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SOME NEW CLASSES OF GENERALIZED HERMITE-BASED APOSTOL-EULER AND APOSTOL-GENOCCHI POLYNOMIALS

Abstract. In this paper, we introduce a new class of generalized Apostol-Hermite-Euler polynomials and Apostol-Hermite-Genocchi polynomials and derive some implicit summation formulae by applying the generating functions. These results extend some known summations and identities of generalized Hermite-Euler polynomials studied by Dattoli et al, Kurt and Pathan.

Key words: Hermite polynomials, Apostol-Hermite-Bernoulli polynomials, Apostol-Hermite-Euler polynomials, Apostol-Hermite-Genocchi polynomials, summation formulae.

AMS Mathematics Subject Classification: 05A10, 11B65, 28B99, 11B68.

1. Introduction

The 2-variable Kampe de Feriet generalization of the Hermite polynomials [2] reads

$$H_n(x,y) = n! \sum_{r=0}^{[\frac{n}{2}]} \frac{y^r x^{n-2r}}{r!(n-2r)!}. \tag{1}$$

These polynomials are usually defined by the generating function

$$e^{xt+yt^2} = \sum_{n=0}^{\infty} H_n(x,y) \frac{t^n}{n!} \tag{2}$$

and reduce to the ordinary Hermite polynomials $H_n(x)$ when $y=-1$ and x is replaced by $2x$.

The classical Bernoulli polynomials $B_n(x)$, the classical Euler polynomials $E_n(x)$ and the classical Genocchi polynomials $G_n(x)$, together with their familiar generalizations $B_n^{(\alpha)}(x)$, $E_n^{(\alpha)}(x)$ and $G_n^{(\alpha)}(x)$ of (real or complex) order α are usually defined by means of the following generating functions (see for details [1], [21], pp. 532-533 and [23], p. 61; see also [24] and the references cited therein):

$$\left(\frac{t}{e^t-1}\right)^\alpha e^{xt} = \sum_{n=0}^{\infty} B_n^{(\alpha)} \frac{t^n}{n!} \quad (|\, t \,|< 2\pi; 1^\alpha = 1) \tag{3}$$

$$\left(\frac{2}{e^t+1}\right)^\alpha e^{xt} = \sum_{n=0}^{\infty} E_n^{(\alpha)} \frac{t^n}{n!} \quad (|\, t \,|< \pi; 1^\alpha = 1) \tag{4}$$

and

$$\left(\frac{2t}{e^t+1}\right)^\alpha e^{xt} = \sum_{n=0}^{\infty} G_n^{(\alpha)} \frac{t^n}{n!} \quad (|\, t \,|< \pi; 1^\alpha = 1). \tag{5}$$

So that obviously the classical Bernoulli polynomials $B_n(x)$, the classical Euler polynomials $E_n(x)$ and the classical Genocchi polynomials $G_n(x)$ are given respectively by

$$B_n(x) = B_n^{(1)}(x), E_n(x) = E_n^{(1)}(x).$$

and

$$G_n(x) = G_n^{(1)}(x) \quad (n \in N). \tag{6}$$

For the classical Bernoulli numbers B_n, the classical Euler numbers E_n and the classical Genocchi numbers G_n

$$B_n^1(0) = B_n(0) = B_n, \quad E_n^1(0) = E_n(0) = E_n$$

and

$$G_n^1(0) = G_n(0) = G_n, \tag{7}$$

respectively.

In particular, Luo and Srivastava [8, 9] introduced the generalized Apostol-Bernoulli polynomials $B_n^{(\alpha)}(x;\lambda)$ of order $\alpha \in \mathcal{C}$; Luo [11, 12, 13] introduced the generalized Apostol-Euler polynomials $E_n^{(\alpha)}(x;\lambda)$ of order $\alpha \in \mathcal{C}$ and the generalized Apostol-Genocchi polynomials $G_n^{(\alpha)}(x;\lambda)$ of order $\alpha \in \mathcal{C}$ in [10, 15, 16, 17]. These polynomials are defined, respectively as follows.

Definition 1. *The generalized Apostol-Bernoulli polynomials $B_n^{(\alpha)}(x)$ of order α are defined by means of the generating function*

$$\left(\frac{t}{\lambda e^t - 1}\right)^\alpha = \sum_{n=0}^{\infty} B_n^{(\alpha)}(x;\lambda)\frac{t^n}{n!}, \tag{8}$$
$$(|\, t \,|< 2\pi,\ \ if\ \ \lambda = 1;\ \ |\, t \,|<|\log\lambda\,|,\ \ if\ \ \lambda \neq 1; 1^\alpha = 1)$$

with

$$B_n^{(\alpha)}(x) = B_n^{(\alpha)}(x;1)$$

and

$$B_n^{(\alpha)}(\lambda) = B_n^{(\alpha)}(0;\lambda) \tag{9}$$

where $B_n^{(\alpha)}(\lambda)$ denotes the so called Apostol-Bernoulli numbers of order α.

Definition 2. *The generalized Apostol-Euler polynomials $E_n^{(\alpha)}(x)$ of order α are defined by means of the generating function*

$$\left(\frac{2}{\lambda e^t + 1}\right)^\alpha = \sum_{n=0}^{\infty} E_n^{(\alpha)}(x;\lambda)\frac{t^n}{n!},\ \ (|\, t \,|<|\log(-\lambda)\,|< \pi, 1^\alpha = 1) \tag{10}$$

with

$$E_n^{(\alpha)}(x) = E_n^{(\alpha)}(x;1)$$

and

$$E_n^{(\alpha)}(\lambda) = E_n^{(\alpha)}(0;\lambda) \tag{11}$$

where $E_n^{(\alpha)}(\lambda)$ denotes the so called Apostol-Euler numbers of order α.

Definition 3. *The generalized Apostol-Genocchi polynomials $G_n^{(\alpha)}(x)$ of order α are defined by means of the generating function*

$$\left(\frac{2t}{\lambda e^t + 1}\right)^\alpha = \sum_{n=0}^{\infty} G_n^{(\alpha)}(x;\lambda)\frac{t^n}{n!},\ \ (|\, t \,|<|\log(-\lambda)\,|< \pi, 1^\alpha = 1) \tag{12}$$

with

$$G_n^{(\alpha)}(x) = G_n^{(\alpha)}(x;1),\quad G_n^{(\alpha)}(\lambda) = G_n^{(\alpha)}(0;\lambda) \tag{13}$$

where $G_n^{(\alpha)}(\lambda)$ denotes the so called Apostol-Genocchi numbers of order α.

Recently, Tremblay et al [25, 26] studied a new family of generalized Apostol-Bernoulli, Apostol-Euler and Apostol-Genocchi polynomials of order α in the following form.

Definition 4. *For arbitrary real or complex parameter α and for $a, c \in \Re^{+}$, the generalized Apostol-Bernoulli polynomials $B_n^{[m-1,\alpha]}(x; a, c, \lambda)$, $m \in N$, $\lambda \in \mathcal{C}$ are defined in a suitable neighborhood of $t = 0$ with $\mid t \log(a) \mid < 2\pi$, if $\lambda = 1$ or with $\mid t \log(a) <\mid \log(\lambda) \mid$, if $\lambda \neq 1$ by means of the following generating function:*

$$t^{m\alpha}[A(\lambda, a; t)]^{\alpha} c^{xt} = \sum_{n=0}^{\infty} B_n^{[m-1,\alpha]}(x; a, c, \lambda) \frac{t^n}{n!} \tag{14}$$

where

$$A(\lambda, a; t) = \left(\lambda a^t - \sum_{h=0}^{m-1} \frac{(t \log a)^h}{h!} \right)^{-1}. \tag{15}$$

It is easy to see that if we set $m = 1$, $a = c = e$ in (14), we arrive at the following

$$\left(\frac{t}{\lambda e^t - 1} \right)^{\alpha} e^{xt} = \sum_{n=0}^{\infty} B_n^{(\alpha)}(x; e, e, \lambda) \frac{t^n}{n!}, \quad \mid t \mid < 2\pi, \ 1^{\alpha} = 1 \tag{16}$$

with

$$B_n^{[0,\alpha]}(x, e, e; \lambda) = B_n^{(\alpha)}(x; \lambda). \tag{17}$$

Obviously when we set $\lambda = 1$ and $\alpha = 1$ in (17), we obtain

$$B_n^{[0,1]}(x, e, e; 1) = B_n^{(\alpha)}(x)$$

where $B_n(x)$ are the classical Bernoulli polynomials.

Definition 5. *For arbitrary real or complex parameter α and for the $a, c \epsilon R^{+}$, the Apostol-Euler polynomials $E_n^{[m-1,\alpha]}(x; a, c, \lambda)$, $m \in N$, $\lambda \in \mathcal{C}$ are defined in a suitable neighborhood of $t = 0$ with $\mid t \log a \mid < \mid t \log(-\lambda) \mid$ by means of the generating function*

$$2^{m\alpha}[B(\lambda, a; t)]^{\alpha} c^{xt} = \sum_{n=0}^{\infty} E_n^{[m-1,\alpha]}(x; a, c, \lambda) \frac{t^n}{n!} \tag{18}$$

where

$$B(\lambda, a; t) = \left(\lambda a^t + \sum_{h=0}^{m-1} \frac{(t \log a)^h}{h!} \right)^{-1}. \tag{19}$$

It is easy to see that if we set $m = 1$, $a = c = e$ in (18), we arrive at the following

$$\left(\frac{2}{\lambda e^t + 1}\right)^{\alpha} e^{xt} = \sum_{n=0}^{\infty} E_n^{[0,\alpha]}(x; e, e, \lambda)\frac{t^n}{n!}, \mid t \mid < \pi, 1^{\alpha} = 1 \tag{20}$$

with

$$E_n^{[0,\alpha]}(x, e, e; \lambda) = E_n^{(\alpha)}(x; \lambda).$$

Definition 6. *For arbitrary real or complex parameter α and for the $a, c \in R^+$, the Apostol-Genocchi polynomials $G_n^{[m-1,\alpha]}(x; a, c, \lambda)$, $m \in N$, $\lambda \in \mathcal{C}$ are defined in a suitable neighborhood of $t = 0$ with $\mid t \log a \mid < \mid t \log(-\lambda) \mid$ by means of the generating function*

$$2^{m\alpha} t^{m\alpha} [B(\lambda, a; t)]^{\alpha} c^{xt} = \sum_{n=0}^{\infty} G_n^{[m-1,\alpha]}(x; a, c, \lambda)\frac{t^n}{n!} \tag{21}$$

where $B(\lambda, a; t)$ is given by equation (19). Obviously if we set $m = 1$, $a = c = e$ in (21), we obtain

$$\left(\frac{2t}{\lambda e^t + 1}\right)^{\alpha} e^{xt} = \sum_{n=0}^{\infty} G_n^{[0,\alpha]}(x; e, e, \lambda)\frac{t^n}{n!}, \quad \mid t \mid < \pi, \quad 1^{\alpha} = 1 \tag{22}$$

with

$$G_n^{[0,\alpha]}(x, e, e; \lambda) = G_n^{(\alpha)}(x; \lambda). \tag{23}$$

The popularity of Hermite, Bernoulli and Euler polynomials in number theory, combinatorics and mathematical physics is due in part to the papers of researchers in [3] to [5], [9] to [14], [18], [19], [20], [22] and their generalizations and various extensions which appeared in the literature. In this paper, we propose a further generalization of Apostol-Euler polynomials and Apostol-Genocchi polynomials and we give some properties involving them. For the new class of Apostol-Hermite-Euler polynomials ${}_HE_n^{[\alpha,m-1]}(x, y; a, c, \lambda)$ and Apostol-Hermite-Genocchi polynomials ${}_HG_n^{[\alpha,m-1]}$ $(x, y; a, c, \lambda)$, we modify generating functions given by Tremblay et al [26] and derive some identities.

2. New classes of generalized Hermite-Based, Apostol-Bernoulli, Apostol-Euler and Apostol-Genocchi polynomials

The following definitions provide a natural generalization of the Apostol-Bernoulli polynomials $B_n^{[m-1,\alpha]}(x;\lambda)$, $m \in N$ of order $\alpha\epsilon\mathcal{C}$, Apostol-Euler polynomials $E_n^{[m-1,\alpha]}(x;\lambda)$, $m \in N$ of order $\alpha \in \mathcal{C}$ and Apostol-Genocchi polynomials $G_n^{[m-1,\alpha]}(x;\lambda)$, $m\epsilon N$ of order $\alpha\epsilon\mathcal{C}$.

Definition 7. *For arbitrary real or complex parameter α and for $a, c\varepsilon\Re^+$, the generalized Apostol-Hermite-Bernoulli polynomials ${}_HB_n^{[m-1,\alpha]}(x,y;a,c,\lambda)$ $m \in N$, $\lambda \in \mathcal{C}$ are defined in a suitable neighborhood of $t = 0$ with $\mid t\log(a) <\mid \log(-\lambda) \mid$, by means of the following generating function:*

$$t^{m\alpha}[A(\lambda,a;t)]^{\alpha}c^{xt+yt^2} = \sum_{n=0}^{\infty} {}_HB_n^{[m-1,\alpha]}(x,y;a,c,\lambda)\frac{t^n}{n!} \tag{24}$$

where $A(\lambda,a;t)$ is given by equation (15). It is easy to see that if we set y=0 in (24), we arrive at a recent result of Tremblay et al [26, p. 3, Eq. (1.8)] involving the generalized Apostol-Bernoulli polynomials

$$t^{m\alpha}[A(\lambda,a;t)]^{\alpha}c^{xt} = \sum_{n=0}^{\infty} B_n^{[m-1,\alpha]}(x;a,c,\lambda)\frac{t^n}{n!}. \tag{25}$$

For $c = e$ in (24) gives

$$t^{m\alpha}[A(\lambda,a;t)]^{\alpha}e^{xt+yt^2} = \sum_{n=0}^{\infty} {}_HB_n^{[m-1,\alpha]}(x,y;a,e,\lambda)\frac{t^n}{n!}. \tag{26}$$

Moreover if we set $y = 0$, $m = 1$, $a = c = e$ in (24), we arrive at the following result

$$\left(\frac{t}{\lambda e^t - 1}\right)^{\alpha} e^{xt} = \sum_{n=0}^{\infty} B_n^{[0,\alpha]}(x;e,e,\lambda)\frac{t^n}{n!}, \quad (\mid t \mid < 2\pi, 1^{\alpha} = 1) \tag{27}$$

which is a generating function for the generalized Apostol-Bernoulli polynomials of order α. Thus we have

$$B_n^{[0,\alpha]}(x;e,e,\lambda) = B_n^{[\alpha]}(x;\lambda). \tag{28}$$

Definition 8. *For arbitrary real or complex parameter α and $a, c \in R^+$, the generalized Apostol-Hermite-Euler polynomials ${}_HE_n^{[m-1,\alpha]}(x,y;a,c,\lambda)$,*

$m \in N$, $\lambda \in \mathcal{C}$ are defined in a suitable neighborhood of $t = 0$ with $\mid t \log a \mid < \mid \log(-\lambda) \mid$ by means of generating function

$$2^{m\alpha}[B(\lambda, a; t)]^{\alpha} c^{xt+yt^2} = \sum_{n=0}^{\infty} {}_H E_n^{[m-1,\alpha]}(x, y; a, c, \lambda) \frac{t^n}{n!} \tag{29}$$

where $B(\lambda, a; t)$ is given by equation (19). It is easy to see that if we set y=0 in (29), we arrive at a recent result of Tremblay et al [26, p.3, Eq.(2.1)] involving the generalized Apostol-Euler polynomials

$$2^{m\alpha}[B(\lambda, a; t)]^{\alpha} c^{xt} = \sum_{n=0}^{\infty} E_n^{[m-1,\alpha]}(x; a, c, \lambda) \frac{t^n}{n!}. \tag{30}$$

For $c = e$ in (29) gives

$$2^{m\alpha}[B(\lambda, a; t)]^{\alpha} e^{xt+yt^2} = \sum_{n=0}^{\infty} {}_H E_n^{[m-1,\alpha]}(x, y; a, e, \lambda) \frac{t^n}{n!}. \tag{31}$$

Moreover if we set $y = 0$, $m = 1$, $a = c = e$ in (29), we arrive at the following result

$$\left(\frac{2}{\lambda e^t + 1}\right)^{\alpha} e^{xt} = \sum_{n=0}^{\infty} E_n^{[0,\alpha]}(x; e, e, \lambda) \frac{t^n}{n!}, \quad (\mid t \mid < \pi, 1^{\alpha} = 1) \tag{32}$$

which is a generating function for the generalized Apostol-Euler polynomials of order α. Thus we have

$$E_n^{[0,\alpha]}(x; e, e, \lambda) = E_n^{[\alpha]}(x; \lambda). \tag{33}$$

Definition 9. *For arbitrary real or complex parameter α and $a, c \epsilon R^+$, the generalized Apostol-Hermite-Genocchi polynomials ${}_H G_n^{[m-1,\alpha]}(x, y; a, c, \lambda)$, $m \epsilon N$, $\lambda \epsilon \mathcal{C}$ are defined in a suitable neighborhood of $t = 0$ with $\mid t \log a \mid < \mid \log(-\lambda) \mid$ by means of generating function*

$$2^{m\alpha} t^{m\alpha}[B(\lambda, a; t)]^{\alpha} c^{xt+yt^2} = \sum_{n=0}^{\infty} {}_H G_n^{[m-1,\alpha]}(x, y; a, c, \lambda) \frac{t^n}{n!} \tag{34}$$

where $B(\lambda, a; t)$ is given by equation (19). It is easy to see that if we set $y = 0$ in (34), we arrive at a recent result of Tremblay et al [26, p.5, Eq.(2.4)] involving the generalized Apostol-Genocchi polynomials

$$2^{m\alpha} t^{m\alpha}[B(\lambda, a; t)]^{\alpha} c^{xt} = \sum_{n=0}^{\infty} G_n^{[m-1,\alpha]}(x; a, c, \lambda) \frac{t^n}{n!}. \tag{35}$$

For $c = e$ in (34) gives

$$2^{m\alpha}t^{m\alpha}[B(\lambda,a;t)]^{\alpha}e^{xt+yt^2} = \sum_{n=0}^{\infty} {}_HG_n^{[m-1,\alpha]}(x,y;a,e,\lambda)\frac{t^n}{n!}. \tag{36}$$

Obviously if we set $y = 0$, $m = 1$, $a = c = e$ in (34), we arrive at the following result

$$\left(\frac{2t}{\lambda e^t+1}\right)^{\alpha} e^{xt} = \sum_{n=0}^{\infty} G_n^{[0,\alpha]}(x;e,e,\lambda)\frac{t^n}{n!}, \quad (|\, t \,|< \pi, 1^{\alpha} = 1) \tag{37}$$

which is a generating function for the generalized Apostol-Genocchi polynomials of order α. Thus we have

$$G_n^{[0,\alpha]}(x;,e,e,\lambda) = G_n^{[\alpha]}(x;\lambda). \tag{38}$$

The generalized Apostol-Hermite-Euler polynomials ${}_HE_n^{[m-1,\alpha]}(x,y;a,c,\lambda)$ defined by (25) posses the following interesting properties. These are stated as Theorems 1 to 4 below:

Theorem 1. *The generalized Apostol-Hermite-Euler polynomials ${}_HE_n^{[m-1,\alpha]}(x,y;a,c,\lambda)$ and Apostol-Hermite-Bernoulli polynomials ${}_HB_n^{[m-1,\alpha]}(x,y;a,c,\lambda)$, $\alpha \in N_0$ are related by*

$${}_HB_n^{[m-1,\alpha]}(x,y;a,c,-\lambda) = \frac{(-1)^{\alpha}n!}{2^{m\alpha}(n-m\alpha)!}\,{}_HE_{n-m\alpha}^{[m-1,\alpha]}(x,y;a,c,\lambda) \tag{39}$$

or equivalently by

$${}_HE_n^{[m-1,\alpha]}(x,y;a,c,-\lambda) = \frac{(-2^m)^{\alpha}n!}{(n+m\alpha)!}\,{}_HB_{n+m\alpha}^{[m-1,\alpha]}(x,y;a,c,\lambda) \tag{40}$$

Proof. Considering the generating function (24)

$$t^{m\alpha}[A(-\lambda,a;t)]^{\alpha}c^{xt+yt^2} = \sum_{n=0}^{\infty} {}_HB_n^{[m-1,\alpha]}(x,y;a,c,-\lambda)\frac{t^n}{n!}$$

$$\frac{(-1)^{\alpha}t^{m\alpha}}{2^{m\alpha}}t^{m\alpha}[B(\lambda,a;t)]^{\alpha}c^{xt+yt^2} = \sum_{n=0}^{\infty} {}_HB_n^{[m-1,\alpha]}(x,y;a,c,-\lambda)\frac{t^n}{n!}$$

$$\sum_{n=0}^{\infty} {}_HB_n^{[m-1,\alpha]}(x,y;a,c,-\lambda)\frac{t^n}{n!} = \frac{(-1)^{\alpha}}{2^{m\alpha}}\sum_{n=0}^{\infty} {}_HE_n^{[m-1,\alpha]}(x,y;a,c,\lambda)\frac{t^{n+m\alpha}}{n!}$$

Replacing n by $n-m\alpha$ in R.H.S of above equation, we get

$$\sum_{n=0}^{\infty} {}_HB_n^{[m-1,\alpha]}(x,y;a,c,-\lambda)\frac{t^n}{n!} = \frac{(-1)^{\alpha}}{2^{m\alpha}}\sum_{n=0}^{\infty} {}_HE_{n-m\alpha}^{[m-1,\alpha]}(x,y;a,c,\lambda)\frac{t^n}{(n-m\alpha)!}$$

Comparing the coefficients of t^n on both sides of the above equation, we obtain the result (38). Next consider the generating function (25)

$$2^{m\alpha}[B(\lambda,a;t)]^{\alpha}c^{xt+yt^2}=\sum_{n=0}^{\infty}{}_HE_n^{[m-1,\alpha]}(x,y;a,c,\lambda)\frac{t^n}{n!}$$

$$\frac{(-1)^{\alpha}2^{m\alpha}}{t^{m\alpha}}t^{m\alpha}[A(\lambda,a;t)]^{\alpha}c^{xt+yt^2}=\sum_{n=0}^{\infty}{}_HE_n^{[m-1,\alpha]}(x,y;a,c,-\lambda)\frac{t^n}{n!}$$

$$(-2^m)^{\alpha}\sum_{n=0}^{\infty}{}_HB_n^{[m-1,\alpha]}(x,y;a,c,\lambda)\frac{t^{n-m\alpha}}{n!}=\sum_{n=0}^{\infty}{}_HE_n^{[m-1,\alpha]}(x,y;a,c,-\lambda)\frac{t^n}{n!}$$

Replacing n by $n+m\alpha$ in L.H.S of above equation, we get

$$(-2^m)^{\alpha}\sum_{n=0}^{\infty}{}_HB_{n+m\alpha}^{[m-1,\alpha]}(x,y;a,c,\lambda)\frac{t^n}{(n+m\alpha)!}$$
$$=\sum_{n=0}^{\infty}{}_HE_n^{[m-1,\alpha]}(x,y;a,c,-\lambda)\frac{t^n}{n!}$$

Comparing the coefficients of t^n on both sides of the above equation, we obtain the result (40). ■

For $y=0$ in equation (39) and (40), the result reduces to known result Tremblay et al [26](see also [6]).

Theorem 2. *Let $a,b,c\epsilon R^+$, α an arbitrary complex number and $m\epsilon N$. Then the generalized Apostol-Hermite-Euler polynomials ${}_HE_n^{[m-1,\alpha]}(x,y;a,c,\lambda)$ satisfy the following relations*

(41)
$$ {}_HE_n^{[m-1,\alpha+\beta]}(x+u,y;a,c,\lambda)$$
$$=\sum_{k=0}^{n}\binom{n}{k}{}_HE_k^{[m-1,\alpha]}(x,y;a,c,\lambda)E_{n-k}^{[m-1,\beta]}(u,a,c;\lambda)$$

Proof. Considering the generating function (29) as

$$2^{m\alpha}[B(\lambda,a;t)]^{\alpha+\beta}c^{(x+u)t+yt^2}$$
$$=\sum_{k=0}^{\infty}{}_HE_k^{[m-1,\alpha]}(x,y;a,c,\lambda)\frac{t^k}{k!}\sum_{n=0}^{\infty}E_n^{[m-1,\beta]}(u,a,c,\lambda)\frac{t^n}{n!}$$

(42)
$$\sum_{n=0}^{\infty}{}_HE_n^{[m-1,\alpha+\beta]}(x+u,y;a,c,\lambda)\frac{t^n}{n!}$$
$$=\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}{}_HE_k^{[m-1,\alpha]}(x,y;a,c,\lambda)E_n^{[m-1,\beta]}(u,a,c,\lambda)\frac{t^{n+k}}{n!k!}$$

Replacing n by $n-k$ in R.H.S of above equation, we get

$$\sum_{n=0}^{\infty} {}_HE_n^{[m-1,\alpha+\beta]}(x+u,y;a,c,\lambda)\frac{t^n}{n!}$$
$$=\sum_{n=0}^{\infty}\sum_{k=0}^{n}\binom{n}{k}{}_HE_k^{[m-1,\alpha]}(x,y;a,c,\lambda)E_{n-k}^{[m-1,\beta]}(u,a,c,\lambda)\frac{t^n}{n!}$$

Finally equating the coefficients of $\frac{t^n}{n!}$, we get the result (41).

For $y=0$ in equation (41), the result reduces to known result of Tremblay et al [26]. ■

Theorem 3. *The generalized Apostol-Hermite-Euler polynomials* ${}_HE_n^{[m-1,\alpha]}$ $(x,y;a,c,\lambda)$ *satisfies the following recurrence relation*

$$\lambda{}_HE_n^{[m-1,\alpha]}(x+1,y;a,c,\lambda)+{}_HE_n^{[m-1,\alpha]}(x,y;a,c,\lambda)$$
$$=2\sum_{k=0}^{n}\binom{n}{k}{}_HE_k^{[m-1,\alpha]}(x,y;a,c,\lambda)E_{n-k}^{(-1)}(0,a,\lambda). \tag{43}$$

Proof. Let

$$\lambda{}_HE_n^{[m-1,\alpha]}(x+1,y;a,c,\lambda)+{}_HE_n^{[m-1,\alpha]}(x,y;a,c,\lambda)$$
$$=\left(2^m a^t\right)^{\alpha}c^{xt+yt^2}(\lambda c^t+1)$$
$$=22^{m\alpha}[B(\lambda,a;t)]^{\alpha}c^{xt+yt^2}\left(\frac{2}{\lambda a^t+1}\right)^{(-1)}$$
$$=2\sum_{k=0}^{\infty}{}_HE_k^{[m-1,\alpha]}(x,y;a,c,\lambda)\frac{t^k}{k!}\sum_{n=0}^{\infty}E_n^{(-1)}(0,a;\lambda)\frac{t^n}{n!}$$
$$=2\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}{}_HE_k^{[m-1,\alpha]}(x,y;a,c,\lambda)E_n^{(-1)}(0,a;\lambda)\frac{t^{n+k}}{n!k!}.$$

Replacing n by $n-k$ in R.H.S of above equation, we get

$$\sum_{n=0}^{\infty}\left(\lambda{}_HE_n^{[m-1,\alpha]}(x+1,y;a,c,\lambda)+{}_HE_n^{[m-1,\alpha]}(x,y;a,c,\lambda)\right)\frac{t^n}{n!}$$
$$=\sum_{n=0}^{\infty}\left(2\sum_{k=0}^{n}{}_HE_k^{[m-1,\alpha]}(x,y;a,c,\lambda)E_{n-k}^{(-1)}(0,a;\lambda)\right)\frac{t^n}{(n-k)!k!}.$$

Comparing the coefficients of $\frac{t^n}{n!}$ in the above equation, we get the result (43).

For $y=0$ in equation (43), the result reduces to known result Tremblay et al [26]. ■

Remark 1. *Setting $y=0$, $m=1$ and $b=c=e$ in (43) and using (29), we find*

$$\lambda E_n^{\alpha}(x+1;\lambda)+E_n^{\alpha}(x;\lambda)=2\sum_{k=0}^{n}\binom{n}{k}E_k^{(\alpha)}(x;\lambda)E_{n-k}^{(-1)}(0;\lambda) \tag{44}$$

Using the well known result (see [9])

$$E_n^{\alpha+\beta}(x+y;\lambda)=2\sum_{k=0}^{n}\binom{n}{k}E_k^{(\alpha)}(x;\lambda)E_{n-k}^{(\beta)}(y;\lambda) \tag{45}$$

equation (44) becomes the familiar relation for the generalized Apostol-Euler polynomials (see [9])

$$\lambda E_n^{\alpha}(x+1;\lambda)+E_n^{\alpha}(x;\lambda)=2E_n^{(\alpha-1)}(x;\lambda). \tag{46}$$

Theorem 4. *Let $a,b\in R$, α and β arbitrary complex numbers, $m\epsilon N$. Then the generalized Apostol-Hermite-Euler polynomials ${}_HE_n^{[\alpha,m-1]}(x,y;a,c,\lambda)$ satisfy the following relation*

$$\begin{aligned}&{}_HE_n^{[\alpha+\beta,m-1]}(x_1+x_2,y_1+y_2;a,c,\lambda)\\&=\sum_{k=0}^{n}\binom{n}{k}{}_HE_{n-k}^{[\alpha,m-1]}(x_1,y_1;a,c,\lambda){}_HE_k^{[\beta,m-1]}(x_2,y_2;a,c,\lambda).\end{aligned} \tag{47}$$

Proof. Use definition (25) to get

$$\begin{aligned}&\sum_{n=0}^{\infty}{}_HE_n^{[\alpha+\beta,m-1]}(x_1+x_2,y_1+y_2;a,c,\lambda)\frac{t^n}{n!}\\&=2^{m\alpha}[B(\lambda,a;t)]^{\alpha+\beta}c^{(x_1+x_2)t+(y_1+y_2)t^2}\\&=\left(\sum_{n=0}^{\infty}{}_HE_n^{[\alpha,m-1]}(x_1,y_1;a,c,\lambda)\frac{t^n}{n!}\right)\left(\sum_{k=0}^{\infty}{}_HE_k^{[\beta,m-1]}(x_2,y_2;a,c,\lambda)\frac{t^k}{k!}\right)\\&=\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}{}_HE_n^{[\alpha,m-1]}(x_1,y_1;a,c,\lambda){}_HE_k^{[\beta,m-1]}(x_2,y_2;a,c,\lambda)\frac{t^{n+k}}{n!k!}.\end{aligned}$$

Replacing n by $n-k$ in R.H.S of above equation, we get

$$\begin{aligned}L.H.S.\ =\ &\sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}\right){}_HE_{n-k}^{[\alpha,m-1]}(x_1,y_1;a,c,\lambda)\\&\times{}_HE_k^{[\beta,m-1]}(x_2,y_2;a,c,\lambda)\frac{t^n}{(n-k)!k!}.\end{aligned}$$

Comparing the coefficients of $\frac{t^n}{n!}$ in above equation, we get the desired result (47). For m=1 in equation (47), the result reduces to a known result of Gaboury et al [6., p.7, Eq. 3.6]. ■

3. New classes of Apostol-Hermite-Genocchi polynomials

Now let us shift our focus on some interesting properties for the generalized Apostol-Hermite-Genocchi polynomials ${}_HG_n^{[\alpha,m-1]}(x,y;a,c,\lambda)$ defined by (35). These are stated as Theorem 5 to Theorem 9 below:

Theorem 5. *The generalized Apostol-Hermite-Genocchi polynomials* ${}_HG_n^{[\alpha,m-1]}(x,y;a,c,\lambda)$, *the generalized Apostol-Hermite-Bernoulli polynomials* ${}_HB_n^{[\alpha,m-1]}(x,y;a,c,\lambda)$ *and the generalized Apostol-Hermite-Euler polynomials* ${}_HE_n^{[\alpha,m-1]}(x,y;a,c,\lambda)$ *are related by*

$$(48)\quad {}_HG_n^{[\alpha,m-1]}(x,y;a,c,-\lambda) = (-2^m)^\alpha\, {}_HB_n^{[\alpha,m-1]}(x,y;a,c,\lambda), \quad (\alpha \in \mathcal{C})$$

or equivalently

$$(49)\qquad {}_HG_n^{[\alpha,m-1]}(x,y;a,c,\lambda) = \frac{n!}{(n-m\alpha)!}\, {}_HE_{n-m\alpha}^{[\alpha,m-1]}(x,y;a,c,\lambda),$$

$n, \alpha, m \in N$, $n \geq m\alpha$.

Proof. Using definition (24)

$$t^{m\alpha}[A(\lambda,a;t)]^\alpha c^{xt+yt^2} = \sum_{n=0}^{\infty} {}_HB_n^{[\alpha,m-1]}(x,y;a,c,\lambda)\frac{t^n}{n!}$$

$$t^{m\alpha}[B(-\lambda,a;t)]^\alpha c^{xt+yt^2} = (-2^m)^\alpha \sum_{n=0}^{\infty} {}_HB_n^{[\alpha,m-1]}(x,y;a,c,\lambda)\frac{t^n}{n!}$$

$$\sum_{n=0}^{\infty} {}_HG_n^{[\alpha,m-1]}(x,y;a,c,-\lambda)\frac{t^n}{n!} = (-2^m)^\alpha \sum_{n=0}^{\infty} {}_HB_n^{[\alpha,m-1]}(x,y;a,c,\lambda)\frac{t^n}{n!}$$

Comparing the coefficients of $\frac{t^n}{n!}$ on both sides, we get the desired result (48). Next using definition (26)

$$2^{m\alpha}t^{m\alpha}[B(\lambda,a;t)]^\alpha c^{xt+yt^2} = \sum_{n=0}^{\infty} {}_HG_n^{[\alpha,m-1]}(x,y;a,c,\lambda)\frac{t^n}{n!}$$

$$2^{m\alpha}t^{m\alpha}[B(\lambda,a;t)]^\alpha c^{xt+yt^2} = \sum_{n=0}^{\infty} {}_HG_n^{[\alpha,m-1]}(x,y;a,c,\lambda)\frac{t^n}{n!}$$

$$\sum_{n=0}^{\infty} {}_H E_n^{[\alpha,m-1]}(x,y;a,c,\lambda)\frac{t^{n+m\alpha}}{n!} = \sum_{n=0}^{\infty} {}_H G_n^{[\alpha,m-1]}(x,y;a,c,\lambda)\frac{t^n}{n!}$$

Replace n by $n - m\alpha$ in L.H.S of the above equation, we get

$$\sum_{n=m\alpha}^{\infty} {}_H E_{n-m\alpha}^{[\alpha,m-1]}(x,y;a,c,\lambda)\frac{t^n}{(n-m\alpha)!} = \sum_{n=0}^{\infty} {}_H G_n^{[\alpha,m-1]}(x,y;a,c,\lambda)\frac{t^n}{n!}$$

Comparing the coefficients of t on both sides, we get the result (49).

For $y = 0$ in equation (48) and (49), the result reduces to known result of Tremblay et al [26]. ■

Theorem 6. *Let $a, c\epsilon R$, α an arbitrary complex number and $m\epsilon N$, then the generalized Apostol-Hermite-Genocchi polynomials ${}_H G_n^{[\alpha,m-1]}(x,y;a,c,\lambda)$ satisfy the following relations*

$$(50)\qquad {}_H G_n^{[\alpha+\beta,m-1]}(x+u,y;a,c,\lambda) = \sum_{k=0}^{n} \binom{n}{k} {}_H G_k^{[m-1,\alpha]}(x,y;a,c,\lambda) G_{n-k}^{[m-1,\beta]}(u,a,c,\lambda).$$

Proof. Using definition (26)

$$\sum_{n=0}^{\infty} {}_H G_n^{[\alpha+\beta,m-1]}(x+u,y;a,c,\lambda)\frac{t^n}{n!}$$
$$= 2^{m\alpha}t^{m\alpha}[B(\lambda,a;t)]^{\alpha}c^{xt+yt^2}2^{m\alpha}t^{m\alpha}[B(\lambda,a;t)]^{\beta}c^{ut}$$
$$= \sum_{n=0}^{\infty}\sum_{k=0}^{\infty} {}_H G_k^{[\alpha,m-1]}(x,y;a,c,\lambda) G_n^{[\beta,m-1]}(u,a,c,\lambda)\frac{t^{n+k}}{n!}.$$

Replacing n by $n - k$ in above equation, we have

$$L.H.S. = \sum_{n=0}^{\infty}\left(\sum_{k=0}^{n} {}_H G_k^{[\alpha,m-1]}(x,y;a,c,\lambda) G_{n-k}^{[\beta,m-1]}(u,a,c,\lambda)\right)\frac{t^n}{(n-k)!k!}$$

Finally equating the coefficients of $\frac{t^n}{n!}$, we get the result (50).

For $y = 0$ in equation (50), the result reduces to known result of Tremblay et al [26]. ■

Theorem 7. *The generalized Apostol-Hermite-Genocchi polynomials ${}_H G_n^{[m-1,\alpha]}(x,y;a,c,\lambda)$ satisfy the following recurrence relation*

$$(51)\qquad \lambda {}_H G_n^{[m-1,\alpha]}(x+1,y;a,c,\lambda) + {}_H G_n^{[m-1,\alpha]}(x,y;a,c,\lambda) = 2n\sum_{k=0}^{n} \binom{n-1}{k} {}_H G_k^{[m-1,\alpha]}(x,y;a,c,\lambda) G_{n-1-k}^{(-1)}(0,a,\lambda)$$

Proof. Let us write

$$\begin{aligned}
L.H.S &= 2^{m\alpha}t^{m\alpha}[B(\lambda,a;t)]^{\alpha}c^{xt+yt^2}(\lambda a^t+1)\\
&= 2t2^{m\alpha}t^{m\alpha}[B(\lambda,a;t)]^{\alpha}c^{xt+yt^2}\left(\frac{2t}{\lambda a^t+1}\right)^{(-1)}\\
&= 2t\sum_{k=0}^{\infty}{}_HG_k^{[m-1,\alpha]}(x,y;a,c,\lambda)\frac{t^k}{k!}\sum_{n=0}^{\infty}G_n^{(-1)}(0,a;\lambda)\frac{t^n}{n!}\\
&= 2\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}{}_HG_k^{[m-1,\alpha]}(x,y;a,c,\lambda)G_n^{(-1)}(0,a;\lambda)\frac{t^{n+k+1}}{n!k!}.
\end{aligned}$$

Replacing n by $n-k-1$ in R.H.S of above equation, we get

$$\begin{aligned}
&\sum_{n=0}^{\infty}\left(\lambda_HG_n^{[m-1,\alpha]}(x+1,y;a,c,\lambda)+{}_HG_n^{[m-1,\alpha]}(x,y;a,c,\lambda)\right)\frac{t^n}{n!}\\
&=\sum_{n=0}^{\infty}\left(2n\sum_{k=0}^{n}{}_HG_k^{[m-1,\alpha]}(x,y;a,c,\lambda)G_{n-1-k}^{(-1)}(0,a;\lambda)\right)\frac{t^n}{(n-1-k)!k!}.
\end{aligned}$$

Comparing the coefficients of $\frac{t^n}{n!}$ in the above equation, we get the result (51).

For $y=0$ in equation (51), the result reduces to known result of Tremblay et al [26]. ■

Remark 2. Setting $y=0$, $m=1$ and b=c=e in (51) and using (34), we find

(52) $$\lambda G_n^{\alpha}(x+1;\lambda)+G_n^{\alpha}(x;\lambda)=2n\sum_{k=0}^{n}\binom{n-1}{k}G_k^{(\alpha)}(x;\lambda)E_{n-1-k}^{(-1)}(0;\lambda$$

Using the well known result (see [9])

(53) $$G_n^{\alpha+\beta}(x+y;\lambda)=\sum_{k=0}^{n}\binom{n}{k}G_k^{(\alpha)}(x;\lambda)G_{n-k}^{(\beta)}(y;\lambda)$$

equation (52) becomes the familiar relation for the generalized Apostol-Genocchi polynomials (see [9])

(54) $$\lambda G_n^{\alpha}(x+1;\lambda)+G_n^{\alpha}(x;\lambda)=2nG_{n-1}^{(\alpha-1)}(x;\lambda).$$

Theorem 8. *Let $a, b, c, p, q \in R$, α an arbitrary complex number and $m\epsilon N$, then the generalized Apostol-Hermite-Genocchi polynomials ${}_HG_n^{[\alpha,m-1]}(x, y; a, c, \lambda)$ satisfy the following relation*

$$\begin{aligned}
{}_HG_n^{[\alpha+\beta,m-1]}&(px, qy; a, c, \lambda) \\
&= n! \sum_{k=0}^{n} \sum_{j=0}^{[\frac{k}{2}]} {}_HG_{n-k}^{[m-1,\alpha]}(x, y; a, c, \lambda)((p-1)x \ln c)^k \\
&\times ((q-1)y \ln c)^j \frac{1}{(n-k-2j)!j!}
\end{aligned} \tag{55}$$

Proof. Using definition (26)

$$\begin{aligned}
&\sum_{n=0}^{\infty} {}_HG_n^{[\alpha+\beta,m-1]}(px, qy; a, c, \lambda)\frac{t^n}{n!} \\
&\quad = 2^{m\alpha}t^{m\alpha}[B(\lambda, a; t)]^{\alpha} c^{xt+yt^2} c^{(p-1)xt} c^{(q-1)yt^2} \\
&\quad = \left(\sum_{n=0}^{\infty} {}_HG_n^{[\alpha,m-1]}(x, y; a, c, \lambda)\frac{t^n}{n!}\right) \\
&\quad\times \left(\sum_{k=0}^{\infty}((p-1)x \ln c)^k \frac{t^k}{k!}\right) \left(\sum_{j=0}^{\infty}((q-1)y \ln c)^j \frac{t^{2j}}{j!}\right) \\
&\quad = \left(\sum_{n=0}^{\infty} {}_HG_n^{[\alpha,m-1]}(x, y; a, c, \lambda)\frac{t^n}{n!}\right) \\
&\quad\times \sum_{k=0}^{\infty}\sum_{j=0}^{\infty}((p-1)x \ln c)^k((q-1)y \ln c)^j \frac{t^{k+2j}}{k!j!}
\end{aligned}$$

Replacing k by $k-2j$ in above equation, we have

$$\begin{aligned}
L.H.S. &= \left(\sum_{n=0}^{\infty} {}_HG_n^{[\alpha,m-1]}(x, y; a, c, \lambda)\frac{t^n}{n!}\right) \\
&\times \sum_{k=0}^{\infty}\sum_{j=0}^{[\frac{k}{2}]}((p-1)x \ln c)^{k-2j}((q-1)y \ln c)^j \frac{t^k}{(k-2j)!j!} \\
&= \sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\sum_{j=0}^{[\frac{k}{2}]} {}_HG_n^{[\alpha,m-1]}(x, y; a, c, \lambda)((p-1)x \ln c)^{k-2j} \\
&\times ((q-1)y \ln c)^j \frac{t^{n+k}}{(k-2j)!j!n!}
\end{aligned}$$

Replacing n by $n-k$ in above equation, we have

$$L.H.S. = \sum_{n=0}^{\infty}\sum_{k=0}^{n}\sum_{j=0}^{[\frac{k}{2}]} {}_{H}G_{n-k}^{[\alpha,m-1]}(x,y;a,c,\lambda)((p-1)x\ln c)^{k-2j} \times ((q-1)y\ln c)^{j}\frac{t^n}{(n-k-2j)!j!k!}$$

Finally equating the coefficients of $\frac{t^n}{n!}$, we get the result (3.8). For $m=1$ in equation (3.8), the result reduces to a known result of Gaboury et al [6, p.10.,Eq.3.16]. ■

Theorem 9. *Let $a, b \in R$, α and β arbitrary complex number $m \epsilon N$ then the generalized Apostol-Hermite-Genocchi polynomials ${}_{H}G_{n}^{[\alpha,m-1]}(x,y;a,c,\lambda)$ satisfy the following relation*

$$ {}_{H}G_{n}^{[\alpha+\beta,m-1]}(x_1+x_2,y_1+y_2;a,c,\lambda) = \sum_{k=0}^{n}\binom{n}{k} {}_{H}G_{n-k}^{[\alpha,m-1]}(x_1,y_1;a,c,\lambda){}_{H}G_{k}^{[\beta,m-1]}(x_2,y_2;a,c,\lambda)$$

Proof. Use definition (25) to get

$$\begin{aligned} L.H.S &= 2^{m\alpha}t^{m\alpha}[B(\lambda,a;t)]^{\alpha+\beta}c^{(x_1+x_2)t+(y_1+y_2)t^2} \\ &= \left(\sum_{n=0}^{\infty} {}_{H}G_{n}^{[\alpha,m-1]}(x_1,y_1;a,c,\lambda)\frac{t^n}{n!}\right) \\ &\times \left(\sum_{k=0}^{\infty} {}_{H}G_{k}^{[\beta,m-1]}(x_2,y_2;a,c,\lambda)\frac{t^k}{k!}\right) \\ &= \sum_{n=0}^{\infty}\sum_{k=0}^{\infty} {}_{H}G_{n}^{[\alpha,m-1]}(x_1,y_1;a,c,\lambda){}_{H}G_{k}^{[\beta,m-1]}(x_2,y_2;a,c,\lambda)\frac{t^{n+k}}{n!k!} \\ &= \sum_{n=0}^{\infty}\sum_{k=0}^{n} \\ &\times {}_{H}G_{n-k}^{[\alpha,m-1]}(x_1,y_1;a,c,\lambda){}_{H}G_{k}^{[\beta,m-1]}(x_2,y_2;a,c,\lambda)\frac{t^n}{(n-k)!k!} \end{aligned}$$

Comparing the coefficients of $\frac{t^n}{n!}$ in above equation, we get the desired result (56). For $m=1$ in equation (56), the result reduces to a known result of Gaboury et al [6, p.7, Eq. 3.6]. ■

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