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NEW STABILITY RESULTS OF PICARD ITERATION FOR CONTRACTIVE TYPE MAPPINGS

Abstract. There exists several concepts of stability for fixed point iterative methods in literature. The aim of this paper is to compare two such concepts, namely one due to Harder and the second one due to Rus, in the class of contractive mappings.

Key words: fixed point, stability, limit shadowing property.

AMS Mathematics Subject Classification: 54H25, 47H10.

1. Introduction

For a metric space (X, d) and a self mapping $T : X \to X$, with $Fix(T) \neq 0$, let the Picard iteration procedure $\{x_n\}_{n=0}^{\infty}$ defined by $x_{n+1} = Tx_n$, $n = 0, 1, \ldots$, with an arbitrary $x_0 \in X$.

In concrete solving problems, instead of the theoretical sequence $\{x_n\}_{n=0}^{\infty}$, we obtain an approximative sequence $\{y_n\}_{n=0}^{\infty}$, because of rounding errors and numerical approximations of functions.

Numerical stability of Picard iteration was approached by convergence of $\{y_n\}_{n=0}^{\infty}$ to the fixed point of T and Ostrowski[19] established the first stability result for a fixed point iteration procedure.

Harder and Hicks[10] introduced the concept of stability for general fixed point iteration procedures and made a systematical study by obtaining stability results that extend Ostrowski's theorem to mappings satisfying more general contractive conditions for various fixed point iteration procedures.

Definition 1 ([10]). *Let (X, d) be a metric space and $T : X \to X$ a mapping, $x_0 \in X$ and assume that iteration procedure $x_{n+1} = f(T, x_n)$, $n = 0, 1, 2, \ldots$, respectively the sequence $\{x_n\}_{n=0}^{\infty}$ converges to a fixed point p of T.*

Let $\{y_n\}_{n=0}^{\infty}$ be an arbitrary sequence in X and set

$$\epsilon_n = d(y_{n+1}, f(T, y_n)), \quad n = 0, 1, 2, \ldots$$

We shall say that the fixed point iteration procedure is T-stable or stable with respect to T if and only if

$$\lim_{n\to\infty} \epsilon_n = 0 \;\Leftrightarrow\; \lim_{n\to\infty} y_n = p.$$

Harder and Hicks [12] showed that Picard iteration is T-stable for mappings satisfying various contractive definitions and Rhoades [21], [22] extended some of these results to independent contractive definitions and also proved stability theorems for additional iteration procedures.

One of the most general contractive definition for which corresponding stability results have been obtained in arbitrary Banach spaces appears to be the following class of mappings: for (X, d) a metric space, $T : X \to X$ is supposed to satisfy the condition

$$d\,(Tx, Ty) \le ad(x, y) + Ld(x, Tx), \tag{1}$$

for some $a \in [0, 1)$, $L \ge 0$ and for all $x, y \in X$. This condition appears in [16] and other related results may be found in [14], [21], [22].

Taking into account the notions of stability in the case of difference equations, dynamical systems, differential equations, operator theory and numerical analysis, Rus [23] unified these notions and introduced a new concept of stability for fixed point iterations.

Our aim in this paper is to compare theoretically and practically the two concepts of stability for fixed point iterations, namely the one given by Definition 1 (Harder) and the second one introduced by Rus.

The main result shows that the stability in the sense of Rus [23] is more general than the concept introduced by Harder [10]. To illustrate our theoretical results, we give some examples of contractive mappings for which Picard iteration is not stable in the sense of Harder but which is stable in the sense of Rus.

2. New stability concept for Picard iterative procedure

Eirola, Nevanlinna and Pilyugin [9] introduced the notion of *limit shadowing property* and Rus [23] adopted it, in order to introduce a new concept of stability for fixed point iteration procedures which appears to be more general than the notion of stability introduced by Harder [10].

Definition 2 ([9]). *The operator T has the limit shadowing property with respect to Picard iteration, if*

$$y_n \in X, \quad n \in \mathbb{N}, \quad d(y_{n+1}, Ty_n) \to 0 \quad as \quad n \to \infty$$

imply that there exists $x_0 \in X$, such that

$$d(y_n, T^n x_0) \to 0 \quad as \quad n \to \infty.$$

Definition 3 ([23])**.** *Picard iteration is stable with respect to an operator T if it is convergent with respect to T and the operator T has the limit shadowing property with respect to this iterative procedure.*

In the following, we study the relationship between the two stability definitions, the one of Harder [10] and the other one due to Rus [23].

Proposition 1. *Let (X,d) be a metric space and $T : X \to X$ be a mapping. Let $x_0 \in X$ and let use assume that the Picard iteration procedure $x_{n+1} = Tx_n$, $n = 0,1,2,\ldots$, converges to a fixed point p of T.*

Suppose that Picard iteration is stable in the sense of Harder. Then, it is also stable in the sense of Rus.

Proof. Let (X,d) be a metric space and $T : X \to X$ a mapping, $x_0 \in X$ and let us assume that the iteration procedure $x_{n+1} = Tx_n$, $n = 0,1,2,\ldots$, converges to a fixed point p of T.

Let $\{y_n\}_{n=0}^{\infty}$ be an arbitrary sequence in X and set

$$\epsilon_n = d\left(y_{n+1}, f\left(T, y_n\right)\right), \quad n = 0,1,2,\ldots$$

According to Definition 1, the fixed point iteration procedure is T-stable if and only if

$$\lim_{n\to\infty} \epsilon_n = 0 \;\;\Leftrightarrow\;\; \lim_{n\to\infty} y_n = p.$$

Now, according to Definition 3 of Rus, we take $y_n \in X$, $n \in \mathbb{N}$, with $d(y_{n+1}, Ty_n) \to 0$ as $n \to \infty$.

Then, as long as Picard iteration is T-stable in the sense of Harder, there exists $x_0 \in X$, such that $d(y_n, x_n) \le d(y_n, p) + d(p, x_n) \to 0$, as $n \to \infty$, so Picard iteration is also T-stable in the sense of Rus. ∎

Corollary 1. *Let (X,d) be a metric space and $T : X \to X$ be a mapping satisfying the contraction condition of Zamfirescu, i.e., there exists real numbers α, β, γ, satisfying $0 \le \alpha < 1$, $0 \le \beta, \gamma < \frac{1}{2}$, such that, for each $x, y \in X$, at least one of the following is true:*

(a) $d(Tx,Ty) \le \alpha d(x,y)$;

(b) $d(Tx,Ty) \le \beta\left[d(x,Tx) + d(y,Ty)\right]$;

(c) $d(Tx,Ty) \le \gamma\left[d(x,Ty) + d(y,Tx)\right]$.

Let $x_0 \in X$ and let use assume that the Picard iteration procedure $x_{n+1} = Tx_n$, $n = 0,1,2,\ldots$, converges to a fixed point p of T.

Suppose that Picard iteration is stable in the sense of Harder. Then, it is also stable in the sense of Rus and in this case, we obtain a stability result corresponding to fixed point theorem of Zamfirescu [27].

Corollary 2. *Let (X, d) be a metric space and $T : X \to X$ be a mapping satisfying Kannan's contraction condition, i.e., there exists $a \in [0, 1)$ such that for all $x, y \in X$,*

$$d(Tx, Ty) \leq a[d(x, Tx) + d(y, Ty)].$$

Let $x_0 \in X$ and let use assume that the Picard iteration procedure $x_{n+1} = Tx_n$, $n = 0, 1, 2, \ldots$, converges to a fixed point p of T.

Suppose that Picard iteration is stable in the sense of Harder. Then, it is also stable in the sense of Rus and in this case, we obtain a stability result corresponding to fixed point theorem of Kannan [13].

Corollary 3. *Let (X, d) be a metric space and $T : X \to X$ be a mapping satisfying Chatterjea's contraction condition, i.e., there exists $a \in \left[0, \frac{1}{2}\right)$ such that for all $x, y \in X$,*

$$d(Tx, Ty) \leq a[d(x, Ty) + d(y, Tx)].$$

Let $x_0 \in X$ and let use assume that the Picard iteration procedure $x_{n+1} = Tx_n$, $n = 0, 1, 2, \ldots$, converges to a fixed point p of T.

Suppose that Picard iteration is stable in the sense of Harder. Then, it is also stable in the sense of Rus and in this case, we obtain a stability result corresponding to fixed point theorem of Chatterjea [8].

Remark 1. The converse of Proposition 1 is not generally true, as shown by the following example.

Example 1. Let $T : [0, 1] \to [0, 1]$ be identity mapping on $[0, 1]$, that is, $Tx = x$, for each $x \in [0, 1]$, where $[0, 1]$ is endowed with the usual metric. Every point in $[0, 1]$ is a fixed point of T and T is nonexpansive, but not a contraction.

Harder [12] showed in this case that Picard iteration is not T-stable. Let now study the stability in sense of Rus. For any $y_n \in X$, with $n \in \mathbb{N}$, we have to prove that $\lim_{n\to\infty} d(y_{n+1}, Ty_n) = 0$ implies that there exists $x_0 \in X$, such that $\lim_{n\to\infty} d(y_n, T^n x_0) = 0$.

Indeed, for any $y_n \in [0, 1]$, we get $Ty_n = y_n$, and suppose that

$$\lim_{n\to\infty} d(y_{n+1}, Ty_n) = \lim_{n\to\infty} d(y_{n+1}, y_n) = 0.$$

Now, there exists $x_0 \in X$, where $x_0 = l := \lim_{n\to\infty} y_n$ such that

$$\lim_{n\to\infty} d(y_n, T^n x_0) = \lim_{n\to\infty} d(y_n, x_0) = 0.$$

Hence, Picard iteration is stable in the sense of Rus.

Corollary 4. *Let (X, d) be a metric space and $T : X \to X$ a mapping, $x_0 \in X$ and let use assume that the sequence $\{x_n\}_{n=1}^{\infty}$ converges to a fixed point p of T.*

If Picard iteration procedure is stable in the sense of Harder, then the fixed point is unique.

Proof. Suppose that $Fix(T) = \{p, q\}$, with $p \neq q$. For the sequence $\{y_n\}_{n=0}^{\infty}$, $y_n = q$, with $Ty_n = q$, we have that $\lim_{n\to\infty} d(y_{n+1}, Ty_n) = 0$, but $\lim_{n\to\infty} y_n = q \neq p$.

So, Picard iteration procedure is stable in the sense of Harder if and only if $Fix(T) = \{p\}$. ∎

Remark 2. Corollary 4 has been suggested by Professor I.A. Rus (private communication).

3. Stability results for mappings satisfying certain contraction conditions

According to above stability definitions of Rus [23], in the following we study the stability of Picard iterative procedure with respect to mappings satisfying some particular contraction conditions.

We precede with a useful result in the sequel.

Lemma 1 ([6])**.** *Let $\{a_n\}_{n=0}^{\infty}$, $\{b_n\}_{n=0}^{\infty}$ be sequences of nonnegative numbers and a constant h, $0 \leq h < 1$, so that*

$$a_{n+1} \leq ha_n + b_n, \quad n \geq 0.$$

- *If* $\lim_{n\to\infty} b_n = 0$, *then* $\lim_{n\to\infty} a_n = 0$.
- *If* $\sum_{n=0}^{\infty} b_n < \infty$, *then* $\sum_{n=0}^{\infty} a_n < \infty$.

A generalized contraction condition introduced by Berinde [4], named *almost contraction* condition has some surprising properties: it ensures the convergence of Picard iteration to a fixed point and under adequate conditions, an unique fixed point, but it does not require the sum of the coefficients on the right side of the contractive condition to be less than 1.

In a metric space (X, d), a self mapping $T : X \to X$ is called an *almost contraction* if there exists two constants $\delta \in [0, 1)$ and $L \geq 0$ such that

$$d(Tx, Ty) \leq \delta d(x, y) + Ld(y, Tx),$$

for any $x, y \in X$. Here, $\delta + L$ is not restricted to be less than 1.

Almost contractions have a very similar behavior to that of Banach contractions, which explains their name, except for the fact that the fixed point is generally not unique.

In order to ensure this uniqueness, Berinde [4] considered another condition, similar to the above one, namely

$$d(Tx, Ty) \leq \delta_u d(x, y) + L_u d(x, Tx), \tag{2}$$

for any $x, y \in X$, where $\delta_u \in [0, 1)$ and $L_u \geq 0$ are constants.

Note that (2) has been used by Osilike [15], [16], Osilike and Udomene [18] in order to establish several stability results.

Berinde [5] also proved the existence of coincidence points and common fixed points for a large class of almost contractions in cone metric spaces.

Moreover, Berinde [3] proved the existence of coincidence points and common fixed points of noncommuting almost contractions in metric spaces and a method for approximating the coincidence points or the common fixed points is also constructed, for which both a priori and a posteriori error estimates are obtained.

Using this condition, we obtain the following stability result:

Theorem 1. *Let (X, d) be a metric space and $T : X \to X$ be a self mapping satisfying the contraction condition (2), i.e., for some $\delta_u \in [0, 1)$ and $L_u \geq 0$. For all $x, y \in X$, we have*

$$d(Tx, Ty) \leq \delta_u d(x, y) + L_u d(x, Tx).$$

Then, the associated Picard iteration is T-stable in the sense of Definition 3.

Proof. Osilike [16] established the stability in the sense of Harder for Picard iteration and using a mapping satisfying (2).

Further, by Proposition 1, stability in the sense of Harder involve stability in the sense of Rus, so, we get the conclusion. ■

Example 2. Let $X = \left\{0, \frac{1}{2}, \frac{1}{2^2}, ...\right\}$ with the usual metric. Define $T : X \to X$ by $T(0) = \frac{1}{2}$, $T\left(\frac{1}{2^n}\right) = \frac{1}{2^{n+1}}$, $n = 1, 2, 3, \ldots$

Babu, Sandhya and Kameswari [1] proved that T satisfies the almost contraction condition (2), with $\delta = \frac{1}{2}$ and $L = 1$, when $\delta + L = \frac{3}{4} > 1$.

Because T has no fixed points, Picard iteration is not stable in the sense of Harder. Now, we study the stability in the sense of Rus.

For an arbitrary sequence $\{y_n\}_{n=0}^{\infty} \in X$, with $\lim_{n\to\infty} d(y_{n+1}, Ty_n) = 0$, where $\lim_{n\to\infty} y_n := l$, there obviously exists $x_0 \in X$, with $\lim_{n\to\infty} x_n = l$, such that $\lim_{n\to\infty} d(y_n, x_n) = 0$.

Because Picard iteration is also convergent with respect to T, then it is stable in the sense of Rus.

Babu, Sandhya and Kameswari [1] found a different contractive condition that ensures the uniqueness of fixed points of almost contractions: if there exists $\delta \in (0,1)$ and $L \geq 0$, such that for all $x, y \in X$,

$$d(Tx, Ty) \leq \delta d(x,y) + L \min\{d(x,Tx), d(y,Ty), d(x,Ty), d(y,Tx)\}. \tag{3}$$

Using this condition, we obtain the following stability result:

Theorem 2. *Let (X,d) be a metric space and a self mapping $T : X \to X$, satisfying the almost contraction condition (3), i.e., there exists $\delta \in (0,1)$ and $L \geq 0$, such that*

$$d(Tx, Ty) \leq \delta d(x,y) + L \min\{d(x,Tx), d(y,Ty), d(x,Ty), d(y,Tx)\},$$

for all $x, y \in X$.

Then, the associated Picard iteration is T-stable in the sense of Harder.

Proof. Let the Picard iteration with the initial value $x_0 \in X$, $\{x_n\}_{n=1}^{\infty}$, which converges to a fixed point p of T, see [1].

Let $\{y_n\}_{n=0}^{\infty}$ be an arbitrary sequence in X, satisfying condition

$$\lim_{n\to\infty} d(y_{n+1}, Ty_n) = 0.$$

The fixed point iteration is T-stable in the sense of Harder, if this implies

$$\lim_{n\to\infty} d(y_n, p) = 0.$$

We have

$$\begin{aligned} d(y_{n+1}, p) &\leq d(y_{n+1}, Ty_n) + d(Ty_n, Tx_n) + d(Tx_n, p) \\ &\leq d(y_{n+1}, Ty_n) + \delta d(x_n, y_n) + L \min\{d(x_n, Tx_n), \\ &\qquad d(y_n, Ty_n), d(x_n, Ty_n), d(y_n, Tx_n)\} + d(Tx_n, p). \end{aligned}$$

We discuss four cases.

Case 1.

$$\min\{d(x_n, Tx_n), d(y_n, Ty_n), d(x_n, Ty_n), d(y_n, Tx_n)\} := d(x_n, Tx_n).$$

Then, $d(y_{n+1}, p) \leq \epsilon_n + \delta d(x_n, y_n)$, where $\epsilon_n := d(y_{n+1}, Ty_n) + L d(x_n, Tx_n)$ $+ d(Tx_n, p) \to 0$, as $n \to \infty$, and applying Lemma 1 for $\delta \in (0,1)$, we get the conclusion.

Case 2.

$$\min\{d(x_n, Tx_n), d(y_n, Ty_n), d(x_n, Ty_n), d(y_n, Tx_n)\} := d(y_n, Ty_n).$$

As $d(y_n, Ty_n) \leq d(x_n, Tx_n)$, then, $d(y_{n+1}, x_{n+1}) \leq d(y_{n+1}, Ty_n) + \delta d(x_n, y_n) + Ld(y_n, Ty_n) + d(Tx_n, p) \leq d(y_{n+1}, Ty_n) + \delta d(x_n, y_n) + Ld(x_n, Tx_n) + d(Tx_n, p) \leq \epsilon'_n + \delta d(x_n, y_n)$, where $\epsilon'_n := d(y_{n+1}, Ty_n) + Ld(x_n, Tx_n) + d(Tx_n, p) \to 0$, as $n \to \infty$, and applying again Lemma 1 for $\delta \in (0, 1)$, we get the conclusion.

Case 3.

$$\min \{d(x_n, Tx_n), d(y_n, Ty_n), d(x_n, Ty_n), d(y_n, Tx_n)\} := d(x_n, Ty_n).$$

As $d(x_n, Ty_n) \leq d(x_n, Tx_n)$, we follow the same steps as in above case in order to get the conclusion.

Case 4.

$$\min \{d(x_n, Tx_n), d(y_n, Ty_n), d(x_n, Ty_n), d(y_n, Tx_n)\} := d(y_n, Tx_n).$$

As $d(y_n, Tx_n) \leq d(x_n, Tx_n)$, we follow the same steps as in above case in order to get the conclusion.

In a similar way, we treat the last two cases.

Therefore, the fixed point iteration procedure is stable with respect to T, in the sense of Harder. ■

Corollary 5. *Let (X, d) be a metric space and a self mapping $T : X \to X$, satisfying the almost contraction condition (3), i.e., there exists $\delta \in (0, 1)$ and $L \geq 0$, such that*

$$d(Tx, Ty) \leq \delta d(x, y) + L \min \{d(x, Tx), d(y, Ty), d(x, Ty), d(y, Tx)\},$$

for all $x, y \in X$.

Then, the associated Picard iteration is T-stable in the sense of Rus, provided it is T-stable in the sense of Harder.

4. Examples

In the following, we give some examples of mappings satisfying certain contractive conditions for which the associated Picard iteration is not stable in the sense of Harder but it is actually stable in the sense of Rus.

Example 3 ([20]). Let $T : [0, 2] \to [0, 2]$ be given by

$$Tx = \begin{cases} \frac{x}{2}, & x \in [0, 1), \\ 2, & x \in [1, 2], \end{cases}$$

where $[0, 2]$ is endowed with the usual metric. T has two fixed points, $Fix(T) = \{0, 2\}$.

Păcurar [20] showed that T is an almost contraction, i.e., there exists the constants $\delta = \frac{1}{2} \in [0,1)$ and $L = 3 \geq 0$, such that, for any $x, y \in [0,2]$, we have that

$$d(Tx, Ty) \leq \delta d(x,y) + Ld(y, Tx).$$

Note that $\delta + L = \frac{7}{2} > 1$.

In the following, we show that Picard iteration is not T-stable in sense of Harder but it is T-stable in sense of Rus.

Let $x_0 \in X$ and assume that Picard iteration procedure $\{x_n\}_{n=1}^{\infty}$, given by $x_{n+1} = Tx_n$, $n = 0, 1, 2, \ldots$, converges to a fixed point p of T.

Let $\{y_n\}_{n=0}^{\infty}$ be an arbitrary sequence in X and set

$$\lim_{n\to\infty} \epsilon_n = \lim_{n\to\infty} d\left(y_{n+1}, Ty_n\right) = 0.$$

According to Definition 1 of Harder, fixed point iteration procedure is T-stable if and only if

$$\lim_{n\to\infty} \epsilon_n = 0 \quad \Leftrightarrow \quad \lim_{n\to\infty} y_n = p.$$

Let $x_0 \in [0,1)$, so $x_n = \frac{1}{2^n} x_0$, with $\lim_{n\to\infty} x_n = 0 = p$. Then, $Tx_n = \frac{1}{2^{n+1}} x_0$.

Now, for an arbitary $\{y_n\}_{n=0}^{\infty}$, with $\lim_{n\to\infty} d\left(y_{n+1}, Ty_n\right) = 0$, let $y_n = \frac{2n-1}{n} \in [1,2]$, with $Ty_n = 2$. Indeed, we have that

$$\lim_{n\to\infty} d\left(y_{n+1}, Ty_n\right) = \lim_{n\to\infty} d\left(\frac{2n+1}{n+1}, 2\right) = 0.$$

Then, $\lim_{n\to\infty} d\left(y_n, p\right) = \lim_{n\to\infty} d\left(\frac{2n-1}{n}, 0\right) = 2 \neq 0$, so the Picard iteration is not T-stable in sense of Harder.

On the other hand, according to Definition 3 of Rus, Picard iteration is Rus-stable if $y_n \in X$, $n \in \mathbb{N}$, $d(y_{n+1}, Ty_n) \to 0$ as $n \to \infty$ implies that there exists $x_0 \in X$, such that $d(y_n, T^n x_0) \to 0$ as $n \to \infty$. We discuss two cases.

Case 1. If $y_n \in [0,1)$, then $y_n = \frac{1}{2^n} y_0$, with $Ty_n = \frac{1}{2^{n+1}} y_0$.

So, $\lim_{n\to\infty} d(y_{n+1}, Ty_n) = \lim_{n\to\infty} d\left(\frac{1}{2^{n+1}} y_0, \frac{1}{2^{n+1}} y_0\right) = 0$ and therefore, there exists $x_0 \in X$ such that

$$\lim_{n\to\infty} d(y_n, x_n) = \lim_{n\to\infty} d\left(\frac{1}{2^{n+1}} y_0, \frac{1}{2^{n+1}} x_0\right) = \lim_{n\to\infty} \frac{1}{2^{n+1}} d\left(y_0, x_0\right) = 0.$$

Case 2. If $y_n \in [1,2]$, then $y_n = 2 = Ty_n$.

So, $d(y_{n+1}, Ty_n) = d(y_{n+1}, 2)$ and from $\lim_{n\to\infty} d(y_{n+1}, Ty_n) = 0$ we obtain that $\{y_n\}_{n=0}^{\infty}$ converges to 2. Now, just take $x_0 \in [1,2]$ arbitrary, to get $x_n = 2$, $n \geq 0$, and hence, $\lim_{n\to\infty} d(y_n, x_n) = 0$. as required.

Therefore, the Picard iteration is T-stable in sense of Rus.

Example 4 ([20]). Let $T:[0,1]\to[0,1]$ be given by

$$Tx=\begin{cases}\frac{2}{3}x, & x\in\left[0,\frac{1}{2}\right),\\ \frac{2}{3}x+\frac{1}{3}, & x\in\left[\frac{1}{2},1\right],\end{cases}$$

where $[0,1]$ is endowed with the usual metric.

T has two fixed points, $Fix(T)=\{0,1\}$.

Păcurar [20] showed that T is an almost contraction, i.e., there exists the constants $\delta=\frac{2}{3}\in[0,1)$ and $L=6\geq 0$, such that, for any $x,y\in[0,1]$, we have that

$$d(Tx,Ty)\leq\delta d(x,y)+Ld(y,Tx).$$

Note that $\delta+L=6+\frac{2}{3}>1$.

In the following, we show that Picard iteration is not T-stable in sense of Harder but it is T-stable in sense of Rus.

Let $x_0\in X$ and assume that Picard iteration procedure $\{x_n\}_{n=1}^{\infty}$, given by $x_{n+1}=Tx_n$, $n=0,1,2,\ldots$, converges to a fixed point p of T.

Let $\{y_n\}_{n=0}^{\infty}$ be an arbitrary sequence in X and set

$$\lim_{n\to\infty}\epsilon_n=\lim_{n\to\infty}d\left(y_{n+1},Ty_n\right)=0.$$

According to Definition 1 of Harder, fixed point iteration procedure is T-stable if and only if

$$\lim_{n\to\infty}\epsilon_n=0\quad\Leftrightarrow\quad\lim_{n\to\infty}y_n=p.$$

Let $x_0\in\left[0,\frac{1}{2}\right)$, so $x_n=\left(\frac{2}{3}\right)^n x_0$, with $\lim_{n\to\infty}x_n=0=p$.

Now, for an arbitary $\{y_n\}_{n=0}^{\infty}$, with $\lim_{n\to\infty}d\left(y_{n+1},Ty_n\right)=0$, let $y_n=\frac{n-1}{n}\in[\frac{1}{2},1]$, with $\lim_{n\to\infty}y_n=1$ and $Ty_n=\frac{2}{3}y_n+\frac{1}{3}$.

Indeed, we have that

$$\begin{aligned}\lim_{n\to\infty}d\left(y_{n+1},Ty_n\right)&=\lim_{n\to\infty}d\left(\frac{n}{n+1},\frac{2}{3}y_n+\frac{1}{3}\right)\\&=\lim_{n\to\infty}d\left(\frac{n}{n+1},\frac{2}{3}\cdot\frac{n-1}{n}+\frac{1}{3}\right)=0.\end{aligned}$$

Then, $\lim_{n\to\infty}d\left(y_n,p\right)=\lim_{n\to\infty}d\left(\frac{n-1}{n},0\right)=1\neq 0$, so the Picard iteration is not T-stable in sense of Harder.

Now, according to Definition 3 of Rus, if $y_n\in X$, $n\in\mathbb{N}$, $d(y_{n+1},Ty_n)\to 0$ as $n\to\infty$ implies that there exists $x_0\in X$, such that $d(y_n,T^nx_0)\to 0$ as $n\to\infty$. We discuss two cases.

Case 1. If $y_n\in\left[0,\frac{1}{2}\right)$, then $Ty_n=\frac{2}{3}y_n$ and by

$$\lim_{n\to\infty}d(y_{n+1},Ty_n)=\lim_{n\to\infty}d\left(y_{n+1},\frac{2}{3}y_n\right)=0,$$

we obtain that $\lim_{n\to\infty} y_n = 0$.

Indeed, by $\left|y_{n+1} - \frac{2}{3}y_n\right| \to 0$, as $n \to \infty$, we have $y_{n+1} - \frac{2}{3}y_n = \alpha_n$, with $\alpha_n \to 0$, as $n \to \infty$. Then, $y_{n+1} = \frac{2}{3}y_n + \alpha_n$, so $y_{n+1} \le \frac{2}{3}y_n + \alpha_n$, and applying Lemma 1, we get $\lim_{n\to\infty} y_n = 0$.

There exists $x_0 \in \left[0, \frac{1}{2}\right)$, such that

$$\lim_{n\to\infty} d(y_n, x_n) = \lim_{n\to\infty} d\left(\left(\frac{2}{3}\right)^n y_0, \left(\frac{2}{3}\right)^n x_0\right) = \lim_{n\to\infty} \left(\frac{2}{3}\right)^n d\left(y_0, x_0\right) = 0.$$

Case 2. If $y_n \in \left[\frac{1}{2}, 1\right]$, then $Ty_n = \frac{2}{3}y_n + \frac{1}{3}$.

So, from $\lim_{n\to\infty} d(y_{n+1}, Ty_n) = \lim_{n\to\infty} d(y_{n+1}, \frac{2}{3}y_n + \frac{1}{3}) = 0$ it results that $\lim_{n\to\infty} y_n = 1$ and therefore, there exists $x_0 \in \left[\frac{1}{2}, 1\right]$, with $\lim_{n\to\infty} x_n = 1$, such that

$$\lim_{n\to\infty} d(y_n, x_n) = \lim_{n\to\infty} d\left(\left(\frac{2}{3}\right)^n y_0 + 1 - \left(\frac{2}{3}\right)^n, \left(\frac{2}{3}\right)^n x_0 + 1 - \left(\frac{2}{3}\right)^n\right) = 0,$$

so, the Picard iteration is T-stable in sense of Rus.

Example 5 ([20])**.** Let $T : [0,1] \to [0,1]$ be given by

$$Tx = \begin{cases} x^2, & x \in \left[0, \frac{1}{4}\right), \\ 0, & x \in \left[\frac{1}{4}, 1\right], \end{cases}$$

where $[0,1]$ is endowed with the usual metric. T has a fixed point at 0.

Păcurar [20] showed that T is an almost contraction, i.e., there exists the constants $\delta = \frac{1}{2} \in [0,1)$ and $L = \frac{1}{3} \ge 0$, such that, for any $x, y \in [0,1]$, we have that

$$d(Tx, Ty) \le \delta d(x,y) + Ld(y, Tx).$$

Note that in this case $\delta + L = \frac{5}{6} < 11$.

In the following, we show that Picard iteration is T-stable in sense of Harder and it is also T-stable in sense of Rus.

Let $x_0 \in X$ and assume that Picard iteration procedure $\{x_n\}_{n=1}^{\infty}$, given by $x_{n+1} = Tx_n$, $n = 0, 1, 2, \ldots$, converges to a fixed point p of T.

Let $\{y_n\}_{n=0}^{\infty}$ be an arbitrary sequence in X and set

$$\lim_{n\to\infty} \epsilon_n = \lim_{n\to\infty} d\left(y_{n+1}, Ty_n\right) = 0.$$

According to Definition 1 of Harder, fixed point iteration procedure is T-stable if and only if

$$\lim_{n\to\infty} \epsilon_n = 0 \quad \Leftrightarrow \quad \lim_{n\to\infty} y_n = p.$$

Let $x_0 \in \left[0, \frac{1}{4}\right)$, so $x_n = (x_0)^{2n}$, with $\lim_{n\to\infty} x_n = 0 = p$.

Now, for an arbitary $\{y_n\}_{n=0}^{\infty}$, we discuss two cases.

Case 1. If $y_n \in \left[\frac{1}{4}, 1\right]$, then $Ty_n = 0$ and from $\lim_{n\to\infty} d\left(y_{n+1}, Ty_n\right) = 0$, it results that $\lim_{n\to\infty} y_n = 0$ and this is a contradiction, as long as $y_n \in \left[\frac{1}{4}, 1\right]$.

Case 2. If $y_n \in \left[0, \frac{1}{4}\right)$, then $Ty_n = y_n^2$ and from $\lim_{n\to\infty} d\left(y_{n+1}, Ty_n\right) = = \lim_{n\to\infty} d\left(y_{n+1}, y_n^2\right) = 0$, we obtain that $\lim_{n\to\infty} y_n = 0$.

Indeed, from $\left|y_{n+1} - y_n^2\right| \to 0$, as $n \to \infty$, we have that

$$y_{n+1} = y_n^2 + \alpha_n, \tag{*}$$

with $\alpha_n \to 0$, as $n \to \infty$. Denote $\lim_{n\to\infty} y_n := l$ and by taking to the limit in $(*)$, we get $l = l^2$, so $l = 0$, or $l = 1$.

Because $y_n \in \left[0, \frac{1}{4}\right)$, we have $l = 0$, so $\lim_{n\to\infty} y_n = 0$.

Then, $\lim_{n\to\infty} d\left(y_n, p\right) = 0$, so the Picard iteration is T-stable in sense of Harder.

According to Proposition 1, if Picard iteration is T-stable in the sense of Harder, it is also stable in the sense of Rus.

Example 6 ([20]). Let $T : [0,1] \to [0,1]$ be given by

$$Tx = \begin{cases} \frac{2}{3}, & x \in [0,1), \\ 0, & x = 1, \end{cases}$$

where $[0,1]$ is endowed with the usual metric.

T has one fixed point at $\frac{2}{3}$, $Fix(T) = \left\{\frac{2}{3}\right\}$.

Păcurar [20] showed that T is an almost contraction, i.e., there exists the constants $\delta = \frac{2}{3} \in [0,1)$ and $L \geq \delta \geq 0$, such that, for any $x, y \in [0,1]$, we have that

$$d(Tx, Ty) \leq \delta d(x,y) + Ld(y, Tx).$$

Note that in this case $\delta + L \geq \frac{4}{3} > 1$.

In the following, we show that Picard iteration is T-stable in sense of Harder and hence it is also T-stable in sense of Rus.

Let $x_0 \in X$ and assume that Picard iteration procedure $\{x_n\}_{n=1}^{\infty}$, given by $x_{n+1} = Tx_n$, $n = 0, 1, 2, \ldots$, converges to a fixed point p of T.

Let $\{y_n\}_{n=0}^{\infty}$ be an arbitrary sequence in X and set

$$\lim_{n\to\infty} \epsilon_n = \lim_{n\to\infty} d\left(y_{n+1}, Ty_n\right) = 0.$$

According to Definition 1 of Harder, fixed point iteration procedure is T-stable if and only if

$$\lim_{n\to\infty} \epsilon_n = 0 \quad \Leftrightarrow \quad \lim_{n\to\infty} y_n = p.$$

For any $x_0 \in [0,1]$, $x_n = \frac{2}{3}$, so $\lim_{n\to\infty} x_n = \frac{2}{3} = p$.

Now, for an arbitary $\{y_n\}_{n=0}^{\infty}$, we discuss two cases.

Case 1. If $y_n = 1$, then $Ty_n = 0$ and then $\lim_{n\to\infty} d(y_{n+1}, Ty_n) = 1 \neq 0$ and it is a contradiction.

Case 2. If $y_n \in [0,1)$, then $Ty_n = \frac{2}{3}$ and from $\lim_{n\to\infty} d(y_{n+1}, Ty_n) = 0$, it results that $\lim_{n\to\infty} y_n = \frac{2}{3}$.

Then, $\lim_{n\to\infty} d(y_n, p) = 0$, so the Picard iteration is T-stable in sense of Harder.

According to Proposition 1, if Picard iteration is T-stable in the sense of Harder, it is also stable in the sense of Rus.

Example 7 ([20])**.** Let $T : [0,1] \to [0,1]$ be given by

$$Tx = \begin{cases} 0, & x \in \left[0, \frac{1}{2}\right], \\ \frac{x}{2}, & x \in \left(\frac{1}{2}, 1\right], \end{cases}$$

where $[0,1]$ is endowed with the usual metric.

T has one fixed point at $\frac{1}{2}$, $Fix(T) = \left\{\frac{1}{2}\right\}$.

Păcurar [20] showed that T is an almost contraction, i.e., there exists two constants $\delta_u = \frac{1}{2} \in [0,1)$ and $L_u = 1 \geq 0$, such that, for any $x, y \in [0,1]$, we have that

$$d(Tx, Ty) \leq \delta_u d(x,y) + L_u d(x, Tx).$$

Note that in this case $\delta + L = \frac{3}{2} > 1$.

In the following, we show that Picard iteration is T-stable in sense of Harder and it is also T-stable in sense of Rus.

Let $x_0 \in X$ and assume that Picard iteration procedure $\{x_n\}_{n=1}^{\infty}$, given by $x_{n+1} = Tx_n$, $n = 0, 1, 2, \ldots$, converges to a fixed point p of T.

Let $\{y_n\}_{n=0}^{\infty}$ be an arbitrary sequence in X and set

$$\lim_{n\to\infty} \epsilon_n = \lim_{n\to\infty} d(y_{n+1}, Ty_n) = 0.$$

According to Definition 1 of Harder, fixed point iteration procedure is T-stable if and only if

$$\lim_{n\to\infty} \epsilon_n = 0 \quad \Leftrightarrow \quad \lim_{n\to\infty} y_n = p.$$

For any $x_0 \in [0,1]$, $x_n = \frac{2}{3}$, so $\lim_{n\to\infty} x_n = \frac{2}{3} = p$.

For any $x_0 \in [0,1]$, we have that $\lim_{n\to\infty} x_n = 0 = p$.

Now, for an arbitary $\{y_n\}_{n=0}^{\infty}$, we discuss two cases.

Case 1. If $y_n \in \left(\frac{1}{2}, 1\right]$, then $Ty_n = \frac{y_n}{2}$ and by $\lim_{n\to\infty} d(y_{n+1}, Ty_n) = 0$, it results that $\lim_{n\to\infty} y_n = 0$ and it is a contradiction, as long as $y_n \in \left(\frac{1}{2}, 1\right]$.

Case 2. If $y_n \in \left[0, \frac{1}{2}\right]$, then $Ty_n = 0$ and by $\lim_{n\to\infty} d(y_{n+1}, Ty_n) = 0$, we obtain that $\lim_{n\to\infty} y_n = 0$.

Hence, $\lim_{n\to\infty} d\left(y_n, p\right) = 0$, so the Picard iteration is T-stable in sense of Harder.

According to Proposition 1, if Picard iteration is T-stable in the sense of Harder, it is also stable in the sense of Rus.

Example 8 ([12])**.** Let $T : [0,1] \to [0,1]$ be given by

$$Tx = \begin{cases} \frac{1}{2}, & x \in \left[0, \dfrac{1}{2}\right], \\ 0, & x \in \left(\dfrac{1}{2}, 1\right], \end{cases}$$

where $[0,1]$ is endowed with the usual metric. T is continuous at each point of $[0,1]$ except at $\frac{1}{2}$.

T has an unique fixed point at $\frac{1}{2}$, $Fix(T) = \left\{\frac{1}{2}\right\}$. For each $x, y \in [0,1]$, with $x \neq y$, T satisfies the condition

$$d(Tx, Ty) < \max\left\{d(x, Tx), d(y, Ty)\right\},$$

and also we showed that the associated Picard iteration is not T-stable in the sense of Harder, by using a divergent sequence $\{y_n\}_{n=0}^{\infty} = \frac{1}{2}, \frac{1}{4}, \frac{1}{2} + \frac{1}{4^2}, \frac{1}{4^3}, \frac{1}{2} + \frac{1}{4^4}, \frac{1}{4^5}, \cdots$.

In the following, we prove that it is stable in the sense of Rus.

By Definition 3 of Rus, for any $y_n \in [0,1]$, we have that $\lim_{n\to\infty} d(y_{n+1}, Ty_n) = 0$ and it implies that there exists $x_0 \in X$, such that $\lim_{n\to\infty} d(y_n, T^n x_0) = 0$.

From $\lim_{n\to\infty} d(y_{n+1}, Ty_n) = 0$, it results that $y_n \in \left[0, \frac{1}{2}\right]$ and hence, $Ty_n = \frac{1}{2}$ and $\lim_{n\to\infty} y_n = \frac{1}{2}$.

Now, for any $x_0 \in [0,1]$, we have $x_n = \frac{1}{2}$, $n \geq 2$, and so $\lim_{n\to\infty} x_n = \frac{1}{2}$.

Hence,

$$\lim_{n\to\infty} d(y_n, T^n x_0) = \lim_{n\to\infty} d(y_n, x_n) = 0,$$

so, Picard iteration is T-stable in the sense of Rus.

Example 9 ([12])**.** Let $T : [0,1] \to [0,1]$ be given by

$$Tx = \begin{cases} 0, & x \in \left[0, \frac{1}{2}\right], \\ \frac{1}{2}, & x \in \left(\frac{1}{2}, 1\right], \end{cases}$$

where $[0,1]$ is endowed with the usual metric. T is continuous at every point of $[0,1]$ except at $\frac{1}{2}$.

T has an unique fixed point at 0, $Fix(T) = \{0\}$.

For each $x, y \in [0,1]$, with $x \neq y$, T satisfies the condition

$$d(Tx, Ty) < \max\{d(x, Ty), d(y, Tx)\},$$

and also showed that the associated Picard iteration is not T-stable in the sense of Harder, using $\{y_n\}_{n=0}^{\infty}$, with $y_n = \frac{n+2}{2n}, \quad n \geq 1$.

In the following, we prove that it is stable in the sense of Rus.

According to Definition 3 of Rus, for any $y_n \in [0,1]$, we have to prove that $\lim_{n\to\infty} d(y_{n+1}, Ty_n) = 0$ implies that there exists $x_0 \in X$, such that

$$\lim_{n\to\infty} d(y_n, T^n x_0) = 0.$$

We discuss two cases.

Case 1. If $y_n \in \left[0, \frac{1}{2}\right]$, then $Ty_n = 0$, and hence from $\lim_{n\to\infty} d(y_{n+1}, Ty_n) = 0$, it results that $\lim_{n\to\infty} y_n = 0$.

Case 2. If $y_n \in \left(\frac{1}{2}, 1\right]$, then $Ty_n = \frac{1}{2}$, and hence from $\lim_{n\to\infty} d(y_{n+1}, Ty_n) = 0$, it results that $\lim_{n\to\infty} y_n = \frac{1}{2}$.

Now, definitely, there exists $x_0 \in [0,1]$, such that

$$\lim_{n\to\infty} d(y_n, T^n x_0) = \lim_{n\to\infty} d(y_n, x_n) = 0,$$

so, Picard iteration is T-stable in the sense of Rus.

Example 10 ([12]). Let $T : \mathbb{R} \to \left\{0, \frac{1}{4}, \frac{1}{2}\right\}$ be defined by

$$Tx = \begin{cases} \frac{1}{2}, & x < 0, \\ \frac{1}{4}, & x \in \left[0, \dfrac{1}{2}\right], \\ 0, & x > \frac{1}{2}, \end{cases}$$

where $\mathbb{R}$ is endowed with the usual metric. T is continuous at every point in $\mathbb{R}$ except at 0 and $\frac{1}{2}$.

The only fixed point of T is $\frac{1}{4}$, $Fix(T)$. For each $x, y \in \mathbb{R}$, with $x \neq y$, T satisfies the condition

$$d(Tx, Ty) < \max\left\{d(x, y), \frac{d(x, Tx) + d(y, Ty)}{2}, \frac{d(x, Ty) + d(y, Tx)}{2}\right\},$$

and also showed that the associated Picard iteration is not T-stable in the sense of Harder by using the sequence $\{y_n\}_{n=0}^{\infty}$ of real numbers $y_n = \frac{1}{2} + \frac{1}{n}$, for each positive odd integer and $y_n = -\frac{1}{n}$, for each positive even integer.

In the following, we prove that it is stable in the sense of Rus. According to Definition 3 of Rus, for any $y_n \in \mathbb{R}$, we have that $\lim_{n\to\infty} d(y_{n+1}, Ty_n) = 0$

and it implies that there exists $x_0 \in \mathbb{R}$, such that $\lim_{n\to\infty} d(y_n, T^n x_0) = 0$. We discuss three cases.

Case 1. If $y_n < 0$, then $Ty_n = \frac{1}{2}$, so, from $d(y_{n+1}, Ty_n) = d(y_{n+1}, \frac{1}{2}) \to 0$, as $n \to \infty$, it results that $\lim_{n\to\infty} y_n = \frac{1}{2}$, and this is a contradiction, as long as $y_n < 0$.

Case 2. If $y_n > \frac{1}{2}$, then $Ty_n = 0$, so, from $d(y_{n+1}, Ty_n) = d(y_{n+1}, 0) \to 0$, as $n \to \infty$, it results that $\lim_{n\to\infty} y_n = 0$, and this is another contradiction, as long as $y_n > \frac{1}{2}$.

Case 3. If $y_n \in \left[0, \dfrac{1}{2}\right]$, then $Ty_n = \frac{1}{4}$, so, from $d(y_{n+1}, Ty_n) = d(y_{n+1}, \frac{1}{4}) \to 0$, as $n \to \infty$, it results that $\lim_{n\to\infty} y_n = \frac{1}{4}$.

Now, definitely, there exists $x_0 \in \mathbb{R}$, such that $\lim_{n\to\infty} x_n = \frac{1}{4}$ and $\lim_{n\to\infty} d(y_n, T^n x_0) = \lim_{n\to\infty} d(y_n, x_n) = 0$, so, Picard iteration is T-stable in the sense of Rus.

5. Concluding remarks

A fixed point iteration procedure which is stable in the sense of Harder is also stable in the sense of Rus. But the reverse is not generally true, because Harder stability implies the uniqueness of fixed point, while the new one of Rus does not.

The stability of a fixed point iteration procedure in the sense of Rus may imply stability in the sense of Harder, if and only if the iterative procedure converges to the fixed point.

On the other hand, there are many examples of mappings that satisfy certain contractive conditions and for which the associated Picard iteration is not stable in the sense of Harder but it is actually stable in the sense of Rus.

In above examples, there are some nonexpansive mappings and almost contractions for which the associated Picard iteration is stable in the sense of Rus but it is not stable in the sense of Harder.

Open problem: Study the stability in the sense of Rus for general nonexpansive mappings as well as for general almost contractions (that do not satisfy a certain uniqueness condition).

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