$

(ii) If $\alpha = 1$, then we have

$$\mathcal{S}_{\alpha,\beta} = \begin{cases} cl^c\left(\left(n^{\beta}\right)_n\right), & \text{if } \beta < 0, \\ \varnothing, & \text{if } \beta \ge 0. \end{cases}$$

(iii) If $\alpha > 1$, then we have

$$\mathcal{S}_{\alpha,\beta} = \begin{cases} cl^c\left(\left(n^{\beta}\right)_n\right), & \text{if } \beta \le 0, \\ \varnothing, & \text{if } \beta > 0. \end{cases}$$

Corollary 9. *$\mathcal{S}_{\alpha,\beta} \neq \varnothing$ if and only if $\beta \le \alpha - 1 < 0$, or $\alpha = 1$ and $\beta < 0$, or $\alpha > 1$ and $\beta \le 0$.*

Example 5. As a direct consequence of the preceding, notice that the (SSE) $\left(s^0_{(n^{-\alpha})_n}\right)_{\Delta} + s^{(c)}_{1/x} = c$ is equivalent to $x_n \to L$ $(n \to \infty)$ with $L > 0$, for all $\alpha > 1$.

7.4. On the (SSE) $\left(\ell^p_{(n^{-\alpha})_n}\right)_{\Delta} + s^{(c)}_{1/x} = s^{(c)}_{\left(n^{-\beta}\right)_n}$

In the next corollary we deal with the statement for reals α and β and $p > 1$: the condition $n^{\beta}y_n \to l_1$ holds if and only if there are u, $v \in \omega$ with $y = u + v$ such that $\sum_{k=1}^{\infty}\left(k^{\alpha}\left|u_k - u_{k-1}\right|\right)^p < \infty$ and $x_n v_n \to l_1$ $(n \to \infty)$ for all $y \in \omega$, and for some scalars l_1, l_2. This is equivalent to the (SSE)

$$\left(\ell^p_{(n^{-\alpha})_n}\right)_{\Delta} + s^{(c)}_{1/x} = s^{(c)}_{\left(n^{-\beta}\right)_n}. \tag{23}$$

We obtain the next result.

Corollary 10. *Let α and β be reals and let S^c_p be the set of all the solutions of the (SSE) determined by (23). Then we have* (*i*) *if* $\alpha q \geq 1$, *then we have*

$$S_p^c = \begin{cases} cl^c\left(\left(n^\beta\right)_n\right), & \text{if } \beta < 0, \\ \varnothing, & \text{if } \beta \geq 0. \end{cases}$$

(*ii*) *If* $\alpha q < 1$, *then we have*

$$S_p^c = \begin{cases} cl^c\left(\left(n^\beta\right)_n\right), & \text{if } \alpha - \beta \geq \frac{1}{q}, \\ \varnothing, & \text{if } \alpha - \beta < \frac{1}{q}. \end{cases}$$

Proof. The proof comes from the fact that

$$\sigma_n \sim \frac{n^{(\beta-\alpha)q+1}}{1-\alpha q} \quad (n \to \infty)$$

if $\alpha q \neq 1$; and $\sigma_n \sim n^{\beta q} \ln n$ $(n \to \infty)$ if $\alpha q = 1$. Then it can easily be seen that $\sigma \in \ell_\infty$ if and only if $\alpha - \beta \geq 1/q$ for $\alpha q < 1$, or $\beta < 0$ for $\alpha q \geq 1$. We conclude by Theorem 1. ■

We deal for reals β with the statement: $n^\beta y_n \to l_1$ if and only if $y = u+v$ with

$$\sum_{k=1}^{\infty} \left(\frac{|u_k - u_{k-1}|}{k}\right)^2 < \infty \text{ and } x_n v_n \to l_2 \ (n \to \infty) \text{ for all } y$$

and for some scalars l_1, l_2. This statement is equivalent to the (SSE) defined by $\left(\ell^2_{(n)_n}\right)_\Delta + s_{1/x}^{(c)} = s_{\left(n^{-\beta}\right)_n}^{(c)}$ and this (SSE) has solutions if and only if $\beta \leq -3/2$.

Example 6. Notice that the set of all the solutions of the (SSE) defined by $\left(\ell^2_{\left(1/\sqrt{n}\right)_n}\right)_\Delta + s_x^{(c)} = s_{(\ln n)_n}^{(c)}$ are determined by $\lim_{n\to\infty} (x_n / \ln n) > 0$. This result comes from the equivalence $\sum_{k=1}^n \left(1/\sqrt{k}\right)^2 \sim \ln n$ $(n \to \infty)$.

8. Solvability of the (SSE) of the form $(E_a)_\Delta + s_x^0 = s_b^0$

In this section we solve the (SSE) $(E_a)_\Delta + s_x^0 = s_b^0$ where $E = c$, or ℓ_∞. For $E = c$, the solvability of the previous (SSE) consists in determining the set of all positive sequences $x = (x_n)_n$ that satisfy the next statement. For every y the condition $y_n/b_n \to 0$ $(n \to \infty)$ holds if and only if there are two sequences u, v, with $y = u + v$ such that $(u_n - u_{n-1})/a_n \to l$ and $v_n/x_n \to 0$ $(n \to \infty)$ for some scalar l. Here also we may consider the (SSE)

$\left(s_a^{(c)}\right)_\Delta + s_x^0 = s_b^0$ as a perturbed equation of the equation $s_x^0 = s_b^0$, which is equivalent to $K_1 \leq x_n/b_n \leq K_2$ for all n and for some K_1, $K_2 > 0$. We obtain the equivalence of these two equations under some conditions on a and b.

8.1. Solvability of the (SSE) $(E_a)_\Delta + s_x^0 = s_b^0$ where $E = c$, or ℓ_∞ in the general case

To prove the next result we need a lemma.

Lemma 8. *Let $b \in U^+$ and let T be a triangle. Then we have*

$$s_b^0 \subset Ts_1 \text{ if and only if } s_b \subset Ts_1.$$

Proof. We have $s_b^0 \subset Ts_1$ if and only if

$$T^{-1}D_b \in (c_0, s_1)\,. \tag{24}$$

Since $(c_0, s_1) = (s_1, s_1)$ the condition in (24) is equivalent to $s_b \subset Ts_1$. This concludes the proof. ■

Theorem 2. *The set S_E^0 of all the solutions of the (SSE) $(E_a)_\Delta + s_x^0 = s_b^0$ where $E = c$, or ℓ_∞ is determined by*

$$S_E^0 = \begin{cases} cl^\infty(b)\,, & \text{if } \sigma \in c_0, \\ \varnothing, & \text{if } \sigma \notin c_0. \end{cases}$$

Proof. Let $x \in S_E^0$. Then we have the inclusion $(E_a)_\Delta + s_x^0 \subset s_b^0$. This implies $(E_a)_\Delta \subset s_b^0$ and $D_{1/b}\Sigma D_a \in (E, c_0)$. This implies

$$D_{1/b}\Sigma D_a \in (c, c_0)$$

since $E \supset c$ and

$$\sigma_n \to 0 \;\; (n \to \infty)\,. \tag{25}$$

Now we have $s_x^0 \subset s_b^0$ and

$$x \in s_b. \tag{26}$$

Then we consider the (SSIE) defined by

$$s_b^0 \subset (E_a)_\Delta + s_x^0. \tag{27}$$

The (SSIE) in (27) with $E \subset s_1$ implies $s_b^0 \subset Ts_1$ with $T = \Sigma D_a + D_x$. So the inclusion in (27) implies

$$s_b \subset (s_a)_\Delta + s_x$$

by Lemma 8. Since $(s_a)_\Delta = \Sigma s_a$ there are h, $k \in s_1$ such that

$$\frac{b_n}{x_n}\left(1 - \frac{1}{b_n}\sum_{k=1}^{n} a_k h_k\right) = k_n \text{ for all } n.$$

We have

$$\left|\frac{1}{b_n}\sum_{k=1}^{n} a_k h_k\right| \leq K\sigma_n \text{ for all } n \text{ and for some } K > 0,$$

and from the condition in (25) we deduce $b/x \in s_1$. Using the condition in (26) we conclude $x \in S_E^0$ implies $x \in cl^\infty(b)$. Conversely, assume $x \in cl^\infty(b)$ and (25) holds. Since $1/b \in c_0$, we have $(E_a)_\Delta \subset s_b^0$ for $E = c_0$, or s_1, we obtain

$$(E_a)_\Delta + s_x^0 = (E_a)_\Delta + s_b^0 = s_b^0,$$

and $x \in S_E^0$. We conclude $S_E^0 = cl^\infty(b)$. ∎

For $a = e$ we easily obtain the next result.

Corollary 11. (*i*) *The set $S(E_\Delta, c_0)$ of all the solutions of the (SSE) $E_\Delta + s_x^0 = s_b^0$ where $E = c$, or ℓ_∞ is determined by*

$$S(E_\Delta, c_0) = \begin{cases} cl^\infty(b) & \text{if } (n/b_n)_n \in c_0, \\ \varnothing & \text{if } (n/b_n)_n \notin c_0. \end{cases}$$

Example 7. The equation $E_\Delta + s_x^0 = s_{(n^\alpha)_n}^0$ where $E = c$, or ℓ_∞, has solutions if and only if $\alpha > 1$. So the equation $E_\Delta + s_x^0 = c_0$ has no solution, and the solutions of the equation $E_\Delta + s_x^0 = s_{(n^2)_n}^0$ are determined by $K_1 n^2 \leq x_n \leq K_2 n^2$ for all n and for some K_1, $K_2 > 0$.

8.2. Applications to the solvability of (SSE) of the form $(E_a)_\Delta + s_x^0 = s_b^0$ for particular sequences a and b

Consider the (SSE) determined by

$$\left(s_{(n^{-\alpha})_n}^{(c)}\right)_\Delta + s_{1/x}^0 = s_{(n^{-\beta})_n}^0 \tag{28}$$

$$\left(s_R^{(c)}\right)_\Delta + s_x^0 = s_{\overline{R}}^0, \tag{29}$$

$$\left(s_{(n^{-\alpha})_n}^{(c)}\right)_\Delta + s_{1/x}^0 = s_{1/R}^0, \tag{30}$$

$$\left(s_{1/R}^{(c)}\right)_\Delta + s_{1/x}^0 = s_{(n^{-\beta})_n}^0, \tag{31}$$

with reals α and β and R, $\overline{R} > 0$. We obtain the following.

Proposition 1. (*i*) *The (SSE) defined in (28) has solutions if and only if $\beta < \alpha - 1 < 0$, or $\alpha \geq 1$ and $\beta < 0$.*

(*ii*) *The (SSE) defined in (29) has solutions if and only if $R \leq 1 < \overline{R}$, or $1 < R < \overline{R}$.*

(*iii*) *The (SSE) defined in (30) has solutions if and only if $R < 1 < \alpha$, or $R < \alpha = 1$, or α and $R < 1$.*

(*iv*) *The (SSE) defined in (31) has solutions if and only if $R = 1$ and $\beta < -1$, or $R > 1$ and $\beta < 0$.*

Proof. (*i*) It can easily be seen that $\sigma_n \sim n^{\beta-\alpha+1}/(1-\alpha)$ $(n \to \infty)$ for $\alpha < 1$; $\sigma_n \sim n^{\beta} \ln n$ $(n \to \infty)$ for $\alpha = 1$; and $\sigma_n \sim Kn^{\beta}$ $(n \to \infty)$ for $\alpha > 1$. We conclude the (SSE) defined by (28) has solutions if and only if $\sigma_n = o(1)$ $(n \to \infty)$, that is, for $\beta < \alpha - 1 < 0$, or $\alpha \geq 1$ and $\beta < 0$.

(*ii*) We have

$$\sigma_n \sim \frac{1}{R-1}\frac{R^{n+1}}{\overline{R}^n} \ (n \to \infty) \text{ for } R > 1;$$

$$\sigma_n \sim \frac{R}{1-R}\frac{1}{\overline{R}^n} \ (n \to \infty) \text{ for } R < 1;$$

and

$$\sigma_n \sim \frac{n}{\overline{R}^n} \ (n \to \infty) \text{ for } R = 1;$$

and we conclude as above.

(*iii*) This result is a direct consequence of the following. We have $\sigma_n \sim KR^n$ $(n \to \infty)$ for $\alpha > 1$ and for some $K > 0$; $\sigma_n \sim R^n \ln n$ $(n \to \infty)$ for $\alpha = 1$; and $\sigma_n \sim n^{1-\alpha}R^n/(1-\alpha)$ $(n \to \infty)$ for $\alpha < 1$. We conclude by Theorem 2.

(*iv*) Here we have $\sigma_n \sim n^{\beta}/(R-1)$ $(n \to \infty)$ for $R > 1$; $\sigma_n \sim n^{\beta+1}$ $(n \to \infty)$ for $R = 1$; and $\sigma_n \sim n^{\beta}R^{-n}/(1-R)$ $(n \to \infty)$ for $R < 1$ and we conclude by Theorem 2. ■

Example 8. The (SSE) $\left(s_{1/2}^{(c)}\right)_{\Delta} + s_x^0 = s_{\overline{R}}^0$ has solutions if and only if $\overline{R} > 1$.

Example 9. Let τ, τ' be reals. Then the system of (SSE) defined by

$$\begin{cases} c_{\Delta} + s_x^0 = s^0_{(n^{\tau})_n}, \\ \left(s_{1/2}^{(c)}\right)_{\Delta} + s_x^0 = s^0_{\left(n^{\tau'}\right)_n}, \end{cases}$$

where x is the unknown, has solutions if and only if $\tau = \tau' > 1$. Then x is a solution of the system if and only if $x_n \sim Cn^{\tau}$ $(n \to \infty)$ for some $C > 0$.

This is a direct consequence of Proposition 1 (iv) and of the elementary fact that $s_{(n^\tau)_n} = s_{\left(n^{\tau'}\right)_n}$ if and only if $\tau = \tau'$.

Example 10. Let S_c^0 be the set of all positive sequences that satisfy the following statement. For every y the condition $y_n/n \to 0$ $(n \to \infty)$ holds if and only if there are two sequences u, v, with $y = u + v$ such that $\sqrt{n}\,(u_n - u_{n-1}) \to L$ and $x_n v_n \to 0$ $(n \to \infty)$ for some scalar L. By Proposition 1 (i), we have $x \in S_c^0$ if and only if $K_1/n \le x_n \le K_2/n$ for all n and for some K_1, $K_2 > 0$.

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