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FASCICULI MATHEMATICI

Nr 56 2016

DOI:10.1515/fascmath-2016-0004

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SOME ADVANCES IN THE THEORY OF QUASI-PSEUDOMETRIC TYPE SPACES

Abstract. In this paper, we extend most of the results proved in [4]. In particular, we give some topological properties of the quasi-pseudometric type spaces. Moreover, some fixed point and common fixed point theorems are obtained in the setting of quasi-pseudometric spaces, introduced some months ago by Kazeem et al in [4].

Key words: quasi-pseudometric type spaces, fixed point, left K-completeness.

AMS Mathematics Subject Classification: 47H09.

Symmetric spaces were introduced in 1931 by Wilson [6], as metric-like spaces lacking the triangle inequality. Several fixed point results in such spaces were obtained. In the same dynamics, cone metric spaces were introduced by Huang [3] and many fixed point results concerning mappings in these spaces have also been established. In [5], M. A. Khamsi connected this concept with a generalised form of metric that he named *metric type*. Namely, he observed that if $d(x, y)$ is a cone metric, then $D(x, y) = \|d(x, y)\|$ is symmetric with some special properties, particularly in the case when the underlying cone is normal. Recently in [4], Kazeem et al. discussed the newly introduced notion of *quasi-pseudometric type spaces* as a logical equivalent to metric type spaces when the initial distance-like function is not symmetric. Some fixed point results of mappings on such spaces were discussed as well in [4]. It is the aim of this article to continue the study of quasi-pseudometric spaces by proving several other fixed point and common fixed point results, hence extending the fixed point results of [4] to a class of mappings satisfying more general contractive conditions.

In this section, we recall briefly some elementary definitions from the asymmetric topology which are necessary for a good understanding of the work below. For recent results and detailed explanations for the concepts in the theory of asymmetric spaces, the reader is referred to [2, 4, 7, 8].

Definition 1. *Let E be a real Banach space with norm $\|.\|$ and P be a subset of E. Then P is called a cone if and only if*

(a) *P is closed, nonempty and $P \neq \{\theta\}$, where θ is the zero vector in E;*
(b) *for any $a, b \geq 0$, and $x, y \in P$, we have $ax + by \in P$;*
(c) *for $x \in P$, if $-x \in P$, then $x = \theta$.*

Given a cone P in a Banach space E, we define on E a partial order $\preceq$ with respect to P by

$$x \preceq y \Longleftrightarrow y - x \in P.$$

We also write $x \prec y$ whenever $x \preceq y$ and $x \neq y$, while $x \ll y$ will stand for $y - x \in Int(P)$ (where $Int(P)$ designates the interior of P).

The cone P is called **normal** if there is a number $C > 0$, such that for all $x, y \in E$, we have

$$\theta \preceq x \preceq y \Longrightarrow \|x\| \leq C\|y\|.$$

The least positive number satisfying this inequality is called the **normal constant** of P. Therefore, we shall then say that P is a K-normal cone to indicate the fact that the normal constant is K.

Definition 2 (Compare [4]). *Let X be a nonempty set. Suppose the mapping $q : X \times X \to E$ satisfies*

($q1$) *$\theta \preceq q(x, y)$ for all $x, y \in X$;*
($q2$) *$q(x, y) = \theta = q(y, x)$ if and only if $x = y$;*
($q3$) *$q(x, z) \preceq q(x, y) + q(y, z)$ for all $x, y, z \in X$.*

Then, q is called a **quasi-cone metric** on X, and (X, q) is called a **quasi-cone metric space**.

Definition 3 (Compare [4]). *A sequence in a quasi-cone metric space (X, q) is called*

(a) ***Q-Cauchy** or **bi-Cauchy** if for every $c \in X$ with $c \gg \theta$, there exists $n_0 \in \mathbb{N}$ such that*

$$\forall\ n, m \geq n_0 \quad q(x_n, x_m) \ll c;$$

(b) ***left(right) Cauchy** if for every $c \in X$ with $c \gg \theta$, there exists $n_0 \in \mathbb{N}$ such that*

$$\forall\ n, m : n_0 \leq m \leq n \quad q(x_m, x_n) \ll c\ (q(x_n, x_m) \ll c \text{ resp.}).$$

Remark 1. A sequence is Q-Cauchy if and only if it is both left and right Cauchy.

Definition 4. (a) *In a quasi-cone metric space (X, q), we say that the sequence (x_n) **left converges** to $x \in X$ if for every $c \in E$ with $\theta \ll c$ there exists N such that for all $n > N$, $q(x_n, x) \ll c$.*

(b) *Similarly, in a quasi-cone metric space* (X, q)*, we say that a sequence* (x_n) ***right converges*** *to* $x \in X$ *if for every* $c \in E$ *with* $\theta \ll c$ *there exists* N *such that for all* $n > N$*,* $q(x, x_n) \ll c$*.*

(c) *Finally, in a quasi-cone metric space* (X, q)*, we say that the sequence* (x_n) ***converges*** *to* $x \in X$ *if for every* $c \in E$ *with* $\theta \ll c$ *there exists* N *such that for all* $n > N$*,* $q(x_n, x) \ll c$ *and* $q(x, x_n) \ll c$*.*

Definition 5. *A quasi-cone metric space* (X, q) *is called*

(a) ***left complete*** (*resp.* ***right complete***) *if every left Cauchy* (*resp. right Cauchy*) *sequence in* X *left* (*resp. right*) *converges.*

(b) ***bicomplete*** *if every* Q*-Cauchy sequence converges.*

Remark 2. A quasi-cone metric space (X, q) is bicomplete if and only if it is left complete and right complete.

Definition 6. *Let* (X, q) *be a quasi-cone metric space. A function* $f : X \to X$ *is said to be* ***Lipschitzian*** *if there exists some* $\kappa \in \mathbb{R}$ *such that*

$$q(f(x), f(y)) \preceq \kappa\, q(x, y) \qquad \forall\, x, y \in X.$$

The smallest constant which satisfies the above inequality is called the ***Lipschitiz constant*** *of* f *and is denoted* $Lip(f)$*. In particular* f *is said to be* ***contractive*** *if* $Lip(f) \in [0, 1)$ *and* ***nonexpansive*** *if* $Lip(f) \leq 1$*.*

Definition 7 (Compare [1])**.** *Let* f *and* g *be self maps on a set* X*. If* $w = fx = gx$ *for some* $x \in X$*, then* x *is called a* ***coincidence point*** *of* f *and* g*, and* w *is called the* ***point of coincidence*** *of* f *and* g*.*

Definition 8. *Let* f *and* g *be self maps on a nonempty set* X*. We say that* f *and* g *are* **weakly compatible** *if they commute at their coincidence point, that is there exists* $x_0 \in X$ *such that* $fx_0 = gx_0$ *then* $gfx_0 = fgx_0$*.*

We also give the following proposition that we take from [1] by omitting the proof.

Proposition 1 (Compare [1])**.** *Let* f *and* g *be weakly compatible self maps on a set* X*. If* f *and* g *have a unique point of coincidence* $w = fx = gx$*, then* w *is the unique common fixed point of* f *and* g*.*

we also have the following important characterization

Lemma 1. *Let* (X, q) *be a quasi-cone metric space,* P *be a* K*-normal cone and* (x_n) *be a sequence in* X*. Then* (x_n) *is a bi-Cauchy sequence if and only if* $q(x_n, x_m) \longrightarrow \theta$ *as* $n, m \longrightarrow \infty$*.*

We now connect the notion of quasi-cone metric to the one of quasi-pseudo-metric type space via the following theorem.

Theorem 1 (Compare [4] Theorem 28). *Let (X, q) be a quasi-cone metric space over the Banach space E with the K-normal cone P. The mapping $Q : X \times X \to [0, \infty)$ defined by $Q(x, y) = \|q(x, y)\|$ satisfies the following properties*

$(Q1)$ $Q(x, x) = 0$ *for any* $x \in X$*;*

$(Q2)$ $Q(x, y) \leq K\big(Q(x, z_1) + Q(z_1, z_2) + \cdots + Q(z_n, y)\big)$, *for any points* $x, y, z_i \in X,\ i = 1, 2, \ldots, n.$

We are therefore led to the following definition.

Definition 9 ([4]). *Let X be a non empty set, and let the function $D : X \times X \to [0, \infty)$ satisfy the following properties:*

$(D1)$ $D(x, x) = 0$ *for any* $x \in X$*;*

$(D2)$ $D(x, y) \leq \alpha\big(D(x, z_1) + D(z_1, z_2) + \cdots + D(x_n, y)\big)$ *for any points* $x, y, z_i \in X$*,* $i = 1, 2, \ldots, n$ *and some constant* $\alpha > 0$*.*

*Then (X, D, α) is called a quasi-pseudometric type space. Moreover, if $D(x, y) = 0 = D(y, x) \Longrightarrow x = y$, then D is said to be a T_0-**quasi-pseudometric type space**. The latter condition is referred to as the T_0-condition.*

Remark 3.
- Let D be a quasi-pseudometric type on X, then the map D^{-1} defined by $D^{-1}(x, y) = D(y, x)$ whenever $x, y \in X$ is also a quasi-pseudometric type on X, called the conjugate of D. We shall also denote D^{-1} by D^t or $\bar{D}$.
- It is easy to verify that the function D^s defined by $D^s := D \vee D^{-1}$, i.e. $D^s(x, y) = \max\{D(x, y), D(y, x)\}$ defines a **metric type** (see [5]) on X whenever D is a T_0-quasi-pseudometric type.
- If we substitute the property $(D1)$ by the following property
  $(D3)$: $D(x, y) = 0 \Longleftrightarrow x = y$,
  we obtain a T_0-quasi-pseudometric type space directly. For instance, this could be done if the map D is obtained from quasi-cone metric.

Moreover, for $\alpha = 1$, we recover the classical pseudometric, hence quasi-pseud-metric type spaces generalize quasi-pseudometrics. It is worth mentioning that if (X, D, α) is a pseudometric type space, then for any $\beta \geq \alpha$, (X, D, β) is also a pseudometric type space. We give the following example to illustrate the above comment.

Example 1. Let $X = \{a, b, c\}$ and the mapping $D : X \times X \to [0, \infty)$ defined by $D(a, b) = D(c, b) = 1/5,\ D(b, c) = D(b, a) = D(c, a) = 1/4,\ D(a, c) = 1/2,\ D(x, x) = 0$ for any $x \in X$ and $D(x, y) = D(y, x)$ for any $x, y \in X$. Since

$$\frac{1}{2} = D(a, c) > D(a, b) + D(b, c) = \frac{9}{20},$$

then we conclude that X is not a quasi-pseudometric space. Nevertheless, with $\alpha = 2$, it is very easy to check that $(X, D, 2)$ is a quasi-pseudometric type space.

Definition 10 ([4]). *Let (X, D, α) be a quasi-pseudometric space. The convergence of a sequence (x_n) to x with respect to D, called **D-convergence** or **left-convergence** and denoted by $x_n \overset{D}{\longrightarrow} x$, is defined in the following way*

$$x_n \overset{D}{\longrightarrow} x \Longleftrightarrow D(x, x_n) \longrightarrow 0. \tag{1}$$

*Similarly, the convergence of a sequence (x_n) to x with respect to D^{-1}, called **D^{-1}-convergence** or **right-convergence** and denoted by $x_n \overset{D^{-1}}{\longrightarrow} x$, is defined in the following way*

$$x_n \overset{D^{-1}}{\longrightarrow} x \Longleftrightarrow D(x_n, x) \longrightarrow 0. \tag{2}$$

*Finally, in a quasi-pseudometric space (X, D, α), we shall say that a sequence (x_n) **D^s-converges** to x if it is both left and right convergent to x, and we denote it as $x_n \overset{D^s}{\longrightarrow} x$ or $x_n \longrightarrow x$ when there is no confusion. Hence*

$$x_n \overset{D^s}{\longrightarrow} x \iff x_n \overset{D}{\longrightarrow} x \text{ and } x_n \overset{D^{-1}}{\longrightarrow} x.$$

Definition 11 ([4]). *A sequence (x_n) in a quasi-pseudometric type space (X, D, α) is called*

*(a) **left K-Cauchy** with respect to D if for every $\epsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that*

$$\forall\ n, k : n_0 \leq k \leq n \quad D(x_k, x_n) < \epsilon;$$

*(b) **right K-Cauchy** with respect to D if for every $\epsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that*

$$\forall\ n, k : n_0 \leq k \leq n \quad D(x_n, x_k) < \epsilon;$$

*(c) **D^s-Cauchy** if for every $\epsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that*

$$\forall n, k \geq n_0 \quad D(x_n, x_k) < \epsilon.$$

Remark 4.

- A sequence is left K-Cauchy with respect to d if and only if it is right K-Cauchy with respect to D^{-1}.
- A sequence is d^s-Cauchy if and only if it is both left and right K-Cauchy.

Definition 12 ([4]). *A quasi-pseudometric space (X, D, α) is called **left-complete** provided that any left K-Cauchy sequence is D-convergent.*

Definition 13 ([4]). *A quasi-pseudometric space (X, D, α) is called **right-complete** provided that any right K-Cauchy sequence is D-convergent.*

Definition 14 ([4]). *A T_0-quasi-pseudometric space (X, D, α) is called **bicomplete** provided that the metric D^s on X is complete.*

2. First results

In [4], Kazeem et al. proved the following:

Theorem 2. *Let (X, q) be a bicomplete quasi-cone metric space, P a K-normal cone. Suppose that a mapping $T : X \to X$ satisfies the contractive condition*

$$q(Tx, Ty) \preceq k\ q(x, y) \quad \text{for all} \quad x, y \in X,$$

where $k \in [0, 1)$. Then T has a unique fixed point. Moreover for any $x \in X$, the orbit $\{T^n x, n \geq 0\}$ converges to the fixed point.

We start by an application of the above the theorem

Theorem 3. *Let (X, q) be a bicomplete quasi-cone metric space, P a K-normal cone. Let $T : X \to X$ be a map such that for every $n \in \mathbb{N}$, there is $\lambda_n \in (0, 1)$ such that*

$$q(T^n x, T^n y) \preceq \lambda_n\ q(x, y) \quad \text{for all} \quad x, y \in X.$$

and let $\lim_{n\to 0} \lambda_n = 0$. Then T has a unique fixed point $\omega \in X$.

Proof. Take λ such that $0 < \lambda < 1$. Since $\lim_{n\to 0} \lambda_n = 0$, there exists $n_0 \in \mathbb{N}$ such that $\lambda_n < \lambda$ for each $n \geq n_0$. Then $q(T^n x, T^n y) \preceq \lambda_n\ q(x, y)$ for all $x, y \in X$ whenever $n \geq n_0$. In other words, for any $m \geq n_0$, $g = T^m$ satisfies

$$q(gx, gy) \preceq k\ q(x, y) \quad \text{for all} \quad x, y \in X.$$

Theorem 2 implies that g has a unique fixed point, say ω. Then $T^m\omega = \omega$, implying that $T^{m+1}\omega = T(T^m\omega) = T^m(T\omega) = T\omega$ and $T\omega$ is also a fixed point of $g = T^m$. Since the fixed point is unique, it follows that $T\omega = \omega$ and ω is the unique fixed point of T. ■

We now state below a generalization of this theorem.

Theorem 4. *Let (X, q) be a bicomplete quasi-cone metric space, P a K-normal cone. Suppose that a mapping $T : X \to X$ is such that for every $n \in \mathbb{N}$, T^n is Lipschitzian and that $\sum_{n=0}^{\infty} Lip(T^n) < \infty$. Then T has a unique fixed point $x^* \in X$.*

Proof. Since for any $n \in \mathbb{N}$, T^n is Lipschitzian, hence there exists $k_n := Lip(T^n) \geq 0$ such that

$$q(T^n x, T^n y) \preceq k_n \; q(x, y) \quad \text{for all} \quad x, y \in X.$$

Now let $x \in X$. For any $n, h \in \mathbb{N}$, we have

$$q(T^n x, T^{n+h} x) \preceq k_n \; q(x, T^h x) \preceq k_n \left[\sum_{i=0}^{h-1} q(T^i x, T^{i+1} x) \right]. \tag{3}$$

Hence

$$q(T^n x, T^{n+h} x) \preceq k_n \left(\sum_{i=0}^{h-1} k_i \right) q(x, Tx), \tag{4}$$

since

$$q(T^i, T^{i+1} x) \preceq k_i \; q(x, Tx), \quad \text{for all} \quad i \in \mathbb{N}.$$

Since $\sum_{n=0}^{\infty} Lip(T^n)$ is convergent, then $\lim_{n \to 0} Lip(T^n) = 0$ and therefore inequality (4) entails that

$$\|q(T^n x, T^{n+h} x)\| \leq K k_n \left(\sum_{i=0}^{h-1} k_i \right) \|q(x, Tx)\| \longrightarrow 0 \;\; \text{as} \;\; n \longrightarrow \infty. \tag{5}$$

Similarly, one shows that

$$\|q(T^{n+h} x, T^n x)\| \leq K k_n \left(\sum_{i=0}^{h-1} k_i \right) \|q(Tx, x)\| \longrightarrow 0 \;\; \text{as} \;\; n \longrightarrow \infty. \tag{6}$$

From relations (5) and (6), we conclude that $(T^n x)$ is a bi-Cauchy sequence. Since (X, q) is bicomplete, there exists $x^* \in X$ such that $(T^n x)$ converges to x^*. First let us show that x^* is a fixed point of T.

On one side, we have

$$\begin{aligned} q(T^{n-1} x, x^*) &\preceq q(T^{n-1} x, T^n x) + q(T^n x, x^*) \\ &\preceq k_{n-1} q(x, Tx) + q(T^n x, x^*), \end{aligned} \tag{7}$$

and on the other side

$$\begin{aligned} q(x^*, T^{n-1} x) &\preceq q(x^*, T^n x) + q(T^n x, T^{n-1} x) \\ &\preceq k_{n-1} q(Tx, x) + q(x^*, T^n x), \end{aligned} \tag{8}$$

From (7), we have that

$$\begin{aligned} q(Tx^*, x^*) &\preceq q(Tx^*, T^n x) + q(T^n x, x^*) \\ &\preceq k_1 q(x^*, T^{n-1}x) + q(T^n x, x^*) \to \theta \quad \text{as} \quad n \to \infty, \end{aligned}$$

i.e

$$q(Tx^*, x^*) = \theta.$$

In the same manner, from (8), we have that

$$q(x^*, Tx^*) = \theta.$$

Hence

$$q(Tx^*, x^*) = \theta = q(x^*, Tx^*).$$

This implies, using property $(q2)$ that $Tx^* = x^*$. So x^* is a fixed point of T. Moreover, if z^* is a fixed point of T, then for all $n \geq 1$, we have

$$q(x^*, z^*) = q(T^n x^*, T^n z^*) \preceq k_n q(x^*, z^*),$$

and

$$q(z^*, x^*) = q(T^n z^*, T^n x^*) \preceq k_n q(z^*, x^*).$$

Since $\lim_{n\to 0} Lip(T^n) = 0$, hence $\|q(x^*, z^*)\| = 0 = \|q(z^*, x^*)\|$ and $x^* = z^*$. Therefore the fixed point is unique. ∎

In the next section, we give some topological properties of quasi-pseudometric type spaces. Most of them deal with sequences and follow closely the classical properties of sequences pseudometric spaces.

3. Topology on Quasi-pseudometric type spaces and fixed point results

3.1. Some topological properties. Let (X, D, α) be a quasi-pseudometric type space. Then for each $x \in X$ and $\epsilon > 0$, the set

$$B_D(x, \epsilon) = \{y \in X : D(x, y) < \epsilon\}$$

denotes the open ϵ-ball at x with respect to D. It should be noted that the collection

$$\{B_D(x, \epsilon) : x \in X, \epsilon > 0\}$$

yields a base for the topology $\tau(D)$ induced by D on X. In a similar manner, for each $x \in X$ and $\epsilon \geq 0$, we define

$$C_D(x, \epsilon) = \{y \in X : D(x, y) \leq \epsilon\},$$

known as the closed ϵ-ball at x with respect to D.

Also the collection

$$\{D_{d^{-1}}(x,\epsilon) : x \in X, \epsilon > 0\}$$

yields a base for the topology $\tau(D^{-1})$ induced by D^{-1} on X. The set $C_D(x,\epsilon)$ is $\tau(D^{-1})$-closed, but not $\tau(D)$-closed in general.

The balls with respect to D are often called *forward balls* and the topology $\tau(D)$ is called *forward topology*, while the balls with respect to D^{-1} are often called *backward balls* and the topology $\tau(D^{-1})$ is called *backward topology*.

The topology $\tau(D)$ of a quasi-pseudometric type space (X, D, α) can be defined starting with starting from the family $\Pi_D(x)$ of neighbourhoods of an arbitrary point $x \in X$.

$$\begin{aligned} V \in \Pi_D(x) &\Longleftrightarrow \exists\ \epsilon > 0 \text{ such that } B_D(x,\epsilon) \subset V \\ &\Longleftrightarrow \exists\ \epsilon' > 0 \text{ such that } C_D(x,\epsilon) \subset V. \end{aligned}$$

To see the equivalence in the above definition, we can take for instance $\epsilon' = \epsilon/3$.

The following proposition contains some simple properties of convergent sequences.

Proposition 2. *Let (x_n) be a sequence in quasi-pseudometric type space (X, D, α).*

(a) *If (x_n) is D-convergent to x and D^{-1}-convergent to y, then $D(x,y) = 0$.*

(b) *If (x_n) is D-convergent to x and $D(y,x) = 0$, then (x_n) is also D-convergent to y.*

Proof.

(a) Letting $n \to \infty$ in the inequality

$$D(x,y) \leq \alpha[D(x,x_n) + D(x_n,y)],$$

one obtains $D(x,y) = 0$.

(b) The result follows from the relations

$$D(x_n,y) \leq \alpha[D(y,x) + D(x,x_n)] = \alpha D(x,x_n) \to 0.$$

■

Also, the following simple remarks concerning sequences in quasi-pseudometric type spaces are true.

Proposition 3. *Let (x_n) be as sequence in a quasi-pseudometric type space (X, D, α).*

(a) *If (x_n) is left K-Cauchy and has a subsequence which is $\tau(D)$-convergent to x, then (x_n) is $\tau(D)$-convergent to x.*

(*b*) *If* (x_n) *is left* K*-Cauchy and has a subsequence which is* $\tau(D^{-1})$*-convergent to* x*, then* (x_n) *is* $\tau(D^{-1})$*-convergent to* x*.*

Proof. (*a*) Suppose that (x_n) is left K-Cauchy and (x_{n_k}) is a subsequence of (x_n) such that $\lim_{k\to\infty} D(x, x_{n_k}) = 0$. For $\epsilon > 0$ choose n_0 such that $n_0 \leq m \leq n$ implies $D(x_m, x_n) < \epsilon/\alpha$, and let $k_0 \in \mathbb{N}$ be such that $n_{k_0} \geq n_0$ and $D(x, x_{n_k}) < \epsilon/\alpha$ for all $k \geq k_0$. Then, for $n \geq n_{k_0}$, $D(x, x_n) \leq \alpha[D(x, x_{n_{k_0}} + D(x_{n_{k_0}}, x_n)] < 2\epsilon$.

(*b*) Reasoning similarly, for $n \geq n_{k_0}$ let $k \in \mathbb{N}$ such that $n_k \geq n$. Then $D(x_n, x) \leq \alpha[D(x_n, x_{n_k}) + D(x_{n_k}, x)] < 2\epsilon$. ■

The proof f the following proposition is trivial and shall then be omitted.

Proposition 4. *If a sequence* (x_n) *in a quasi-pseudometric type space* (X, D, α)*, satisfies*

$$\sum_{n=0}^{\infty} D(x_n, x_{n+1}) < \infty,$$

then (x_n) *is left* K*-Cauchy.*

Definition 15. *A subset* Y *of a quasi-pseudometric type space* (X, D, α) *is called precompact if for every* $\epsilon > 0$ *there exists a finite subset* Z *of* Y *such that*

$$Y \subset \cup\{B_D(z, \epsilon) : z \in Z\}. \tag{9}$$

If for every $\epsilon > 0$ *there exists a finite subset* Z *of* X *such that* (9) *holds, then the set* Y *is called outside precompact. One obtains the same notions if one works with closed balls* $C_D(z, \epsilon)$ $z \in Z$*.*

Obviously a precompact set is outside precompact, but the converse is not true. We then have the following characterization.

Proposition 5. *Let* (X, D, α) *be a quasi-pseudometric type space. A subset* Y *of* X *is precompact if and only if for every* $\epsilon > 0$ *there is a finite subset* $\{x_1, x_2, \cdots, x_n\} \subset X$ *such that* $Y \subset \cup_{i=1}^n B_D(x_i, \epsilon)$ *and* $Y \cap B_{D^{-1}}(x_i, \epsilon) \neq \emptyset$ *for all* $i = 1, 2, \cdots, n$*.*

Proof. For $\epsilon > 0$, let $\{x_1, x_2, \cdots, x_n\} \subset X$ such that the conditions hold for $\epsilon/2\alpha$. If $y_i \in Y \cap B_{D^{-1}}(x_i, \epsilon/2\alpha)$, $i = 1, 2, \cdots, n$, then $Y \subset \cup_{i=1}^n B_D(x_i, \epsilon)$.

Indeed, for any $y \in Y$ there exists $k \in \{1, 2, \cdots, n\}$ such that $D(x_k, y) < \epsilon/2\alpha$, implying

$$D(y_k, y) \leq \alpha[D(y_k, x_k) + D(x_k, y)] = \alpha[D^{-1}(x_k, y_k) + D(x_k, y)] < \epsilon.$$

■

3.2. Fixed point results. We start with the following lemma and repeat the proof as it is in [4].

Lemma 2 (Compare [4] Lemma 38). *Let (y_n) be a sequence in a quasi-pseudometric type space (X, D, α) such that*

$$D(y_n, y_{n+1}) \leq \lambda D(y_{n-1}, y_n) \tag{10}$$

for some $\lambda > 0$ with $\lambda < \min\{1, 1/\alpha\}$. Then (y_n) is left K-Cauchy.

Proof. Let $m < n \in \mathbb{N}$. From the condition $(D2)$ in the definition of a quasi-pseudometric type, we can write:

$$\begin{aligned} D(y_m, y_n) &\leq \alpha[D(y_m, y_{m+1}) + D(y_{m+1}, y_n)] \\ &\leq \alpha D(y_m, y_{m+1}) + \alpha^2 D(y_{m+1}, y_{m+2}) + \alpha^2 D(y_{m+2}, y_n) \\ &\ \vdots \\ &\leq \alpha D(y_m, y_{m+1}) + \alpha^2 D(y_{m+1}, y_{m+2}) + \cdots \\ &\quad + \alpha^{n-m-1} D(y_{n-2}, y_{n-1}) + \alpha^{n-m} D(y_{n-1}, y_n). \end{aligned}$$

From (10) and $\lambda < \frac{1}{\alpha}$, the above becomes

$$\begin{aligned} D(y_m, y_n) &\leq (\alpha\lambda^m + \alpha^2\lambda^{m+1} + \cdots + \alpha^{n-m}\lambda^{n-1}) D(y_0, y_1) \\ &\leq \alpha\lambda^m (1 + \alpha\lambda + \cdots + (\alpha\lambda)^{n-1-m}) D(y_0, y_1) \\ &\leq \frac{\alpha\lambda^m}{1 - \alpha\lambda} D(y_0, y_1) \longrightarrow 0 \text{ as } m \longrightarrow \infty. \end{aligned}$$

■

It follows that (y_n) is left K-Cauchy. Similarly,

Lemma 3. *Let (y_n) be a sequence in a quasi-pseudometric type space (X, D, α) such that*

$$D^{-1}(y_n, y_{n+1}) \leq \lambda D^{-1}(y_{n-1}, y_n) \tag{11}$$

for some $\lambda > 0$ with $\lambda < \min\{1, 1/\alpha\}$. Then (y_n) is right K-Cauchy.

We now state our first fixed point result.

Theorem 5. *Let (X, D, α) be a T_0-quasi-pseudometric type space. Suppose that $f, g : X \to X$ are mappings such that*

$$D(fx, fy) \leq k\ D(gx, gy) \quad \text{for all} \quad x, y \in X, \tag{12}$$

where $k < \min\{1, 1/\alpha\}$. If the range of g contains the range of f and $g(X)$ is bicomplete, then f and g have a unique point of coincidence. Moreover if f and g are weakly compatible, then f and g have a unique common fixed point.

Proof. Take an arbitrary $x_0 \in X$. Choose a point x_1 in X such that $f(x_0) = g(x_1)$. This can be done, since $f(X) \subset g(X)$. Iterating this process, once x_n is chosen in X, we can obtain x_{n+1} in X such that $f(x_n) = g(x_{n+1})$. Then

$$\begin{aligned} D(gx_n, gx_{n+1}) &= D(fx_{n-1}, fx_n) \leq kD(gx_{n-1}, gx_n) \\ &\leq k^2 D(gx_{n-2}, gx_{n-1}) \leq \ldots \leq \ k^n D(gx_0, gx_1). \end{aligned}$$

i.e.

$$D(gx_n, gx_{n+1}) \leq \ k^n D(gx_0, gx_1).$$

Similarly,

$$D(gx_{n+1}, gx_n) \leq \ k^n D(gx_1, gx_0).$$

Hence (gx_n) is a bi-Cauchy sequence. Since $g(X)$ is bicomplete, there exists $x^* \in g(X)$ such that (gx_n) D^s-converges to x^*. In other words, there is a $p^* \in X$ such that (gx_n) converges to $g(p^*) = x^*$.

Moreover

$$D(gx_n, fp^*) = D(fx_{n-1}, fp^*) \leq kD(gx_{n-1}, gp^*) \longrightarrow 0, \text{ as } n \longrightarrow,$$

In the same way, we establish that $D(fp^*, gx_n) \longrightarrow 0$ as $n \longrightarrow \infty$, to then conclude that $gx_n \longrightarrow fp^*$. The uniqueness of the limit implies that $fp^* = gp^*$. We finish the proof by showing that f and g have a unique point of coincidence. For this, assume $z^* \in X$ is a point such that $fz^* = gz^*$.

Now

$$D(gz^*, gp^*) = D(fz^*, fp^*) \leq kD(gz^*, gp^*),$$

which gives $D(gz^*, gp^*) = 0$. On the other hand, by the same reasoning, it also clear that $D(gp^*, gz^*) = 0$. By property the T_0-condition, $gz^* = gp^*$. From Proposition 1, f and g have a unique common fixed point. ■

Theorem 6. *Let (X, D, α) be a T_0-quasi-pseudometric type space. Suppose that $f, g : X \to X$ are mappings such that Suppose that mappings $f, g : X \to X$ satisfy the contractive condition*

$$D(fx, fy) \leq k \ [D(fx, gy) + D(gx, fy)] \ \textit{ for all } \ x, y \in X,$$

where $k \geq 0$ such that $\frac{k}{1-k} < \min\{1, 1/\alpha\}$. If the range of g contains the range of f and $g(X)$ is bicomplete, then f and g have a unique coincidence point in X. Moreover if f and g are weakly compatible, then f and g have a unique common fixed point.

Take an arbitrary $x_0 \in X$. Choose a point x_1 in X such that $f(x_0) = g(x_1)$. This can be done, since $f(X) \subset g(X)$. Iterating this process, once x_n is chosen in X, we can obtain x_{n+1} in X such that $f(x_n) = g(x_{n+1})$. Then

$$\begin{aligned} D(gx_n, gx_{n+1}) &= D(fx_{n-1}, fx_n) \leq k[D(fx_{n-1}, gx_n) + D(gx_{n-1}, fx_n)] \\ &\leq kD(gx_{n-1}, gx_{n+1}) \\ &\leq k[D(gx_{n-1}, gx_n) + D(gx_n, gx_{n+1})], \end{aligned}$$

which entails that

$$D(gx_n, gx_{n+1}) \leq \frac{k}{1-k}(gx_{n-1}, gx_n).$$

Similarly,

$$D(gx_{n+1}, gx_n) \leq \frac{k}{1-k}D(gx_n, gx_{n-1}).$$

Hence (gx_n) is a bi-Cauchy sequence. Since $g(X)$ is bicomplete, there exists $x^* \in g(X)$ such that (gx_n) D^s-converges to x^*. In other words, there is a $p^* \in X$ such that (gx_n) converges to $g(p^*) = x^*$.

Moreover since

$$D(gx_n, fp^*) = D(fx_{n-1}, fp^*) \leq k[D(fx_{n-1}, gp^*) + D(gx_{n-1}, fp^*)],$$

we get that

$$D(gp^*, fp^*) \leq kD(gp^*, fp^*)$$

which implies that $D(gp^*, fp^*) = 0$.

In the same way, we establish that $D(fp^*, gp^*) = 0$, to then conclude that $fp^* = gp^*$.

We finish the proof by showing that f and g have a unique point of coincidence. For this, assume $z^* \in X$ is a point such that $fz^* = gz^*$. Now

$$\begin{aligned} D(gz^*, gp^*) = D(fz^*, fp^*) &\leq k[D(fz^*, gp^*) + D(gz^*, fp^*)] \\ &\leq 2kD(gz^*, gp^*), \end{aligned}$$

which gives $D(gz^*, gp^*) = 0$. On the other hand, by the same reasoning, it also clear that $D(gp^*, gz^*) = 0$. Therefore $gz^* = gp^*$. From Proposition 1, f and g have a unique common fixed point.

Theorem 7. *Let (X, D, α) be a T_0-quasi-pseudometric type space. Suppose that $f, g : X \to X$ are mappings such that*

$$D(fx, fy) \leq \lambda D(gx, gy) + \gamma D(fx, gy) \quad \textit{for all} \quad x, y \in X. \tag{13}$$

where λ, γ are positive constants such that $\lambda + 2\gamma < \min\{1, 1/\alpha\}$. If the range of g contains the range of f and $g(X)$ is bicomplete, then f and g have a unique coincidence point in X. Moreover if f and g are weakly compatible, then f and g have a unique common fixed point.

Proof. Take an arbitrary $x_0 \in X$. Choose a point x_1 in X such that $f(x_0) = g(x_1)$. This can be done, since $f(X) \subset g(X)$. Iterating this process, once x_n is chosen in X, we can obtain x_{n+1} in X such that $f(x_n) = g(x_{n+1})$. Then

$$\begin{aligned} D(gx_n, gx_{n+1}) = D(fx_{n-1}, fx_n) &\leq \lambda D(gx_{n-1}, gx_n) + \gamma D(fx_{n-1}, gx_n) \\ &\leq \lambda D(gx_{n-1}, gx_n). \end{aligned}$$

Therefore (gx_n) is a left K-Cauchy sequence. In a similar manner, we establish that (gx_n) is also a right K-Cauchy sequence. Hence (gx_n) is a bi-Cauchy sequence. Since $g(X)$ is bicomplete, there exists $x^* \in g(X)$ such that (gx_n) D^s-converges to x^*. In other words, there is a $p^* \in X$ such that (gx_n) converges to $g(p^*) = x^*$.

Moreover since

$$D(gx_n, fp^*) = D(fx_{n-1}, fp^*) \leq \lambda D(gx_{n-1}, gp^*) + \gamma D(fx_{n-1}, gp^*)$$

we get that $D(gp^*, fp^*) = 0$. On the other hand, by the same reasoning, it is also clear that $D(fp^*, gp^*) = 0$. Hence $fp^* = gp^*$.

We finish the proof by showing that f and g have a unique point of coincidence. For this, assume $z^* \in X$ is a point such that $fz^* = gz^*$. Now

$$\begin{aligned} D(gz^*, gp^*) = D(fz^*, fp^*) &\leq \lambda D(gz^*, gp^*) + \gamma D(fz^*, gp^*) \\ &\leq (\lambda + \gamma) D(gz^*, gp^*), \end{aligned}$$

which gives $D(gz^*, gp^*) = 0$. On the other hand, by the same reasoning, it also clear that $D(gp^*, gz^*) = 0$. Hence $gz^* = gp^*$. From Proposition 1, f and g have a unique common fixed point. ■

We now give an example to illustrate Theorems 5, 7.

Example 2. Let $X = \mathbb{R}$, $D(x, y) = \max\{x - y, 0\}$ whenever $x, y \in \mathbb{R}$, $f(x) = 2x^2 + 4x + 1$ and $g(x) = 3x^2 + 6x + 2$. Then it easy to see that

$$f(X) = g(X) = [1, \infty) \text{ is bicomplete.}$$

All the conditions of Theorems 5, 7 are satisfied. Indeed:

- for Theorem 5, take $k \in \left[\frac{2}{3}, 1\right)$
- for Theorem 7, take $\lambda \in \left[\frac{2}{3}, 1\right)$, $\gamma = 0$.

f and g become weakly compatible and we obtain a unique point of coincidence and a unique common fixed point $-1 = f(-1) = g(-1)$.

Corollary 1. *Let* (X, D, α) *be a* T_0*-quasi-pseudometric type space. Suppose that mappings* $f, g : X \to X$ *satisfy the contractive condition*

$$D(fx, fy) \leq \alpha[D(gx, gy) + D(fx, gy)] \text{ for all } x, y \in X. \tag{14}$$

where $0 < \alpha < \min\{1, 1/3\alpha\}$*. If the range of* g *contains the range of* f *and* $g(X)$ *is bicomplete, then* f *and* g *have a unique coincidence point in* X*. Moreover if* f *and* g *are weakly compatible, then* f *and* g *have a unique common fixed point.*

Theorem 8. *Let* (X, D, α) *be a* T_0*-quasi-pseudometric type space. Suppose that* $f, g : X \to X$ *are mappings such that*

$$D(fx, fy) \leq \lambda D(gx, gy) + \gamma D(gx, fy) \text{ for all } x, y \in X. \tag{15}$$

where λ, γ *are positive constants such that* $\lambda + 2\gamma < \min\{1, 1/\alpha\}$*. If the range of* g *contains the range of* f *and* $g(X)$ *is bicomplete, then* f *and* g *have a unique coincidence point in* X*. Moreover if* f *and* g *are weakly compatible, then* f *and* g *have a unique common fixed point.*

Corollary 2. *Let* (X, D, α) *be a* T_0*-quasi-pseudometric type space. Suppose that mappings* $f, g : X \to X$ *satisfy the contractive condition*

$$D(fx, fy) \leq \lambda[D(gx, gy) + D(gx, fy)] \text{ for all } x, y \in X. \tag{16}$$

where $0 < \lambda < \min\{1, 1/3\alpha\}$*. If the range of* g *contains the range of* f *and* $g(X)$ *is bicomplete, then* f *and* g *have a unique coincidence point in* X*. Moreover if* f *and* g *are weakly compatible, then* f *and* g *have a unique common fixed point.*

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Received on 14.08.2015 and, in revised form, on 13.04.2016.