

N. RAJESH AND P. SUDHA

PROPERTIES OF ω -CONTINUOUS FUNCTIONS

ABSTRACT. In [9] the authors, introduced the notion of ω -continuity and investigated its fundamental properties. In this paper, we investigate some more properties of this type of continuity.

KEY WORDS: topological spaces, ω -open sets, ω -continuous functions, ω -quotient functions.

AMS Mathematics Subject Classification: 54D10.

1. Introduction

Generalized open sets play a very important role in General Topology and they are now the research topics of many topologists worldwide. Indeed a significant theme in General Topology and Real Analysis concerns the variously modified forms of continuity, separation axioms, compactness, connectedness etc. by utilizing generalized open sets. One of the most well known notions and also an inspiration source is the notion of semiopen sets was introduced by Levine in 1963. In 1995, Sundaram and Sheik John [9] introduced the concepts of ω -open sets and ω -continuity. This notion was further studied by Sheik John and Sundaram in [8] and Noiri and Popa in [4]. In this paper, we obtain further properties of ω -continuity in topological spaces.

2. Preliminaries

Throughout this paper (X, τ) and (Y, σ) (or simply X and Y) represent topological spaces on which no separation axioms are assumed unless otherwise mentioned. For a subset A of a space (X, τ) , $Cl(A)$ and $Int(A)$ denote the closure of A and the interior of A , respectively.

We recall the following definitions, which are useful in the sequel.

Definition 1. *A subset A of a space (X, τ) is called semiopen [2] if $A \subset Cl(Int(A))$. The complement of a semiopen set is called a semiclosed set [1].*

Definition 2. A subset A of a space (X, τ) is called ω -closed [9] if $Cl(A) \subset U$ whenever $A \subset U$ and U is semiopen in X . The complement of an ω -closed set is called an ω -open set.

The family of all ω -open (resp. ω -closed) sets of (X, τ) is denoted by $\omega(\tau)$ (resp. $\omega C(X)$). We set $\omega O(X, x) = \{U : U \in \omega(\tau) \text{ and } x \in U\}$. In [9] shown that the set $\omega(\tau)$ forms a topology, which is finer than τ .

Definition 3. The intersection of all ω -closed sets containing A is called the ω -closure [7] of A and is denoted by $\omega Cl(A)$. A set A is ω -closed if and only if $\omega Cl(A) = A$ [7].

3. Properties of ω -continuous functions

Definition 4. A function $f : (X, \tau) \rightarrow (Y, \sigma)$ is called :

- (1) an ω -continuous [9] at a point $x \in X$ if for each open subset V in Y containing $f(x)$, there exists a $U \in \omega(X, x)$ such that $f(U) \subset V$;
- (2) an ω -continuous [9] if it has this property at each point of X .

Theorem 1 ([7]). The following statements are equivalent for a function $f : (X, \tau) \rightarrow (Y, \sigma)$:

- (1) f is ω -continuous;
- (2) $f : (X, \omega(\tau)) \rightarrow (Y, \sigma)$ is continuous;
- (3) for every open set V of Y , $f^{-1}(V)$ is ω -open in X ;
- (4) for every closed set V of Y , $f^{-1}(V)$ is ω -closed in X .

Lemma 1 ([9]). Let $A \subset B \subset X$, A be ω -open in B and B an open set in (X, τ) , then $A \in \omega(\tau)$.

Theorem 2. Let $f : (X, \tau) \rightarrow (Y, \sigma)$ be a function and $\Lambda = \{U_i : i \in I\}$ be a cover of X such that $U_i \in \omega(\tau)$ for each $i \in I$. If $f|_{U_i}$ is continuous for each $i \in I$, then f is ω -continuous.

Proof. Suppose that V is any open subset of (Y, σ) . Since $f|_{U_i}$ is ω -continuous for each $i \in I$, it follows that $(f|_{U_i})^{-1}(V)$ is open in U_i . We have $f^{-1}(V) = \cup_{i \in I} (f^{-1}(V) \cap U_i) = \cup_{i \in I} (f|_{U_i})^{-1}(V)$. Then by Lemma 1, we obtain $f^{-1}(V) \in \omega(\tau)$, which means that f is ω -continuous. ■

Theorem 3. Let $f : (X, \tau) \rightarrow (Y, \sigma)$ be a function and $x \in X$. If there exists an open set U of X such that $x \in U$, and the restriction of f to U is ω -continuous at x , then f is ω -continuous at x .

Proof. Suppose that F is an open subset of (Y, σ) containing $f(x)$. Since $f|_U$ is ω -continuous at x , there exists an ω -open set V of U containing x such that $f(V) = (f|_U)(V) \subset F$. Since U is open in X containing x , it

follows from Lemma 1 that $V \in \omega(\tau)$ containing x . Thus, f is ω -continuous at x . ■

Theorem 4. *A sequence $\{g_n : X \rightarrow R\}$ of functions converges uniformly if and only if for every $\epsilon > 0$, there exists a natural number N such that for all natural numbers n and m with $n \geq m \geq N$ for all $x \in X$, we have $|g_n(x) - g_m(x)| < \epsilon$.*

Theorem 5. *Let $\{f_n : X \rightarrow R\}$ be a sequence of ω -continuous function such that $|f_n| \leq M_n$ for each n , where $\sum_{n=1}^{\infty} M_n$ is a convergent series of the function $g_n = f_1 + f_2 + \dots + f_n$ is ω -continuous for each n , then the function f defined by $f(x) = \sum_{n=1}^{\infty} f_n(x)$ exists and is ω -continuous on X .*

Proof. For each n , let $g_n = f_1 + f_2 + \dots + f_n$. Let $\epsilon > 0$. Since $\sum_{n=1}^{\infty} M_n$ is a convergent series, there exists a natural number N such that $\sum_{i=N}^{\infty} M_n < \epsilon$. Let n and m be natural numbers such that $n \geq m \geq N$. Then for all $x \in X$, $|g_n(x) - g_m(x)| < |\sum_{i=m}^n f_i(x)| \leq \sum_{i=m}^n |f_i(x)| \leq \sum_{i=N}^{\infty} M_n < \epsilon$. By Theorem 4, this means that the sequence $\{g_n\}$ converges uniformly on X . For each $x \in X$, define $f : X \rightarrow R$ by $f(x) = \lim_{n \rightarrow \infty} g_n(x)$. Since $g_n \rightarrow f$ uniformly on X , it follows that $f(x) = \sum_{n=1}^{\infty} f_n(x)$. It remains to show that f is ω -continuous on X . To this end, let $\epsilon > 0$ and $p \in X$. Since $g_n \rightarrow f$ uniformly on X , there exists a natural number m such that for all $x \in X$, we have $|g_m(x) - f(x)| < \epsilon/3$. Since g_m is ω -continuous, there exists an ω -open set W_ϵ containing p such that for all $z \in W_\epsilon$, we have $|g_m(z) - g_m(p)| < \epsilon/3$. Thus, for all $z \in W_\epsilon$. We have $|f(z) - g(p)| = (|f(z) - g_m(z)| + |g_m(z) - g_m(p)| + |g_m(p) - f(p)|) < \epsilon$. This means that for every open set $V = (f(p) - \epsilon, f(p) + \epsilon)$ containing $f(p)$, there exists an ω -open set W_ϵ containing p such that $f(W_\epsilon) \subset V$. Therefore, By Theorem 4, f is ω -continuous on X . ■

Theorem 6 ([7]). *If $f : (X, \tau) \rightarrow (Y, \sigma)$ is ω -continuous and if $g : (Y, \sigma) \rightarrow (Z, \eta)$ is continuous, then $g \circ f$ is ω -continuous.*

Theorem 7. *Let $g : (X, \tau) \rightarrow (Y, \sigma)$ be continuous and $h : (X, \tau) \rightarrow (Z, \eta)$ be ω -continuous. Then $f : X \rightarrow Y \times Z$ defined by $f(x) = (g(x), h(x))$ is ω -continuous.*

Proof. Let $A \times B$ be a basic open subset of $Y \times Z$. Then $f^{-1}(A \times B) = g^{-1}(A) \cap h^{-1}(B)$. Since g is continuous and h is ω -continuous, $g^{-1}(A)$ is open and $h^{-1}(B)$ is ω -open. Therefore, $f^{-1}(A \times B)$ is ω -open and hence f is ω -continuous. ■

Remark 1. Every constant function from (X, τ) into (R, σ) is an ω -continuous function.

Theorem 8. Let g and f , respectively, be continuous and ω -continuous real valued functions on X . Then each of the following is true:

- (1) $g + f$ is an ω -continuous function,
- (2) gf is an ω -continuous function,
- (3) $|f|$ is an ω -continuous function,
- (4) $\min\{f, g\}$ and $\max\{f, g\}$ are ω -continuous functions,
- (5) If $g(x) \neq 0$ for all $x \in X$, then $\frac{f}{g}$ is ω -continuous,
- (6) If $f(x) \neq 0$ for all $x \in X$, then $\frac{g}{f}$ is ω -continuous,
- (7) f^k is ω -continuous for each positive integer k .

Proof. We prove (1) and (6). $h : R \times R \rightarrow R$ defined as $h(x, y) = x + y$ is continuous. Hence the proof of (1) follows from Theorems 6 and 7. The proof of (6) follows from the facts that $h(x) = \frac{1}{x}$ is continuous, that f is ω -continuous and (2). ■

Definition 5. A subset A of a space (X, τ) is said to be an ω -zero set of X if there exists an ω -continuous function $f : (X, \tau) \rightarrow (R, \sigma)$ such that $A = \{x \in X : f(x) = 0\}$ and a subset is cow-zero-set if it is the complement of an ω -zero-set. Furthermore, if $f : (X, \tau) \rightarrow (R, \sigma)$ is an ω -continuous function, then the set $\omega Z(f) = \{x \in X : f(x) = 0\}$ is called the ω -zero-set of f .

- Remark 2.**
- (1) Every ω -zero-set of a space is ω -closed and hence every cow-zero-set is an ω -open set,
 - (2) Every zero-set of any space is an ω -zero-set.

Example 1. Let $f : (X, \tau) \rightarrow (R, \sigma)$ be a function defined by $f(x) = 1$ for all $x \in X$, where $X = \{a, b, c\}$ and $\tau = \{\emptyset, X, \{a\}\}$. Then the set $\{b, c\}$ is an ω -closed set but not ω -zero-set.

Example 2. Let $f : (X, \tau) \rightarrow (R, \sigma)$ be a function defined by $f(a) = f(b) = 1$ and $f(c) = 0$, where $X = \{a, b, c\}$ and $\tau = \{\emptyset, X, \{a\}\}$. Then f is ω -continuous but not a continuous function. Hence the set $\{c\}$ is an ω -zero-set which is not zero-set.

Lemma 2. If A is an ω -zero-set of a space X , then there exists an ω -continuous function $f : X \rightarrow R$ such that $f \geq 0$ and $A = \omega Z(f)$.

Proof. Since $A = \omega Z(g)$ for some ω -continuous function $g : X \rightarrow R$, by Theorem 8, the function $f = |g| \geq 0$ is ω -continuous and $A = \omega Z(f)$. ■

Lemma 3. *The intersection and union of any finite number of ω -zero-sets is also an ω -zero-set. If $\omega Z(f)$ and $\omega Z(g)$ are ω -zero-sets of f and g , then $\omega Z(f) \cup \omega Z(g) = \omega Z(fg)$, $\omega Z(f) \cap \omega Z(g) = \omega Z(h)$, where $h = f + g$.*

Proof. By Theorem 8, it follows that both fg and $h = f + g$ are ω -continuous. Therefore, $\omega Z(f) \cup \omega Z(g) = \omega Z(fg)$, $\omega Z(f) \cap \omega Z(g) = \omega Z(h)$ are ω -zero-sets. ■

Lemma 4. *If $\alpha \in R$ and $f : X \rightarrow R$ is an ω -continuous function, then the set $A = \{x \in X : f(x) \geq \alpha\}$ as well as $B = \{x \in X : f(x) \leq \alpha\}$ are ω -zero-sets, and hence the sets $\{x \in X : f(x) < \alpha\}$ and $\{x \in X : f(x) > \alpha\}$ are cow-zero-sets.*

Proof. By using Theorem 8, it is easy to see that $A = \omega Z(\min\{f(x)\backslash\alpha, 0\})$ and $B = \omega Z(\max\{f(x)\backslash\alpha, 0\})$ are ω -zero-sets. ■

Lemma 5. *If A and B are disjoint ω -zero-sets the space X , then there exist disjoint cow-zero-sets U and V containing A and B , respectively.*

Proof. Let $A = \omega Z(f)$ and $B = \omega Z(g)$ Then the function $h : X \rightarrow R$ given by $h(x) = \frac{f(x)}{f(x)+g(x)}$ is well defined and in view of Theorem 8 it is ω -continuous, $h(A) = \{0\}$ and $h(B) = \{1\}$. Then by Lemma 4, the sets $\{x \in X : h(x) > \frac{1}{2}\}$ and $\{x \in X : h(x) < \frac{1}{4}\}$ are the required cow-zero (hence ω -open) sets. ■

Definition 6. *A topological space (X, τ) is said to be ω^* -normal if for any pair of disjoint ω -closed subsets F_1 and F_2 of X , there exist disjoint ω -open sets U and V such that $F_1 \subset U$ and $F_2 \subset V$. That is, a topological space (X, τ) is ω^* -normal if and only if (X, τ_ω) is a normal space.*

It is not difficult to prove the following, characterization of an ω^* -normal space:

Theorem 9. *A topological space (X, τ) is an ω^* -normal space if for each pair of ω -open sets U and V in X such that $X = U \cap V$, there exist ω -closed sets A and B which are contained in U and V , respectively and $X = A \cup B$.*

Theorem 10. *If (X, τ) is any topological space, then the following statements are equivalent:*

- (1) *The space X is ω^* -normal.*
- (2) *For each ω -closed set A in X and each ω -open set G in X containing A , there is an ω -open set U such that $A \subseteq U \subseteq \omega Cl(U) \subseteq G$.*
- (3) *For each ω -closed set A and each ω -open set G containing A , there exist ω -open sets $\{U_n, n \in N\}$ such that $A \subseteq \cup\{U_n : n \in N\}$ and $\omega Cl(U_n) \subseteq G$ for each $n \in N$.*

Now, we can establish the following Urysohn's type lemma of ω^* -normality which is important characterization of the ω^* -normal space:

Theorem 11. *Let (X, τ) be any topological space. Then the following statements are equivalent:*

- (1) X is an ω^* -normal space,
- (2) For each ω -closed subset A and ω -open subset B of X such that $A \subseteq B$, there exists an ω -continuous function $f : X \rightarrow I$ such that $f(A) = \{0\}$ and $f(X \setminus B) = \{1\}$,
- (3) For each pair of disjoint ω -closed subsets F and H of X , there exists an ω -continuous function $f : X \rightarrow I$ such that $f(F) = \{0\}$ and $f(H) = \{1\}$.

Proof. (1) \Rightarrow (2): Suppose that B is an ω -open subset of an ω^* -normal space X containing an ω -closed subset A of X . Then by Theorem 10, there exists an ω -open set which we denote by $U_{\frac{1}{2}}$ such that $A \subseteq U_{\frac{1}{2}} \subseteq \omega Cl(U_{\frac{1}{2}}) \subseteq B$. Then $U_{\frac{1}{2}}$ and B are ω -open subsets of X containing the ω -closed sets A and $\omega Cl(U_{\frac{1}{2}})$, respectively. In the same way, there exist ω -open sets, say $U_{\frac{1}{4}}$ and $U_{\frac{3}{4}}$ such that $A \subseteq U_{\frac{1}{4}} \subseteq \omega Cl(U_{\frac{1}{4}}) \subseteq U_{\frac{1}{2}}$ and $\omega Cl(U_{\frac{1}{2}}) \subseteq U_{\frac{3}{4}} \subseteq \omega Cl(U_{\frac{3}{4}}) \subseteq B$. Continuing in this process, for each rational number in the open interval $(0, 1)$ of the form $t = \frac{m}{2^n}$, where $n = 1, 2, \dots$ and $m = 1, 2, \dots, 2^{n-1}$, we obtain ω -open sets of the form U_t such that for each $s < t$ then $A \subseteq U_s \subseteq \omega Cl(U_s) \subseteq U_t \subseteq \omega Cl(U_t)$. We denote the set of all such rational numbers of Ψ , and define $f : X \rightarrow I$ as follows:

$$f(x) = \begin{cases} 1 & \text{if } x \in X \setminus B, \\ \inf\{t : t \in \Psi \text{ and } x \in U_t\} & \end{cases}$$

$f(X \setminus B) = \{1\}$ and if $x \in A$, then $x \in U_t$ for all $t \in \Psi$. Therefore, by the definition of f , we have $f(x) = \inf \Psi = 0$. Hence $f(B) = \{0\}$ and $f(x) \in I$ for all $x \in X$. It remains only to show that f is an ω -continuous function since the intervals of the form $[0, a)$ and $(b, 1]$, where $a, b \in (0, 1)$ form an open subbase of the space I . If $x \in U_t$ for some $t < a$, then $f(x) = \inf\{s : s \in \Psi \text{ and } x \in U_s\} = r \leq t < a$. Thus $0 \leq f(x) < a$. If $f(x) = 0$, then $x \in U_t$ for all $t \in \Psi$. Hence $x \in U_t$ for some $t < a$. If $0 < f(x) < a$, by definition of f we have $f(x) = \{s : s \in \Psi \text{ and } x \in U_s\} < a$ since $a < 1$. Thus $f(x) = t$ for some $t < a$, and hence $x \in U_t$ for some $t < a$. Therefore, we conclude that $0 \leq f(x) < a$ if and only if $x \in U_t$ for some $t < a$. Hence $f^{-1}([0, a)) = \cup\{U_t : t \in \Psi \text{ and } x \in U_t\}$ which is an ω -open subset of X . Also it is easy to assert that: $0 \leq f(x) \leq b$ if and only if $x \in U_t$ for all $t > b$. Let $x \in X$ such that $0 \leq f(x) \leq b$. It is evident that $f(x) < t$ for all $t > b$ which implies that $x \in U_t$ for all $t > b$. For the converse, let $x \in U_t$ for all $t > b$. Then $f(x) \leq t$ for all $t > b$. Thus $f(x) \leq b$ and it is clear

form the definition of f , that $f(x) \geq 0$. This proves our assertion. Since for all $t > b$, there is $r \in \Psi$ such that $t > r > b$. Then $\omega Cl(U_r) \subseteq U_t$. Consequently we have $\cap\{U_t : t \in \Psi \text{ and } t > b\} = \cap\{\omega Cl(U_r) : r \in \Psi \text{ and } r > b\}$. Therefore, $f^{-1}([0, b]) = \{x : 0 \leq f(x) \leq b\} = \cap\{U_t : t \in \Psi \text{ and } t > b\} = \cap\{\omega Cl(U_r) : r \in \Psi \text{ and } r > b\}$. Since $f^{-1}((0, 1]) = f^{-1}(I \setminus [0, b]) = X \setminus f^{-1}([0, b]) = \cup\{X \setminus \omega Cl(U_r) : r \in \Psi \text{ and } r > b\}$ which is ω -open, and this completes the proof of this part.

(2) \Rightarrow (3): Obvious.

(3) \Rightarrow (1): Let A and B be two disjoint ω -closed subsets of X . Then by hypothesis, there exists an ω -continuous function $f : X \rightarrow I$ such that $f(A) = \{0\}$ and $f(B) = \{1\}$. Then the disjoint open sets $[0, \frac{1}{2})$ and $(\frac{1}{2}, 1]$ in I containing $f(A)$ and $f(B)$, respectively. The ω -continuity of f gives that $f^{-1}([0, \frac{1}{2}))$ and $f^{-1}((\frac{1}{2}, 1])$ are disjoint ω -open sets in X containing A and B , respectively. ■

In virtue of Theorem 5, Theorem 11 and the fact that every bounded closed intervals of R are homeomorphic, we can generalize the Tietze Extension Theorem to ω^* -normality which is also an important characterization of ω^* -normal space.

Theorem 12. *A space X is ω^* -normal if and only if every ω -continuous function g on an ω -closed subset of X into any closed interval $[a, b]$ has an ω -continuous extension f over X into $[a, b]$.*

The following result contains the relationship between ω^* -normal and an ω -zero-set:

Proposition 1. *Let X be a space. Then*

- (1) *An ω -zero-set of X is ω -closed and it is the intersection of many countable ω -open sets,*
- (2) *Let H be an ω -closed subset of X which is the intersection of many countable ω -open sets. If X is ω^* -normal, then H is an ω -zero-set.*

Proof. (1). Let F be an ω -zero-set of a space X . Then by Remark 2, F is an ω -closed subset of X . Then by Lemma 2, there exists an ω -continuous function $f : X \rightarrow R$ such that $f \geq 0$ and $F = \omega Z(f)$. Hence $F = \cap\{U_n : n \in Z^+\}$, where $U_n = \{x \in X : f(x) < \frac{1}{n}\}$.

(2). Let H be an ω -closed subset of an ω^* -normal space X such that $H = \cap\{U_n : n \in Z^+\}$, where U_n is an ω -open set for each $n \in Z^+$. Since $H \subseteq U_n$ for each $n \in Z^+$ and X is an ω^* -normal space, for each $n \in Z^+$, there exists an ω -continuous function $f_n : X \rightarrow [0, \frac{1}{3^n}]$ such that $f_n(H) = \{0\}$ and $f_n(X \setminus U_n) = \{\frac{1}{3^n}\}$ by Theorem 11. Since $\sum_{n=0}^{\infty} f_n(x) \leq \sum_{n=0}^{\infty} \frac{1}{3^n}$ and the series $\sum_{n=0}^{\infty} \frac{1}{3^n}$ is absolutely convergent, the function $f : X \rightarrow R$ given by

$f(x) = \sum_{n=0}^{\infty} f_n(x)$ for each $x \in X$ is an ω -continuous function and $H = \omega Z(f)$ by Theorem 5. ■

Definition 7. (1) (i) A filter base Λ is said to be ω -convergent to a point x in X if for any $U \in \omega(\tau)$ containing x , there exists $B \in \Lambda$ such that $B \subset U$.

(ii) A filter base Λ is said to be convergent to a point x in X if for any open set U of X containing x , there exists $B \in \Lambda$ such that $B \subset U$.

Theorem 13. If a function $f : (X, \tau) \rightarrow (Y, \sigma)$ is ω -continuous, then for each point $x \in X$ and each filter base Λ in X ω -converging to x , the filter base $f(\Lambda)$ is convergent to $f(x)$.

Proof. Let $x \in X$ and Λ be any filter base in X ω -converging to x . Since f is ω -continuous, then for any open set V of (Y, σ) containing $f(x)$, there exists $U \in \omega O(X, x)$ such that $f(U) \subset V$. Since Λ is ω -converging to x , there exists a $B \in \Lambda$ such that $B \subset U$. This means that $f(B) \subset V$ and hence the filter base $f(\Lambda)$ is convergent to $f(x)$. ■

Let $\{X_\alpha : \alpha \in \Lambda\}$ and $\{Y_\alpha : \alpha \in \Lambda\}$ be two families of topological spaces with the same index set Λ . The product space of $\{X_\alpha : \alpha \in \Lambda\}$ is denoted by $\Pi \{X_\alpha : \alpha \in \Lambda\}$ (or simply ΠX_α). Let $f_\alpha : X_\alpha \rightarrow Y_\alpha$ be a function for each $\alpha \in \Lambda$. The product function $f : \Pi X_\alpha \rightarrow \Pi Y_\alpha$ is defined by $f(\{x_\alpha\}) = \{f_\alpha(x_\alpha)\}$ for each $\{x_\alpha\} \in \Pi X_\alpha$.

Theorem 14. If a function $f : X \rightarrow \Pi Y_\alpha$ is ω -continuous, then $P_\alpha \circ f : X \rightarrow Y_\alpha$ is ω -continuous for each $\alpha \in \Lambda$, where P_α is the projection of ΠY_α onto Y_α .

Proof. Let V_α be any open set of Y_α . Then, $P_\alpha^{-1}(V_\alpha)$ is open in ΠY_α and hence $(P_\alpha \circ f)^{-1}(V_\alpha) = f^{-1}(P_\alpha^{-1}(V_\alpha))$ is ω -open in X . Therefore, $P_\alpha \circ f$ is ω -continuous. ■

Theorem 15. If a function $f : \Pi X_\alpha \rightarrow \Pi Y_\alpha$ is ω -continuous, then $f_\alpha : X_\alpha \rightarrow Y_\alpha$ is ω -continuous for each $\alpha \in \Lambda$.

Proof. Let V_α be any open set of Y_α . Then $P_\alpha^{-1}(V_\alpha)$ is open in ΠY_α and $f^{-1}(P_\alpha^{-1}(V_\alpha)) = f_\alpha^{-1}(V_\alpha) \times \Pi\{X_\alpha : \alpha \in \Lambda \setminus \{\alpha\}\}$. Since f is ω -continuous, $f^{-1}(P_\alpha^{-1}(V_\alpha))$ is ω -open in ΠX_α . Since the projection P_α of ΠX_α onto X_α is open continuous, $f_\alpha^{-1}(V_\alpha)$ is ω -open in X_α and hence f_α is ω -continuous. ■

Theorem 16. Let $\{X_\alpha : \alpha \in \Sigma\}$ be a family of spaces. Let $X = \Pi_\Sigma X_\alpha$ and let Γ be a finite nonempty subset of Σ . For each $\alpha \in \Gamma$, let $g_\alpha : X_\alpha \rightarrow Y_\alpha$ be ω -continuous. Then $g : X \rightarrow \Pi_\Gamma Y_\alpha$ defined by $g(x) = (g_\alpha(P_\alpha(x)))$ is ω -continuous, where P_α is the projection from X to X_α .

Proof. Let $\Pi_\Gamma V_\alpha$ be a basic open subset of $\Pi_\Gamma Y_\alpha$. Then $g^{-1}(\Pi_\Gamma V_\alpha) = \bigcap_{\alpha \in \Gamma} P_\alpha^{-1}(g_\alpha^{-1}V_\alpha)$. Since for each $\alpha \in \Gamma$, $g_\alpha : X_\alpha \rightarrow Y_\alpha$ is ω -continuous, $g^{-1}(\Pi_\Gamma V_\alpha)$ is a ω -open in X . Therefore g is ω -continuous. ■

Theorem 17. *Let $\{X_\alpha : \alpha \in \Sigma\}$ be a family of topological spaces and Γ be a nonempty and finite subfamily of Σ . For each $\alpha \in \Gamma$, let $g_\alpha : X_\alpha \rightarrow R$ be ω -continuous. Let $X = \Pi_\Sigma X_\alpha$. Then for each of the following cases, $\lambda : X \rightarrow R$, as defined, is ω -continuous.*

- (1) $\lambda(x) = \Sigma_{\alpha \in \Gamma} g_\alpha \circ P_\alpha(x)$.
- (2) $\lambda(x) = \Pi_{\alpha \in \Gamma} g_\alpha \circ P_\alpha(x)$.
- (3) $\lambda(x) = \max\{g_\alpha \circ P_\alpha(x) : \alpha \in \Gamma\}$.
- (4) $\lambda(x) = \min\{g_\alpha \circ P_\alpha(x) : \alpha \in \Gamma\}$.

Proof. We shall give the proof of (3). The proofs of (1), (2) and (4) can be done similarly. Proof of (3). Let $Y_\alpha = R$ for each $\alpha \in \Gamma$. Then the function $h : \Pi Y_\alpha \rightarrow R$ is defined by $h(x) = \max\{P_\alpha(x) : \alpha \in \Gamma\}$ is continuous. If g is defined as in Theorem 16 above, then for each $x \in X$, $(h \circ g)(x) = h((g_\alpha \circ P_\alpha(x))) = \max\{g_\alpha \circ P_\alpha(x) : \alpha \in \Gamma\} = \lambda(x)$. In view of Theorem 16, g is ω -continuous. Hence λ is ω -continuous. ■

Theorem 18. *Let $\{X_\alpha : \alpha \in \Sigma\}$ be a collection of spaces with the property that if $F_\alpha \subseteq X_\alpha$ is closed and $x_\alpha \in X_\alpha \setminus F_\alpha$. Then there is an ω -continuous function $g_\alpha : X_\alpha \rightarrow [0, 1]$ such that $g_\alpha(x_\alpha) = 0$ and $g_\alpha(F_\alpha) = 1$. Let $X = \Pi_\Sigma X_\alpha$. Then for each closed set $F \subseteq X$ and $x \in X \setminus F$, there is a ω -continuous function $g : X \rightarrow [0, 1]$ such that $g(x) = 0$ and $g(F) = 1$.*

Proof. Let $\{X_\alpha : \alpha \in \Sigma\}$ be a collection of spaces satisfying the hypothesis of the theorem and let $X = \Pi_\Sigma X_\alpha$. Let $F \subseteq X$ be a closed subset and $x \in X \setminus F$. There is a nonempty finite $\Gamma \subseteq \Sigma$ and for each $\alpha \in \Gamma$, an open subset V_α of X_α such that $x \in \bigcap_\Gamma P_\alpha^{-1}(V_\alpha) \subseteq X \setminus F$. For each $\alpha \in \Gamma$, $P_\alpha(x) \in V_\alpha$ and $X_\alpha \setminus V_\alpha$ is a closed subset of X_α . Hence there is a ω -continuous function $g_\alpha : X_\alpha \rightarrow [0, 1]$ such that $g_\alpha(P_\alpha(x)) = 0$ and $g_\alpha(X_\alpha \setminus V_\alpha) = 1$. Let $\lambda : X \rightarrow [0, 1]$ be defined as $\lambda(y) = \max\{g_\alpha \circ P_\alpha(y) : \alpha \in \Gamma\}$. Then λ is ω -continuous, in view of Theorem 17. Moreover, $\lambda(x) = \max\{g_\alpha \circ P_\alpha(x) : \alpha \in \Gamma\} = 0$. Suppose $y \in F$. Since $y \notin X \setminus F$, there is a $\mu \in \Gamma$ such that $y \notin P_\mu^{-1}(V_\mu)$, since $y \notin \bigcap_\Gamma P_\alpha^{-1}(V_\alpha)$. For such a μ , $P_\mu(y) \notin V_\mu$. Hence $g_\mu \circ P_\mu(y) = 1$ and $\lambda(y) = 1$. ■

Recall that for a function $f : (X, \tau) \rightarrow (Y, \sigma)$, the subset $\{(x, f(x)) : x \in X\} \subset X \times Y$ is called the graph of f and is denoted by $G(f)$.

Definition 8 ([4]). *A graph $G(f)$ of a function $f : (X, \tau) \rightarrow (Y, \sigma)$ is said to be ω -closed if for each $(x, y) \in (X \times Y) \setminus G(f)$, there exists $U \in \omega\mathcal{O}(X, x)$ and a closed set V of Y containing y such that $(U \times V) \cap G(f) = \emptyset$.*

Lemma 6. *A graph $G(f)$ of a function $f : (X, \tau) \rightarrow (Y, \sigma)$ is ω -closed in $X \times Y$ if and only if for each $(x, y) \in (X \times Y) \setminus G(f)$, there exist $U \in \omega(\tau)$ containing x and a closed set V of Y containing y such that $f(U) \cap V = \emptyset$.*

Proof. The proof is an immediate consequence of Definition 8. ■

Theorem 19. *If $f : (X, \tau) \rightarrow (Y, \sigma)$ is an ω -continuous function and (Y, σ) is a T_1 -space, then $G(f)$ is ω -closed.*

Proof. Let $(x, y) \in (X \times Y) \setminus G(f)$. Then $y \neq f(x)$. Since Y is T_1 , there exists an open set V in Y such that $f(x) \in V$ and $y \notin V$. Since f is ω -continuous, there exists $U \in \omega O(X, x)$ such that $f(U) \subset V$. Therefore, $f(U) \cap (Y \setminus V) = \emptyset$ and $Y \setminus V$ is a closed subset of Y containing y . This shows that $G(f)$ is ω -closed. ■

Now, we recall the following definitions.

Definition 9. *A space (X, τ) is said to be*

- (1) *ω -compact [7], [4] if every ω -open cover of X has a finite subcover;*
- (1) *ω -compact relative to X if every cover of A by ω -open sets of X has a finite subcover.*

Theorem 20. *[4] If a function $f : (X, \tau) \rightarrow (Y, \sigma)$ is ω -continuous and A is ω -compact relative to X , then $f(A)$ is compact in Y .*

Proof. Let $\{H_\alpha : \alpha \in I\}$ be any cover of $f(A)$ by open sets of the subspace $f(A)$. For each $\alpha \in I$, there exists a open set A_α of Y such that $H_\alpha = K_\alpha \cap f(A)$. For each $x \in A$, there exists $\alpha_x \in I$ such that $f(x) \in A_{\alpha_x}$ and there exists $U_x \in \omega(\tau)$ containing x such that $f(U_x) \subset A_{\alpha_x}$. Since the family $\{U_x : x \in K\}$ is a cover of A by ω -open sets of K , there exists a finite subset A_0 of A such that $A \subset \{U_x : x \in A_0\}$. Therefore, we obtain $f(A) \subset \cup\{f(U_x) : x \in A_0\}$ which is a subset of $\cup\{A_{\alpha_x} : x \in A_0\}$. Thus, $f(A) = \cup\{A_{\alpha_x} : x \in A_0\}$ and hence $f(A)$ is compact. ■

Definition 10. *A space (X, τ) is said to be:*

- (1) *countably ω -compact if every ω -open countably cover of X has a finite subcover;*
- (2) *ω -Lindelof if every ω -open cover of X has a countable subcover;*
- (3) *ω -closed compact if every ω -closed cover of X has a finite subcover;*
- (4) *countably ω -closed compact if every countably cover of X by ω -closed sets has a finite subcover.*

Theorem 21. *Let $f : (X, \tau) \rightarrow (Y, \sigma)$ be an ω -continuous surjective function. Then the following statements hold:*

- (1) *If X is ω -Lindelof, then Y is Lindelof;*

(2) If X is countably ω -compact, then Y is countably compact.

Proof. (1). Let $\{V_\alpha : \alpha \in I\}$ be an open cover of Y . Since f is ω -continuous, then $\{f^{-1}(V_\alpha) : \alpha \in I\}$ is an ω -open cover of X . Since X is ω -Lindelof, there exists a countable subset I_0 of I such that $X = \cup\{f^{-1}(V_\alpha) : \alpha \in I_0\}$. Thus, $Y = \cup\{V_\alpha : \alpha \in I_0\}$ and hence Y is Lindelof.

(2). Similar to (1). ■

Theorem 22. Let $f : (X, \tau) \rightarrow (Y, \sigma)$ be an ω -continuous surjective function. Then the following statements hold

- (1) If X is ω -closed compact, then Y is compact;
- (2) If X is ω -closed Lindelof, then Y is Lindelof;
- (13) If X is countably ω -closed compact, then Y is countably compact.

Proof. The proof is similar to Theorem 21. ■

Definition 11. A space (X, τ) is said to be:

- (i) ω - T_1 [3], [5] if for each pair of distinct points x and y of X , there exist ω -open sets U and V containing x and y , respectively such that $y \notin U$ and $x \notin V$.
- (2) ω - T_2 [3], [5] if for each pair of distinct points x and y in X , there exist disjoint ω -open sets U and V in X such that $x \in U$ and $y \in V$.

Recall, that a subset B_x of a topological space (X, τ) is said to be an ω -neighbourhood of a point $x \in X$ [7] if there exists an ω -open set U such that $x \in U \subset B_x$.

Theorem 23 ([4]). If an injective function $f : (X, \tau) \rightarrow (Y, \sigma)$ is ω -continuous and Y is a T_1 -space, then X is an ω - T_1 -space.

Proof. Suppose that Y is T_1 . For any distinct points x and y in X , there exist open sets V and W such that $f(x) \in V$, $f(y) \notin V$, $f(x) \notin W$ and $f(y) \in W$. Since f is ω -continuous, $f^{-1}(V)$ and $f^{-1}(W)$ are ω -open subsets of (X, τ) such that $x \in f^{-1}(V)$, $y \notin f^{-1}(V)$, $x \notin f^{-1}(W)$ and $y \in f^{-1}(W)$. This shows that X is ω - T_1 . ■

Theorem 24 ([4]). If $f : (X, \tau) \rightarrow (Y, \sigma)$ is an ω -continuous injective function and (Y, σ) is a T_2 -space, then (X, τ) is ω - T_2 -space.

Proof. For any pair of distinct points x and y in X , there exist disjoint open sets U and V in Y such that $f(x) \in U$ and $f(y) \in V$. Since f is ω -continuous, $f^{-1}(U)$ and $f^{-1}(V)$ are ω -open subsets of (X, τ) containing x and y , respectively. Therefore, $f^{-1}(U) \cap f^{-1}(V) = \emptyset$ because $U \cap V = \emptyset$. This shows that X is ω - T_2 . ■

Lemma 7 ([7]). The intersection of an open and ω -open subset of (X, τ) is ω -open in (X, τ) .

Theorem 25. *If $f : (X, \tau) \rightarrow (Y, \sigma)$ is a continuous function and $g : (X, \tau) \rightarrow (Y, \sigma)$ is a ω -continuous function and Y is a T_2 -space, then the set $E = \{x \in X : f(x) = g(x)\}$ is ω -closed set in X .*

Proof. If $x \in X \setminus E$, then it follows that $f(x) \neq g(x)$. Since Y is T_2 , there exist disjoint open sets V and W of Y such that $f(x) \in V$ and $g(x) \in W$. Since f is continuous and g is ω -continuous, then $f^{-1}(V)$ is open and $g^{-1}(W)$ is ω -open in X with $x \in f^{-1}(V)$ and $x \in g^{-1}(W)$. Put $A = f^{-1}(V) \cap g^{-1}(W)$. By Lemma 7, A is ω -open in X . Therefore, $f(A) \cap g(A) = \emptyset$ and it follows that $x \notin \omega Cl(E)$. This shows that E is ω -closed in X . ■

References

- [1] CROSSLEY S.G., HILDEBRAND S.K., Semi-closure, *Texas J. Sci.*, 22(1971), 99-112.
- [2] LEVINE N., Semi-open sets and semi-continuity in topological spaces, *Amer. Math. Monthly*, 70(1963), 36-41.
- [3] MAKI H., SUNDARAM P., RAJESH N., Characterizations of ω - T_i topological spaces, where $i \in \{0, \frac{1}{2}, 1, 2\}$, ..., (under preparation).
- [4] NOIRI T., POPA V., A generalization of ω -continuity, *Fasc. Math.*, 45(2010), 71-86.
- [5] NOIRI T., POPA V., On m - D -separation axioms, *The Journal of Math. Univ. Istanbul*, 61-62(2002-2003), 15-28.
- [6] POPA V., NOIRI T., On the definitions of some generalized forms of continuity under minimal conditions, *Mem. Fac. Sci. Kochi Univ. Ser. A Math.*, 22(2001), 9-18.
- [7] SHEIK JOHN M., *A study on generalizations of closed sets and continuous maps in topological spaces*, Ph.D. Thesis, Bharathiyar University, Coimbatore, India (2002).
- [8] SHEIK JOHN M., SUNDARAM P., On decomposition of continuity, *Bull. Allahabad Math. Soc.*, 22(2007), 1-9.
- [9] SUNDARAM P., SHEIK JOHN M., Weakly closed sets and weakly continuous maps in topological spaces, *82nd Indian Science Congress, Calcutta*, (1995), pp. 49.

N. RAJESH

DEPARTMENT OF MATHEMATICS
 RAJAH SERFOJI GOVT. COLLEGE
 THANJAVUR-613005, TAMILNADU, INDIA
e-mail: nrajesh_topology@yahoo.co.in

P. SUDHA

DEPARTMENT OF MATHEMATICS
 PAVENDAR BHARATHIDASAN COLLEGE OF ENGG. AND TECH.
 TRICHY-620024, TAMILNADU, INDIA
e-mail: pg.sudhaganesan@gmail.com

Received on 10.11.2014 and, in revised form, on 06.11.2015.