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ON THE EXISTENCE OF NONTRIVIAL SOLUTIONS FOR NONLOCAL ELLIPTIC KIRCHHOFF TYPE PROBLEMS WITH NONLINEAR BOUNDARY CONDITIONS

Abstract. In this paper, by using the Mountain Pass Lemma, we study the existence of nontrivial solutions for a nonlocal elliptic Kirchhoff type equation together with nonlinear boundary conditions.

Key words: Kirchhoff type problems, Mountain-Pass Lemma, nonlinear boundary conditions.

AMS Mathematics Subject Classification: 35J60, 35J20, 35J25.

1. Introduction and preliminaries

Consider the boundary value problem of Kirchhoff type

$$\begin{cases} -(a+b\int_\Omega |\nabla u|^2 dx)\Delta u = f(x,u), & x\in\Omega, \\ \frac{\partial u}{\partial n} = g(x,u), & x\in\partial\Omega, \end{cases} \tag{1}$$

where Ω is a bounded domain with smooth boundary in R^N for $N=1,2,3$, $a,b>0$, are real numbers, and f, g are Carathéodory functions.

Recently, more researches were done about existence of nontrivial solutions to the classes of the Kirchhoff type problems by Mathematicians (See [3], [4], [5], [6], [7], [8]). Also in [2], Bitao Cheng studied the existence and multiplicity results of nontrivial solutions for nonlocal elliptic Kirchhoff type problem

$$\begin{cases} -(a+b\int_\Omega |\nabla u|^2 dx)\Delta u = f(x,u), & x\in\Omega, \\ u=0, & x\in\partial\Omega, \end{cases} \tag{2}$$

under the following assumptions:

(F1) $\liminf_{|t|\to\infty} \frac{4F(x,t)}{bt^4} > \mu_1$, uniformly in $x\in\Omega$.

(F2) $\limsup_{t\to 0} \frac{2F(x,t)}{at^2} < \lambda_1$, uniformly in $x \in \Omega$.

(F3) $\liminf_{|t|\to\infty} \frac{f(x,t)t-4F(x,t)}{|t|^\tau} > -\alpha$, uniformly in $x \in \Omega$.

where λ_1 is the first of the eigenvalue problem

$$\begin{cases} -\Delta u = \lambda u, & x \in \Omega, \\ u = 0, & x \in \partial\Omega, \end{cases} \tag{3}$$

and μ_1 is the first of the eigenvalue problem

$$\begin{cases} -(\int_\Omega |\nabla u|^2)\triangle u = \mu u^3, & x \in \Omega, \\ u = 0, & x \in \partial\Omega. \end{cases} \tag{4}$$

Motivated by the results of the above works, we are interested in the existence of positive solutions for problem (1). Our main difficulty will be the nonlinearity of $g(x,u)$. To overcome this difficulty, we need to restrict the problem (1) to some assumptions.

Problem (1) is posed in the framework of the Sobolev space $X = H^1(\Omega)$ with the standard norm

$$\|u\|^2 = \int_\Omega |\nabla u|^2 dx.$$

Moreover, a function $u \in X$ is said to be a weak solution of problem (1) if

$$\int_\Omega f(x,u)\, v dx = -(a + b\|u\|^2)[\int_\Omega v\, g(x,u) dx - \int_{\partial\Omega} \nabla u \nabla v ds],$$

for all $v \in H$. It is well known that weak solutions of problem (1) correspond to critical points of the functional $I : X \to \mathbb{R}$

$$I(u) = \frac{a}{2}\|u\|^2 + \frac{b}{4}\|u\|^4 - \int_\Omega F(x,u)dx - \int_{\partial\Omega} G(x,u)dx, \tag{5}$$

where

$$F(x,u) = \int_0^u f(x,t)dt, \qquad G(x,u) = \int_0^u g(x,t)dt.$$

The base of our work is finding critical points by using the mountain pass lemma which we describe below.

Definition 1. *Let X be a Banach space and $I \in C^1(X,R)$. We say that I satisfies the (PS) condition if any sequence $\{u_n\} \subset X$ that $\{I(u_n)\}$ be bounded and $\{I'(u_n)\} \longrightarrow \infty$ as $n \longrightarrow \infty$, possesses a convergent subsequence.*

Lemma 1 (Mountain pass [1])**.** *Let X be a real Banach space and $I \in C^1(X,R^1)$ satisfying (PS) condition. Suppose*

(L1) *there are constants* $a, r > 0$ *such that for any* $u \in X$ *that* $\|u\| = r$, *we have*

$$I(u) \geq a > 0;$$

(L2) *there is* $e \in X$ *such that* $I(e) < 0$;

Then I *possesses a critical value as*

$$C = \inf_{g\in\Gamma} \max_{t\in[0,1]} I(g(t)),$$

where

$$\Gamma = \{g \in C([0,1], X) : g(0) = 0,\ g(1) = e\}.$$

Definition 2. *We say that operator* $J : X \longrightarrow X^*$ *is satisfying in condition* $(S)_+$, *if* $u_n \rightharpoonup u$ *in* X *and* $\limsup_{n\longrightarrow\infty}\langle J(u_n), x_n - x\rangle \leq 0$, *implies* $u_n \to u$ *in* X.

2. Existence theorem

We set

$$\Upsilon(u) = \frac{a}{2}\|u\|^2 + \frac{b}{4}\|u\|^4 \quad \Phi(u) = \int_\Omega F(x,u)dx, \quad \Psi(u) = \int_{\partial\Omega} G(x,u)dx$$

where

$$F(x,u) = \int_0^u f(x,t)dt, \qquad G(x,u) = \int_0^u g(x,t)dt.$$

Note that $\Upsilon' : X \longrightarrow X^*$ such that $\langle \Upsilon'(u), v\rangle = (a + b\|u\|^2)\int_\Omega \nabla u \nabla v dx$, is satisfying in conditions $(S)_+$, and is a homeomorphism.

Also we have $\Phi' : X \longrightarrow X^*$ such that $\langle \Phi'(u), v\rangle = \int_\Omega vg(x,u)dx$, $\forall v \in X$ and for any $v \in X$ we define $F : X^* \to \mathbb{R}$ such that $F(J) = J(v)$. Then $F \in X^{**}$ and F is linear and continuous. Since F is finite rank then it is compact. Now if $\{u_n\}$ be bounded in X then

$$\begin{aligned}\|\Phi'(u_n)\| &= \sup \|\langle \Phi'(u_n), v\rangle\| = \sup_{\|v\|=1} \|\int_\Omega f(x,u_n)vdx\| \\ &\leq \sup_{\|v\|=1} \int_\Omega f(x,u_n)\|v\|dx \leq \int_\Omega \|f\|dx = \|f\||\Omega|,\end{aligned}$$

since $\|f\| = \sup_{x\in\Omega, n\in N} |f(x,u_n)|$ and $\{u_n\} \subset \mathbb{R}$ is bounded and f on $\mathbb{R}$ is continuous, then $\| f \|$ is finite, therefore $\|\Phi'(u_n)\| < \infty$, i.e., $\{\Phi'(u_n)\}$ is bounded. By the compactness of F, $\{F(\Phi'(u_n))\}$ possesses a convergent subsequence, then is satisfying in cauchy condition. It is easy to see that Φ' is compact, so is continuous. Similarly Ψ' is compact and continuous. Then $I \in C^1(X, R)$. We now consider the following assumptions to state our main result:

(H1) there exists $c_1 > 0$, such that $|f(x,t)| \leq c_1 t^p$; $2 < p < 2^*$.

(H2) there exists $c_2 > 0$, such that $|g(x,t)| \leq c_2 t^q$; $2 < q < 2^*$.

(H3) $\lim_{|t|\to\infty} \frac{F(x,t)}{|t|^2} = 0$, uniformly for any x.

(H4) $\lim_{|t|\to\infty} \frac{G(x,t)}{|t|^2} = 0$, uniformly for any x.

(H5) there exists $\Omega' \subset \Omega$ such that $|\Omega'| > 0$, there exists $t_0 > 0$ such that for any $x \in \Omega'$ we have $F(x,t_0) > 0$.

Now we give our main result.

Theorem 1. *Let (H1)-(H5) hold. Then the problem (1) has at least one nontrivial solution in X.*

To prove Theorem 1, we require the following three lemmas:

Lemma 2. *Under the conditions $G_1 - G_5$, the functional defined I in (5) is satisfying in (PS) condition.*

Proof. Let $\{u_n\} \subset X$ be such that $\{I(u_n)\}$ is bounded and $I'(u_n) \longrightarrow 0$. We now show that, $\{u_n\}$ possesses convergent subsequence. As $\{I(u_n)\}$ is bounded, then there exists $K > 0$ such that

$$K \geq I(u_n) = \frac{a}{2} \parallel u \parallel^2 + \frac{b}{4} \parallel u_n \parallel^4 - \int_\Omega F(x,u_n)dx - \int_{\partial\Omega} G(x,u_n)dx.$$

First we claim that $\{u_n\}$ is bounded. If $\{u_n\}$ be unbounded, then contains a subsequence as $\{u_{n_j}\}$, that $\|u_{n_j}\| \to \infty$ as $j \to \infty$. By the Poincar's inequality $\int_\Omega |\nabla u_{n_j}|^2 \to \infty$. So for j large enough we can consider $|\nabla u_{n_j}|^2 \geq 1$. By (G_3) and (G_4) there exists $\{\varepsilon_n\}$ such that $\varepsilon_n \to 0$ as $n \to \infty$, and

$$F(x,u_n) \leq \varepsilon_n |u_n|^2, \qquad G(x,u_n) \leq \varepsilon_n |u_n|^2.$$

Then

$$\begin{aligned} K \geq I(u_{nj}) &= \frac{a}{2} \parallel u_{n_j} \parallel^2 - \varepsilon_n \int_\Omega |u_{n_j}|^2 dx - \varepsilon_n \int_{\partial\Omega} |u_{n_j}|^2 dx \\ &= \frac{a}{2} \parallel u_{n_j} \parallel^2 - \varepsilon_n \left(\int_\Omega |u_{n_j}|^2 dx + \int_{\partial\Omega} |u_{n_j}|^2 dx \right). \end{aligned}$$

On the other hand, by the Poincar's inequality there exists $c_0 > 0$, such that

$$\int_\Omega |u_{n_j}|^2 dx \leq c_0 \int_\Omega |\nabla u_{n_j}|^2 dx.$$

Consequently,

$$K \geq I(u_{n_j}) = \frac{a}{2} \|u_{n_j}\|^2 - \varepsilon_n c_0 \left(\int_\Omega |\nabla u_{n_j}|^2 dx + \frac{1}{c_0} \int_{\partial\Omega} |u_{n_j}|^2 dx \right).$$

If we take

$$\varepsilon_n = \frac{1}{\int_\Omega |\nabla u_{n_j}|^2 dx + \frac{1}{c_0}\int_{\partial\Omega} |u_{n_j}|^2 dx},$$

then $K \geq \frac{a}{2}\|u_{n_j}\|^2 - c_0$. This is contradiction. Therefore, $\{u_n\}$ is bounded in X. So, by the reflexivity of X, $\{u_n\}$ has a week convergent subsequence in X like $\{u_{n_k}\}$. Hence, due to the compactness Φ' and Ψ' we have that

$$\Phi' + \Psi' \to u.$$

Since $I = \Upsilon - \Phi - \Psi$ and $I'(u_{n_k}) \longrightarrow 0$ then

$$\Upsilon'(u_{n_k}) \longrightarrow \Phi'(u_{n_k}) + \Psi'(u_{n_k})$$

Since Υ' is homeomorphism, we can conclude that

$$u_{n_k} \longrightarrow (\Upsilon')^{-1}(u).$$

■

Lemma 3. *There exists $r > 0$ such that for every $u \in X$, with $\|u\| = r$ we have $I(u) > 0$.*

Proof. By (G_1) and (G_2) for every $u \neq 0$, $t \in R$ we see that

$$\begin{aligned} I(tu) &= \frac{a}{2}t^2\|u\|^2 + \frac{b}{4}t^4\|u\|^4 - \int_\Omega F(x,tu)dx - \int_{\partial\Omega} G(x,tu)dx \\ &\geq \frac{a}{2}t^2\|u\|^2 - \max\{c_1,c_2\}|t|^{\max\{p,q\}}[\int_\Omega |u|^2dx + \int_{\partial\Omega}|u|^2dx] \\ &= t^2[\frac{a}{2}\|u\|^2 - \max\{c_1,c_2\}|t|^{\max\{p,q\}-2}(\int_\Omega |u|^2dx + \int_{\partial\Omega}|u|^2dx)]. \end{aligned}$$

Therefore, for every t small enough $I(tu) > 0$. Now for any t that $I(tu) > 0$ we can take $r = \|tu\|$. ■

Lemma 4. *There exists $e \in X$ such that $I(e) < 0$.*

Proof. Define

$$u(x) = \begin{cases} 1, & x \in \Omega, \\ 0, & x \in \Omega - \Omega'. \end{cases}$$

Then $\int_\Omega |\nabla u|^2 dx = 0$. By (G_5) we have

$$I(t_0u) = -\int_\Omega F(x,t_0u)dx - \int_{\partial\Omega} G(x,t_0u)dx.$$

Since $G(x, t_0 u) = G(x, 0) = 0$, for all $x \in \partial\Omega$, and

$$\begin{aligned}\int_\Omega F(x, t_0 u)dx &= \int_{\Omega'} F(x, t_0)dx - \int_{\Omega-\Omega'} F(x, 0)dx \\ &= \int_{\Omega'} F(x, t_0) > 0,\end{aligned}$$

we conclude that $I(t_0 u) < 0$. Therefore by choosing $e = t_0 u$ the lemma is proved. ■

Now, we complete the proof of Theorem 1. By Lemmas 2-4, the conditions of Mountain Pass Lemma are satisfied. Therefore, I has a nontrivial critical point as

$$C = \inf_{g \in \Gamma} \max_{t \in [0,1]} I(g(t)),$$

that

$$\Gamma = \{g \in C([0, 1], X);\ \ g(0) = 0,\ \ g(1) = t_0 u\}.$$

Then the problem (1) has a nontrivial solution and also Lemma 3 implies that C is positive.

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