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FRACTIONAL SELF ADJOINT OPERATOR POINCARÉ AND SOBOLEV TYPE INEQUALITIES

Abstract. We present here many fractional self adjoint operator Poincaré and Sobolev type inequalities to various directions. Initially we give several fractional representation formulae in the self adjoint operator sense. Inequalities are based in the self adjoint operator order over a Hilbert space.

Key words: self adjoint operator, Hilbert space, Poincaré inequality, Sobolev inequality, fractional derivative.

AMS Mathematics Subject Classification: 26A33, 26D10, 26D20, 47A60, 47A67.

1. Background

Let A be a selfadjoint linear operator on a complex Hilbert space $(H; \langle \cdot, \cdot \rangle)$. The Gelfand map establishes a $*-$isometrically isomorphism Φ between the set $C(Sp(A))$ of all continuous functions defined on the spectrum of A, denoted $Sp(A)$, and the C^*-algebra $C^*(A)$ generated by A and the identity operator 1_H on H as follows (see e.g. [6, p. 3]):

For any $f, g \in C(Sp(A))$ and any $\alpha, \beta \in \mathbb{C}$ we have

(i) $\Phi(\alpha f + \beta g) = \alpha \Phi(f) + \beta \Phi(g)$;

(ii) $\Phi(fg) = \Phi(f)\Phi(g)$ (the operation composition is on the right) and $\Phi(\overline{f}) = (\Phi(f))^*$;

(iii) $\|\Phi(f)\| = \|f\| := \sup_{t \in Sp(A)} |f(t)|$;

(iv) $\Phi(f_0) = 1_H$ and $\Phi(f_1) = A$, where $f_0(t) = 1$ and $f_1(t) = t$, for $t \in Sp(A)$.

With this notation we define

$$f(A) := \Phi(f), \text{ for all } f \in C(Sp(A)),$$

and we call it the continuous functional calculus for a selfadjoint operator A.

If A is a selfadjoint operator and f is a real valued continuous function on $Sp(A)$ then $f(t) \geq 0$ for any $t \in Sp(A)$ implies that $f(A) \geq 0$, i.e. $f(A)$ is a positive operator on H. Moreover, if both f and g are real valued continuous functions on $Sp(A)$ then the following important property holds:

(P) $f(t) \geq g(t)$ for any $t \in Sp(A)$, implies that $f(A) \geq g(A)$ in the operator order of $B(H)$ (the Banach algebra of all bounded linear operators from H into itself).

Equivalently, we use (see [5], pp. 7-8):

Let U be a selfadjoint operator on the complex Hilbert space $(H, \langle \cdot, \cdot \rangle)$ with the spectrum $Sp(U)$ included in the interval $[m, M]$ for some real numbers $m < M$ and $\{E_\lambda\}_\lambda$ be its spectral family.

Then for any continuous function $f : [m, M] \to \mathbb{C}$, it is well known that we have the following spectral representation in terms of the Riemann-Stieljes integral:

$$\langle f(U) x, y \rangle = \int_{m-0}^{M} f(\lambda) \, d(\langle E_\lambda x, y \rangle),$$

for any $x, y \in H$. The function $g_{x,y}(\lambda) := \langle E_\lambda x, y \rangle$ is of bounded variation on the interval $[m, M]$, and

$$g_{x,y}(m-0) = 0 \quad \text{and} \quad g_{x,y}(M) = \langle x, y \rangle,$$

for any $x, y \in H$. Furthermore, it is known that $g_x(\lambda) := \langle E_\lambda x, x \rangle$ is increasing and right continuous on $[m, M]$.

We have also the formula

$$\langle f(U) x, x \rangle = \int_{m-0}^{M} f(\lambda) \, d(\langle E_\lambda x, x \rangle), \quad \forall\, x \in H.$$

As a symbol we can write

$$f(U) = \int_{m-0}^{M} f(\lambda) \, dE_\lambda.$$

Above, $m = \min\{\lambda | \lambda \in Sp(U)\} := \min Sp(U)$, $M = \max\{\lambda | \lambda \in Sp(U)\} := \max Sp(U)$. The projections $\{E_\lambda\}_{\lambda \in \mathbb{R}}$, are called the spectral family of A, with the properties:

(a) $E_\lambda \leq E_{\lambda'}$ for $\lambda \leq \lambda'$;

(b) $E_{m-0} = 0_H$ (zero operator), $E_M = 1_H$ (identity operator) and $E_{\lambda+0} = E_\lambda$ for all $\lambda \in \mathbb{R}$.

Furthermore

$$E_\lambda := \varphi_\lambda(U), \quad \forall\, \lambda \in \mathbb{R},$$

is a projection which reduces U, with

$$\varphi_\lambda(s) := \begin{cases} 1, & \text{for} \quad -\infty < s \leq \lambda, \\ 0, & \text{for} \quad \lambda < s < +\infty. \end{cases}$$

The spectral family $\{E_\lambda\}_{\lambda\in\mathbb{R}}$ determines uniquely the self-adjoint operator U and vice versa.

For more on the topic see [7], pp. 256-266, and for more detalis see there pp. 157-266. See also [4].

Some more basics are given (we follow [5], pp. 1-5):

Let $(H; \langle\cdot,\cdot\rangle)$ be a Hilbert space over $\mathbb{C}$. A bounded linear operator A defined on H is selfjoint, i.e., $A = A^*$, iff $\langle Ax, x\rangle \in \mathbb{R}$, $\forall\, x \in H$, and if A is selfadjoint, then

$$\|A\| = \sup_{x\in H:\|x\|=1} |\langle Ax, x\rangle| .$$

Let A, B be selfadjoint operators on H. Then $A \le B$ iff $\langle Ax, x\rangle \le \langle Bx, x\rangle$, $\forall\, x \in H$.

In particular, A is called positive if $A \ge 0$.

Denote by

$$\mathcal{P} := \left\{ \varphi(s) := \sum_{k=0}^{n} \alpha_k s^k | n \ge 0,\ \alpha_k \in \mathbb{C},\ 0 \le k \le n \right\}.$$

If $A \in \mathcal{B}(H)$ is selfadjoint, and $\varphi(s) \in \mathcal{P}$ has real coefficients, then $\varphi(A)$ is selfadjoint, and

$$\|\varphi(A)\| = \max\{|\varphi(\lambda)|, \lambda \in Sp(A)\}.$$

If φ is any function defined on $\mathbb{R}$ we define

$$\|\varphi\|_A := \sup\{|\varphi(\lambda)|, \lambda \in Sp(A)\}.$$

If A is selfadjoint operator on Hilbert space H and φ is continuous and given that $\varphi(A)$ is selfadjoint, then $\|\varphi(A)\| = \|\varphi\|_A$. And if φ is a continuous real valued function so it is $|\varphi|$, then $\varphi(A)$ and $|\varphi|(A) = |\varphi(A)|$ are selfadjoint operators (by [5], p. 4, Theorem 7).

Hence it holds

$$\begin{aligned} \||\varphi(A)|\| &= \||\varphi|\|_A = \sup\{||\varphi(\lambda)||, \lambda \in Sp(A)\} \\ &= \sup\{|\varphi(\lambda)|, \lambda \in Sp(A)\} = \|\varphi\|_A = \|\varphi(A)\|, \end{aligned}$$

that is

$$\||\varphi(A)|\| = \|\varphi(A)\|.$$

For a selfadjoint operator $A \in \mathcal{B}(H)$ which is positive, there exists a unique positive selfadjoint operator $B := \sqrt{A} \in \mathcal{B}(H)$ such that $B^2 = A$, that is $\left(\sqrt{A}\right)^2 = A$. We call B the square root of A.

Let $A \in \mathcal{B}(H)$, then A^*A is selfadjoint and positive. Define the "operator absolute value" $|A| := \sqrt{A^*A}$. If $A = A^*$, then $|A| = \sqrt{A^2}$.

For a continuous real valued function φ we observe the following:

$$|\varphi(A)| \text{ (the functional absolute value) } = \int_{m-0}^{M} |\varphi(\lambda)|\, dE_\lambda$$
$$= \int_{m-0}^{M} \sqrt{(\varphi(\lambda))^2} dE_\lambda = \sqrt{(\varphi(A))^2}$$
$$= |\varphi(A)| \text{ (operator absolute value)},$$

where A is a selfadjoint operator.

That is we have

$$|\varphi(A)| \text{ (functional absolute value) } = |\varphi(A)| \text{ (operator absolute value)}.$$

2. Main results

Let A be a selfadjoint operator in the Hilbert space H with the spectrum $Sp(A) \subseteq [m, M]$, $m < M$; $m, M \in \mathbb{R}$.

In the next we obtain fractional Poincaré and Sobolev type inequalities in the operator order of $\mathcal{B}(H)$ (the Banach algebra of all bounded linear operators from H into itself). All of our functions next in this article are real valued.

We give

Definition 1 ([1], p. 270). *Let $\nu > 0$, $n := \lceil \nu \rceil$ (ceiling of ν), $f \in AC^n([m, M])$ (i.e. $f^{(n-1)}$ is absolutely continuous on $[m, M]$, that is in $AC([m, M])$). We define the left Caputo fractional derivative*

$$(D_{*m}^{\nu} f)(z) := \frac{1}{\Gamma(n-\nu)} \int_m^z (z-t)^{n-\nu-1} f^{(n)}(t)\, dt, \tag{1}$$

which exists almost everywhere for $z \in [m, M]$.

Notice that $D_{*m}^{0} f = f$, and $D_{*m}^{n} f = f^{(n)}$.

We present the operator representation formula

Theorem 1. *Let A be a selfadjoint operator in the Hilbert space H with the spectrum $Sp(A) \subseteq [m, M]$ for some real numbers $m < M$, $\{E_\lambda\}_\lambda$ be its spectral family, I be a closed subinterval on $\mathbb{R}$ with $[m, M] \subset \overset{\circ}{I}$ (the interior of I) and $n \in \mathbb{N}$, with $n := \lceil \nu \rceil$, $\nu > 0$. We consider $f \in AC^n([m, M])$ (i.e. $f^{(n-1)} \in AC([m, M])$, absolutely continuous functions), where $f : I \to \mathbb{R}$.*

Then

$$f(A) = \sum_{k=0}^{n-1} \frac{f^{(k)}(m)}{k!} (A - m1_H)^k + R_n(f, m, M), \tag{2}$$

where

$$R_n(f,m,M) = \frac{1}{\Gamma(\nu)} \int_{m-0}^{M} \left(\int_m^{\lambda} (\lambda - t)^{\nu-1} \left(D_{*m}^{\nu} f\right)(t)\, dt \right) dE_{\lambda}. \tag{3}$$

Proof. We have by left Caputo fractional Taylor's formula [3], p. 54, that

$$f(\lambda) = \sum_{k=0}^{n-1} \frac{f^{(k)}(m)}{k!} (\lambda - m)^k + \frac{1}{\Gamma(\nu)} \int_m^{\lambda} (\lambda - t)^{\nu-1} \left(D_{*m}^{\nu} f\right)(t)\, dt, \tag{4}$$

$\forall\, \lambda \in [m, M]$.

Then we integrate (4) against E_λ to get

$$\begin{aligned} \int_{m-0}^{M} f(\lambda)\, dE_{\lambda} &= \sum_{k=0}^{n-1} \frac{f^{(k)}(m)}{k!} \int_{m-0}^{M} (\lambda - m)^k\, dE_{\lambda} \\ &+ \frac{1}{\Gamma(\nu)} \int_{m-0}^{M} \left(\int_m^{\lambda} (\lambda - t)^{\nu-1} \left(D_{*m}^{\nu} f\right)(t)\, dt \right) dE_{\lambda}. \end{aligned} \tag{5}$$

By the spectral representation theorem we obtain

$$\begin{aligned} f(A) &= \sum_{k=0}^{n-1} \frac{f^{(k)}(m)}{k!} (A - m1_H)^k \\ &+ \frac{1}{\Gamma(\nu)} \int_{m-0}^{M} \left(\int_m^{\lambda} (\lambda - t)^{\nu-1} \left(D_{*m}^{\nu} f\right)(t)\, dt \right) dE_{\lambda}, \end{aligned} \tag{6}$$

proving the claim. ■

Remark 1. In (6) assume that $f^{(k)}(m) = 0$, $k = 0, ..., n-1$. Then

$$f(A) = \frac{1}{\Gamma(\nu)} \int_{m-0}^{M} \left(\int_m^{\lambda} (\lambda - t)^{\nu-1} \left(D_{*m}^{\nu} f\right)(t)\, dt \right) dE_{\lambda}. \tag{7}$$

Therefore it holds

$$\langle f(A)\, x, y \rangle = \frac{1}{\Gamma(\nu)} \int_{m-0}^{M} \left(\int_m^{\lambda} (\lambda - t)^{\nu-1} \left(D_{*m}^{\nu} f\right)(t)\, dt \right) d\langle E_{\lambda} x, y \rangle, \tag{8}$$

$\forall\, x, y \in H$.

The function $g_{x,y}(\lambda) := \langle E_\lambda x, y \rangle$ is of bounded variation on the interval $[m, M]$ and

$$g_{x,y}(m-0) = 0 \ \text{ and } \ g_{x,y}(M) = \langle x, y \rangle, \quad \forall\, x, y \in H. \tag{9}$$

It is also well known that $g_x(\lambda) := \langle E_\lambda x, x\rangle$ is nondecreasing and right continuous on $[m, M]$.

One has

$$\langle f(A)x, x\rangle = \frac{1}{\Gamma(\nu)} \int_{m-0}^{M} \left(\int_m^\lambda (\lambda - t)^{\nu-1} \left(D_{*m}^\nu f\right)(t)\, dt \right) d\langle E_\lambda x, x\rangle, \tag{10}$$

$\forall\ x \in H$.

Remark 2 (all as in Theorem 1, Remark 1)**.** Let $p, q > 1 : \frac{1}{p} + \frac{1}{q} = 1$, with $\nu > \frac{1}{q}$. Then

$$\int_m^\lambda (\lambda - t)^{\nu-1} \left|\left(D_{*m}^\nu f\right)(t)\right| dt \tag{11}$$

$$\leq \left(\int_m^\lambda (\lambda - t)^{p(\nu-1)} dt \right)^{\frac{1}{p}} \left(\int_m^\lambda \left|\left(D_{*m}^\nu f\right)(t)\right|^q dt \right)^{\frac{1}{q}}$$

$$\leq \frac{(\lambda - m)^{\frac{p(\nu-1)+1}{p}}}{(p(\nu-1)+1)^{\frac{1}{p}}} \left(\int_m^M \left|\left(D_{*m}^\nu f\right)(t)\right|^q dt \right)^{\frac{1}{q}} \tag{12}$$

$$= \frac{(\lambda - m)^{\nu - 1 + \frac{1}{p}}}{(p(\nu-1)+1)^{\frac{1}{p}}} \left\| D_{*m}^\nu f \right\|_{q,[m,M]}$$

$$= \frac{(\lambda - m)^{\nu - \frac{1}{q}}}{(p(\nu-1)+1)^{\frac{1}{p}}} \left\| D_{*m}^\nu f \right\|_{q,[m,M]}.$$

We have proved that

$$\left| \int_m^\lambda (\lambda - t)^{\nu-1} \left(D_{*m}^\nu f\right)(t)\, dt \right| \leq \int_m^\lambda (\lambda - t)^{\nu-1} \left|\left(D_{*m}^\nu f\right)(t)\right| dt \tag{13}$$

$$\leq \frac{(\lambda - m)^{\nu - \frac{1}{q}}}{(p(\nu-1)+1)^{\frac{1}{p}}} \left\| D_{*m}^\nu f \right\|_{q,[m,M]},$$

$\forall\ \lambda \in [m, M]$.

Therefore it holds

$$|\langle f(A)x, x\rangle| \overset{(10)}{\leq} \frac{1}{\Gamma(\nu)} \int_{m-0}^{M} \left(\int_m^\lambda (\lambda - t)^{\nu-1} \left(D_{*m}^\nu f\right)(t)\, dt \right) d\langle E_\lambda x, x\rangle \tag{14}$$

$$\leq \frac{\left\| D_{*m}^\nu f \right\|_{q,[m,M]}}{(p(\nu-1)+1)^{\frac{1}{p}} \Gamma(\nu)} \int_{m-0}^{M} (\lambda - m)^{\nu - \frac{1}{q}}\, d\langle E_\lambda x, x\rangle$$

$$= \frac{\left\| D_{*m}^\nu f \right\|_{q,[m,M]}}{(p(\nu-1)+1)^{\frac{1}{p}} \Gamma(\nu)} \left\langle (A - m1_H)^{\nu - \frac{1}{q}} x, x \right\rangle,$$

$\forall x \in H$.

We have proved

Theorem 2. *All as in Theorem 1. Assume further* $f^{(k)}(m) = 0$, $k = 0, 1, ..., n-1$. *Let* $p, q > 1 : \frac{1}{p} + \frac{1}{q} = 1$, *with* $\nu > \frac{1}{q}$. *Then*

$$|\langle f(A)x, x\rangle| \leq \frac{\|D_{*m}^{\nu} f\|_{q,[m,M]}}{(p(\nu-1)+1)^{\frac{1}{p}}\Gamma(\nu)} \left\langle (A - m1_H)^{\nu - \frac{1}{q}} x, x \right\rangle, \tag{15}$$

$\forall x \in H$.

Inequality (15) means that

$$\|f(A)\| \leq \frac{\|D_{*m}^{\nu} f\|_{q,[m,M]}}{(p(\nu-1)+1)^{\frac{1}{p}}\Gamma(\nu)} \left\| (A - m1_H)^{\nu - \frac{1}{q}} \right\| \tag{16}$$

and in particular,

$$f(A) \leq \frac{\|D_{*m}^{\nu} f\|_{q,[m,M]}}{(p(\nu-1)+1)^{\frac{1}{p}}\Gamma(\nu)} (A - m1_H)^{\nu - \frac{1}{q}}. \tag{17}$$

We need

Definition 2. *Let the real valued function* $f \in C([m, M])$, *and we consider*

$$g(t) = \int_m^t f(z)\, dz, \quad \forall\, t \in [m, M], \tag{18}$$

then $g \in C([m, M])$.

We denote by

$$\int_{m1_H}^{A} f := \Phi(g) = g(A). \tag{19}$$

We understand and write that $(r > 0)$

$$g^r(A) = \Phi(g^r) =: \left(\int_{m1_H}^{A} f \right)^r.$$

Clearly $\left(\int_{m1_H}^{A} f \right)^r$ *is a self adjoint operator on* H, *for any* $r > 0$.

We will use

Theorem 3 ([1], p. 451)**.** *Let* $\nu \geq \gamma + 1$, $\gamma \geq 0$, $n := \lceil \nu \rceil$ ($\lceil \cdot \rceil$ *ceiling of number*). *Assume* $f \in C^n([m, M])$ *such that* $f^{(k)}(m) = 0$, $k = 0, 1, ..., n-1$. *Let* $p, q > 1 : \frac{1}{p} + \frac{1}{q} = 1$. *Then*

$$\int_m^\lambda |D_{*m}^\gamma f(t)|^q \, dt \leq \left[\frac{(\lambda - m)^{q(\nu-\gamma)}}{(\Gamma(\nu-\gamma))^q (p(\nu-\gamma-1)+1)^{\frac{q}{p}} q(\nu-\gamma)} \right] \int_m^\lambda |D_{*m}^\nu f(t)|^q \, dt, \tag{20}$$

$\forall \lambda \in [m, M]$.

Note: By Proposition 15.114 ([1], p. 388) we have that $D_{*m}^\nu f, D_{*m}^\gamma f \in C([m, M])$.

Using (20) and properties (P) and (ii), we derive the operator Poincaré inequality:

Theorem 4. *All as in Theorem 3. Then*

$$\int_{m1_H}^A |D_{*m}^\gamma f|^q \leq \left[\frac{(A - m1_H)^{q(\nu-\gamma)}}{(\Gamma(\nu-\gamma))^q (p(\nu-\gamma-1)+1)^{\frac{q}{p}} q(\nu-\gamma)} \right] \left(\int_{m1_H}^A |D_{*m}^\nu f|^q \right). \tag{21}$$

We will use

Theorem 5 ([1], p. 493)**.** *Let* $\nu \geq \gamma + 1$, $\gamma \geq 0$, $n := \lceil \nu \rceil$. *Assume* $f \in C^n([m, M])$ *such that* $f^{(k)}(m) = 0$, $k = 0, 1, ..., n-1$. *Let* $p, q > 1 : \frac{1}{p} + \frac{1}{q} = 1$, $r \geq 1$. *Then*

$$\left(\int_m^\lambda |D_{*m}^\gamma f(t)|^r \, dt \right)^{\frac{1}{r}} \leq \left[\frac{(\lambda - m)^{\nu-\gamma+\frac{1}{r}-\frac{1}{q}}}{(\Gamma(\nu-\gamma))(p(\nu-\gamma-1)+1)^{\frac{1}{p}}} \right] \frac{\left(\int_m^\lambda |D_{*m}^\nu f(t)|^q \, dt \right)^{\frac{1}{q}}}{\left[r\left(\nu-\gamma-\frac{1}{q}\right)+1 \right]^{\frac{1}{r}}}, \tag{22}$$

$\forall \lambda \in [m, M]$.

Applying (22), using properties (P) and (ii), we get the following operator Sobolev type inequality:

Theorem 6. *All as in Theorem 5. Then*

$$\left(\int_{m1_H}^{A}\left|D_{*m}^{\gamma}f\right|^{r}\right)^{\frac{1}{r}} \tag{23}$$

$$\leq \frac{\left(A-m1_H\right)^{\nu-\gamma+\frac{1}{r}-\frac{1}{q}}}{\left(\Gamma\left(\nu-\gamma\right)\right)\left(p\left(\nu-\gamma-1\right)+1\right)^{\frac{1}{p}}}\frac{\left(\int_{m1_H}^{A}\left|D_{*m}^{\nu}f\right|^{q}\right)^{\frac{1}{q}}}{\left[r\left(\nu-\gamma-\frac{1}{q}\right)+1\right]^{\frac{1}{r}}}.$$

Next we follow [1], p. 8.

Definition 3. *Let $\nu>0$, $n:=[\nu]$ (integral part), and $\alpha:=\nu-n$ $(0<\alpha<1)$. Let $f\in C([m,M])$ and define*

$$\left(J_{\nu}^{m}f\right)(z)=\frac{1}{\Gamma(\nu)}\int_{m}^{z}(z-t)^{\nu-1}f(t)\,dt, \tag{24}$$

all $m\leq z\leq M$, where Γ is the gamma function, the left generalized Riemann-Liouville integral. We define the subspace $C_m^{\nu}([m,M])$ of $C^n([m,M])$:

$$C_m^{\nu}([m,M]):=\left\{f\in C^n([m,M]):J_{1-\alpha}^{m}f^{(n)}\in C^1([m,M])\right\}. \tag{25}$$

So let $f\in C_m^{\nu}([m,M])$; we define the left generalized ν-fractional derivative (of Canavati type) of f over $[m,M]$ as

$$D_m^{\nu}f:=\left(J_{1-\alpha}^{m}f^{(n)}\right)'. \tag{26}$$

Notice that

$$\left(J_{1-\alpha}^{m}f^{(n)}\right)(z)=\frac{1}{\Gamma(1-\alpha)}\int_{m}^{z}(z-t)^{-\alpha}f^{(n)}(t)\,dt \tag{27}$$

exists for $f\in C_m^{\nu}([m,M])$, all $m\leq z\leq M$.

Also we notice that $D_m^{\nu}f\in C([m,M])$, $D_m^{n}f=f^{(n)}$, $n\in\mathbb{N}$; $D_m^{0}f=f$.

We need

Theorem 7 ([1], p. 9)**.** *Let $f\in C_m^{\nu}([m,M])$. Then*
(i) for $\nu\geq1$, we have

$$f(\lambda)=f(m)+f'(m)(\lambda-m)+\frac{f''(m)}{2}(\lambda-m)^2+... \tag{28}$$

$$+f^{(n-1)}(m)\frac{(\lambda-m)^{n-1}}{(n-1)!}+\frac{1}{\Gamma(\nu)}\int_{m}^{\lambda}(\lambda-t)^{\nu-1}(D_m^{\nu}f)(t)\,dt,$$

(ii) if $0<\nu<1$ we get

$$f(\lambda)=\frac{1}{\Gamma(\nu)}\int_{m}^{\lambda}(\lambda-t)^{\nu-1}\left(D_m^{\nu}f\right)(t)\,dt, \tag{29}$$

$\forall\ \lambda\in[m,M]$.

We present the following operator representation formula:

Theorem 8. *Theorem 8. Let A be a selfadjoint operator in the Hilbert space H with the spectrum $Sp(A) \subseteq [m, M]$ for some real numbers $m < M$, $\{E_\lambda\}_\lambda$ be its spectral family, $[m, M] \subset (a, b)$ and $n \in \mathbb{N}$, where $n := \lceil \nu \rceil$, $\nu > 0$. We consider $f \in C_m^\nu([m, M])$, where $f : [a, b] \to \mathbb{R}$.*

Then

(i) for $\nu \geq 1$, we have

$$f(A) = \sum_{k=0}^{n-1} \frac{f^{(k)}(m)}{k!} (A - m1_H)^k + R_n^*(f, m, M), \tag{30}$$

where

$$R_n^*(f, m, M) = \frac{1}{\Gamma(\nu)} \int_{m-0}^{M} \left(\int_m^\lambda (\lambda - t)^{\nu-1} (D_m^\nu f)(t)\, dt \right) dE_\lambda. \tag{31}$$

(ii) if $0 < \nu < 1$ we get

$$f(A) = \frac{1}{\Gamma(\nu)} \int_{m-0}^{M} \left(\int_m^\lambda (\lambda - t)^{\nu-1} (D_m^\nu f)(t)\, dt \right) dE_\lambda. \tag{32}$$

Proof. We integrate (28), (29) against E_λ, apply spectral representation theorem. ■

Remark 3. In (30) ($\nu \geq 1$) we assume $f^{(k)}(m) = 0$, $k = 0, 1, ..., n-1$, then

$$f(A) = \frac{1}{\Gamma(\nu)} \int_{m-0}^{M} \left(\int_m^\lambda (\lambda - t)^{\nu-1} (D_m^\nu f)(t)\, dt \right) dE_\lambda. \tag{33}$$

We have

$$\langle f(A) x, x \rangle = \frac{1}{\Gamma(\nu)} \int_{m-0}^{M} \left(\int_m^\lambda (\lambda - t)^{\nu-1} (D_m^\nu f)(t)\, dt \right) d \langle E_\lambda x, x \rangle, \tag{34}$$

$\forall\, x \in H$.

Let $p, q > 1 : \frac{1}{p} + \frac{1}{q} = 1$, with $\nu > \frac{1}{q}$. Then

$$\left| \int_m^\lambda (\lambda - t)^{\nu-1} (D_m^\nu f)(t)\, dt \right| \leq \int_m^\lambda (\lambda - t)^{\nu-1} |(D_m^\nu f)(t)|\, dt \leq \frac{(\lambda - m)^{\nu - \frac{1}{q}}}{(p(\nu-1)+1)^{\frac{1}{p}}} \|D_m^\nu f\|_{q,[m,M]}, \tag{35}$$

$\forall\, \lambda \in [m, M]$.

Hence

$$(36)\quad |\langle f(A)x,x\rangle| \overset{(34)}{\leq} \frac{1}{\Gamma(\nu)} \int_{m-0}^{M} \left| \int_{m}^{\lambda} (\lambda-t)^{\nu-1} \left(D_{m}^{\nu} f\right)(t)\, dt \right| d\langle E_{\lambda}x,x\rangle$$

$$\leq \frac{\left\|D_{m}^{\nu} f\right\|_{q,[m,M]}}{(p(\nu-1)+1)^{\frac{1}{p}} \Gamma(\nu)} \int_{m-0}^{M} (\lambda-m)^{\nu-\frac{1}{q}}\, d\langle E_{\lambda}x,x\rangle$$

$$= \frac{\left\|D_{m}^{\nu} f\right\|_{q,[m,M]}}{(p(\nu-1)+1)^{\frac{1}{p}} \Gamma(\nu)} \left\langle (A-m1_{H})^{\nu-\frac{1}{q}} x,x \right\rangle, \quad \forall x \in H.$$

We have proved

Theorem 9. *All as in Theorem 8. Let $\nu > 0$. In case of $\nu \geq 1$, assume further $f^{(k)}(m) = 0$, $k = 0,1,...,n-1$. Let $p,q > 1 : \frac{1}{p} + \frac{1}{q} = 1$, with $\nu > \frac{1}{q}$. Then*

$$(37)\quad |\langle f(A)x,x\rangle| \leq \frac{\left\|D_{m}^{\nu} f\right\|_{q,[m,M]}}{(p(\nu-1)+1)^{\frac{1}{p}} \Gamma(\nu)} \left\langle (A-m1_{H})^{\nu-\frac{1}{q}} x,x \right\rangle,$$

$\forall x \in H$.

Inequality (37) means that

$$(38)\quad \|f(A)\| \leq \frac{\left\|D_{m}^{\nu} f\right\|_{q,[m,M]}}{(p(\nu-1)+1)^{\frac{1}{p}} \Gamma(\nu)} \left\| (A-m1_{H})^{\nu-\frac{1}{q}} \right\|,$$

and in particular,

$$(39)\quad f(A) \leq \left(\frac{\left\|D_{m}^{\nu} f\right\|_{q,[m,M]}}{(p(\nu-1)+1)^{\frac{1}{p}} \Gamma(\nu)} \right) (A-m1_{H})^{\nu-\frac{1}{q}}.$$

We will use

Theorem 10 ([1], p. 447)**.** *Let $\nu \geq \gamma+1$, $\gamma \geq 0$, $n := [\nu]$. Assume $f \in C_{m}^{\nu}([m,M])$ such that $f^{(k)}(m) = 0$, $k = 0,1,...,n-1$. Let $p,q > 1 : \frac{1}{p} + \frac{1}{q} = 1$. Then*

$$(40)\quad \int_{m}^{\lambda} \left| D_{m}^{\gamma} f(t) \right|^{q} dt$$

$$\leq \left[\frac{(\lambda-m)^{q(\nu-\gamma)}}{(\Gamma(\nu-\gamma))^{q} (p(\nu-\gamma-1)+1)^{\frac{q}{p}} q(\nu-\gamma)} \right] \int_{m}^{\lambda} \left| D_{m}^{\nu} f(t) \right|^{q} dt,$$

$\forall\ \lambda \in [m,M]$.

By Remark 3.4, [1], p. 26, $D_{m}^{\gamma} f \in C([m,M])$.

Using (40) and properties (P) and (ii), we derive the operator Poincaré inequality:

Theorem 11. *All as in Theorem 10. Then*

$$\int_{m1_H}^{A} |D_m^{\gamma} f|^q \leq \left[\frac{(A - m1_H)^{q(\nu-\gamma)}}{(\Gamma(\nu-\gamma))^q (p(\nu-\gamma-1)+1)^{\frac{q}{p}} q(\nu-\gamma)} \right] \left(\int_{m1_H}^{A} |D_m^{\nu} f|^q \right). \tag{41}$$

We will use

Theorem 12 ([1], p. 485)**.** *Let* $\nu \geq \gamma + 1$, $\gamma \geq 0$, $n := [\nu]$. *Assume* $f \in C_m^{\nu}([m, M])$ *such that* $f^{(k)}(m) = 0$, $k = 0, 1, ..., n-1$. *Let* $p, q > 1$: $\frac{1}{p} + \frac{1}{q} = 1$, $r \geq 1$. *Then*

$$\left(\int_m^{\lambda} |D_m^{\gamma} f(t)|^r dt \right)^{\frac{1}{r}} \leq \left[\frac{(\lambda - m)^{\nu-\gamma+\frac{1}{r}-\frac{1}{q}}}{(\Gamma(\nu-\gamma))(p(\nu-\gamma-1)+1)^{\frac{1}{p}}} \right] \frac{\left(\int_m^{\lambda} |D_m^{\nu} f(t)|^q dt \right)^{\frac{1}{q}}}{\left[r\left(\nu-\gamma-\frac{1}{q}\right)+1 \right]^{\frac{1}{r}}}, \tag{42}$$

$\forall\ \lambda \in [m, M]$.

Applying (42), using properties (P) and (ii), we get the following operator Sobolev type inequality:

Theorem 13. *All as in Theorem 12. Then*

$$\left(\int_{m1_H}^{A} |D_m^{\gamma} f|^r \right)^{\frac{1}{r}} \leq \frac{(A - m1_H)^{\nu-\gamma+\frac{1}{r}-\frac{1}{q}}}{(\Gamma(\nu-\gamma))(p(\nu-\gamma-1)+1)^{\frac{1}{p}}} \frac{\left(\int_{m1_H}^{A} |D_m^{\nu} f|^q \right)^{\frac{1}{q}}}{\left[r\left(\nu-\gamma-\frac{1}{q}\right)+1 \right]^{\frac{1}{r}}}. \tag{43}$$

We need

Definition 4 ([2], p. 337)**.** *Let* $f \in AC^n([m, M])$, $n := \lceil \nu \rceil$, $\nu > 0$. *The right Caputo fractional derivative of order* $\nu > 0$, *is given by*

$$\left(D_{M-}^{\nu} f \right)(z) := \frac{(-1)^n}{\Gamma(n-\nu)} \int_z^M (J - z)^{n-\nu-1} f^{(n)}(J)\, dJ, \tag{44}$$

$\forall\ z \in [m, M]$, *which exists a.e. on* $[m, M]$, *and* $D_{M-}^{\nu} f \in L_1([m, M])$.

We notice that $D_{M-}^0 f = f$, $\left(D_{M-}^n f \right)(z) = (-1)^n f^{(n)}(z)$, *for* $n \in \mathbb{N}$.

We need the right Caputo fractional Taylor formula with integral remainder:

Theorem 14 ([2], p. 341). *Let* $f \in AC^n([m, M])$, $\lambda \in [m, M]$, $\nu > 0$, $n = \lceil \nu \rceil$. *Then*

$$f(\lambda) = \sum_{k=0}^{n-1} \frac{f^{(k)}(M)}{k!} (\lambda - M)^k + \frac{1}{\Gamma(\nu)} \int_\lambda^M (J - \lambda)^{\nu-1} \left(D_{M-}^\nu f\right)(J)\, dJ. \tag{45}$$

We present the following operator representation formula:

Theorem 15. *Let* A *be a selfadjoint operator in the Hilbert space* H *with the spectrum* $Sp(A) \subseteq [m, M]$ *for some real numbers* $m < M$, $\{E_\lambda\}_\lambda$ *be its spectral family,* I *be a closed subinterval on* $\mathbb{R}$ *with* $[m, M] \subset \overset{\circ}{I}$ *(the interior of* I*) and* $n \in \mathbb{N}$, *with* $n := \lceil \nu \rceil$, $\nu > 0$. *We consider* $f \in AC^n([m, M])$ *(i.e.* $f^{(n-1)} \in AC([m, M])$*), where* $f : I \to \mathbb{R}$.

Then

$$f(A) = \sum_{k=0}^{n-1} \frac{f^{(k)}(M)}{k!} (A - M1_H)^k + \frac{1}{\Gamma(\nu)} \int_{m-0}^M \left(\int_\lambda^M (J - \lambda)^{\nu-1} \left(D_{M-}^\nu f\right)(J)\, dJ \right) dE_\lambda. \tag{46}$$

Proof. Integrate (45) against E_λ and apply the spectral representation theorem. ■

We make

Remark 4. In (46) assume that $f^{(k)}(M) = 0$, $k = 0, ..., n-1$. Then

$$f(A) = \frac{1}{\Gamma(\nu)} \int_{m-0}^M \left(\int_\lambda^M (J - \lambda)^{\nu-1} \left(D_{M-}^\nu f\right)(J)\, dJ \right) dE_\lambda. \tag{47}$$

We have that

$$\langle f(A)x, x \rangle = \frac{1}{\Gamma(\nu)} \int_{m-0}^M \left(\int_\lambda^M (J - \lambda)^{\nu-1} \left(D_{M-}^\nu f\right)(J)\, dJ \right) d\langle E_\lambda x, x \rangle, \tag{48}$$

$\forall\, x \in H$.

Let $p, q > 1 : \frac{1}{p} + \frac{1}{q} = 1$, with $\nu > \frac{1}{q}$. Then

$$\int_\lambda^M (J - \lambda)^{\nu-1} \left| \left(D_{M-}^\nu f\right)(J) \right| dJ \le \left(\int_\lambda^M (J - \lambda)^{p(\nu-1)}\, dJ \right)^{\frac{1}{p}} \left(\int_\lambda^M \left| \left(D_{M-}^\nu f\right)(J) \right|^q dJ \right)^{\frac{1}{q}} \tag{49}$$

$$\leq \frac{(M-\lambda)^{\frac{p(\nu-1)+1}{p}}}{(p(\nu-1)+1)^{\frac{1}{p}}} \left\| D_{M-}^{\nu} f \right\|_{q,[m,M]} = \frac{(M-\lambda)^{\nu-\frac{1}{q}}}{(p(\nu-1)+1)^{\frac{1}{p}}} \left\| D_{M-}^{\nu} f \right\|_{q,[m,M]}.$$

We have proved that

$$\left| \int_{\lambda}^{M} (J-\lambda)^{\nu-1} \left(D_{M-}^{\nu} f \right)(J)\, dJ \right| \leq \int_{\lambda}^{M} (J-\lambda)^{\nu-1} \left| \left(D_{M-}^{\nu} f \right)(J) \right| dJ \leq \frac{(M-\lambda)^{\nu-\frac{1}{q}}}{(p(\nu-1)+1)^{\frac{1}{p}}} \left\| D_{M-}^{\nu} f \right\|_{q,[m,M]}, \tag{50}$$

$\forall\ \lambda \in [m, M]$.

Therefore it holds

$$|\langle f(A)x, x\rangle| \overset{(48)}{\leq} \frac{1}{\Gamma(\nu)} \times \int_{m-0}^{M} \left(\int_{\lambda}^{M} (J-\lambda)^{\nu-1} \left(D_{M-}^{\nu} f \right)(J)\, dJ \right) d\langle E_\lambda x, x\rangle \overset{(50)}{\leq} \frac{\left\| D_{M-}^{\nu} f \right\|_{q,[m,M]}}{\Gamma(\nu)(p(\nu-1)+1)^{\frac{1}{p}}} \int_{m-0}^{M} (M-\lambda)^{\nu-\frac{1}{q}}\, d\langle E_\lambda x, x\rangle = \frac{\left\| D_{M-}^{\nu} f \right\|_{q,[m,M]}}{(p(\nu-1)+1)^{\frac{1}{p}}\Gamma(\nu)} \left\langle (M1_H - A)^{\nu-\frac{1}{q}} x, x \right\rangle, \quad \forall x \in H. \tag{51}$$

We have proved

Theorem 16. *All as in Theorem 15. Assume further* $f^{(k)}(M) = 0$, $k = 0, 1, ..., n-1$. *Let* $p, q > 1 : \frac{1}{p} + \frac{1}{q} = 1$, *with* $\nu > \frac{1}{q}$. *Then*

$$|\langle f(A)x, x\rangle| \leq \frac{\left\| D_{M-}^{\nu} f \right\|_{q,[m,M]}}{(p(\nu-1)+1)^{\frac{1}{p}}\Gamma(\nu)} \left\langle (M1_H - A)^{\nu-\frac{1}{q}} x, x \right\rangle, \tag{52}$$

$\forall x \in H$.

Inequality (52) means

$$\|f(A)\| \leq \frac{\left\| D_{M-}^{\nu} f \right\|_{q,[m,M]}}{(p(\nu-1)+1)^{\frac{1}{p}}\Gamma(\nu)} \left\| (M1_H - A)^{\nu-\frac{1}{q}} \right\|, \tag{53}$$

and in particular,

$$f(A) \leq \left(\frac{\left\| D_{M-}^{\nu} f \right\|_{q,[m,M]}}{(p(\nu-1)+1)^{\frac{1}{p}}\Gamma(\nu)} \right) (M1_H - A)^{\nu-\frac{1}{q}}. \tag{54}$$

We give the following Poincaré type fractional inequality:

Theorem 17. *Let* $f \in AC^n([m,M])$, $\nu > 0$, $n = \lceil \nu \rceil$. *Assume* $f^{(k)}(M) = 0$, $k = 0, ..., n-1$. *Let* $p, q > 1 : \frac{1}{p} + \frac{1}{q} = 1$, $\nu > \frac{1}{q}$. *Then*

$$\int_w^M |f(\lambda)|^q \, d\lambda \le \frac{(M-w)^{\nu q}}{(p(\nu-1)+1)^{\frac{q}{p}} (\Gamma(\nu))^q \nu q} \int_w^M \left| \left(D_{M-}^{\nu} f \right)(\lambda) \right|^q d\lambda, \tag{55}$$

$\forall \ w \in [m,M]$.

Proof. By the assumption and (45) we have that

$$f(\lambda) = \frac{1}{\Gamma(\nu)} \int_\lambda^M (J-\lambda)^{\nu-1} \left(D_{M-}^{\nu} f \right)(J) \, dJ, \quad \forall \ \lambda \in [m,M]. \tag{56}$$

Hence

$$|f(\lambda)| \le \frac{1}{\Gamma(\nu)} \int_\lambda^M (J-\lambda)^{\nu-1} \left| \left(D_{M-}^{\nu} f \right)(J) \right| dJ, \quad \forall \ \lambda \in [m,M]. \tag{57}$$

As in (49), (50), we get

$$|f(\lambda)| \le \frac{(M-\lambda)^{\nu - \frac{1}{q}}}{(p(\nu-1)+1)^{\frac{1}{p}} \Gamma(\nu)} \left\| D_{M-}^{\nu} f \right\|_{q,[w,M]}, \tag{58}$$

$\forall \ \lambda \in [w,M]$, where $w \in [m,M]$.

Hence it holds

$$|f(\lambda)|^q \le \frac{(M-\lambda)^{\nu q - 1}}{(p(\nu-1)+1)^{\frac{q}{p}} (\Gamma(\nu))^q} \left\| D_{M-}^{\nu} f \right\|_{q,[w,M]}^q, \tag{59}$$

$\forall \ \lambda \in [w,M]$, where $w \in [m,M]$.

Therefore by integration

$$\int_w^M |f(\lambda)|^q \, d\lambda \le \frac{(M-w)^{\nu q}}{(p(\nu-1)+1)^{\frac{q}{p}} (\Gamma(\nu))^q \nu q} \left\| D_{M-}^{\nu} f \right\|_{q,[w,M]}^q, \tag{60}$$

$\forall \ w \in [m,M]$, proving the claim. ■

We need

Definition 5. *Let* $f : [m,M] \to \mathbb{R}$ *be continuous. We consider*

$$g(t) = \int_t^M f(z) \, dz, \quad \forall \ t \in [m,M], \tag{61}$$

then $g \in C([m,M])$.

We denote by

$$\int_A^{M1_H} f := \Phi(g) = g(A). \tag{62}$$

We denote also

$$g^r(A) = \Phi(g^r) =: \left(\int_A^{M1_H} f\right)^r, \quad r > 0. \tag{63}$$

Clearly $\left(\int_A^{M1_H} f\right)^r$ *is a self adjoint operator on* H*, for any* $r > 0$*.*

We present the following operator Poincaré type inequality:

Theorem 18. *All as in Theorem 17. Then*

$$\int_A^{M1_H} |f|^q \leq \frac{(M1_H - A)^{\nu q}}{(p(\nu-1)+1)^{\frac{q}{p}} (\Gamma(\nu))^q \nu q} \left(\int_A^{M1_H} \left|\left(D_{M-}^{\nu} f\right)\right|^q\right). \tag{64}$$

We give the following Sobolev type fractional inequality:

Theorem 19. *All as in Theorem 17, and* $r \geq 1$*. Then*

$$\|f\|_{r,[w,M]} \leq \frac{(M-w)^{\nu - \frac{1}{q} + \frac{1}{r}}}{\left(\nu r - \frac{r}{p} + 1\right)^{\frac{1}{r}} (p(\nu-1)+1)^{\frac{1}{p}} \Gamma(\nu)} \left\|D_{M-}^{\nu} f\right\|_{q,[w,M]}, \tag{65}$$

$\forall\, w \in [m, M]$*.*

Proof. We recall (58):

$$|f(\lambda)| \leq \frac{(M-\lambda)^{\nu - \frac{1}{q}}}{\Gamma(\nu)(p(\nu-1)+1)^{\frac{1}{p}}} \left\|D_{M-}^{\nu} f\right\|_{q,[w,M]}, \tag{66}$$

$\forall\, \lambda \in [w, M]$, where $w \in [m, M]$.

Hence, by $r \geq 1$, we obtain

$$|f(\lambda)|^r \leq \frac{(M-\lambda)^{\nu r - \frac{r}{q}}}{(\Gamma(\nu))^r (p(\nu-1)+1)^{\frac{r}{p}}} \left\|D_{M-}^{\nu} f\right\|_{q,[w,M]}^r, \tag{67}$$

$\forall\, \lambda \in [w, M]$, where $w \in [m, M]$.

Consequently it holds

$$\int_w^M |f(\lambda)|^r d\lambda \leq \frac{(M-w)^{\nu r - \frac{r}{q} + 1}}{\left(\nu r - \frac{r}{q} + 1\right) (p(\nu-1)+1)^{\frac{r}{p}} (\Gamma(\nu))^r} \times \left\|D_{M-}^{\nu} f\right\|_{q,[w,M]}^r, \tag{68}$$

$\forall\, w \in [m, M]$, proving the claim. ■

Next we give an operator Sobolev type inequality:

Theorem 20. *All as in Theorem 19. Then*

$$\left(\int_A^{M1_H} |f|^r\right)^{\frac{1}{r}} \leq \frac{(M1_H - A)^{\nu - \frac{1}{q} + \frac{1}{r}}}{\left(\nu r - \frac{r}{p} + 1\right)^{\frac{1}{r}} (p(\nu - 1) + 1)^{\frac{1}{p}} \Gamma(\nu)} \times \left(\int_A^{M1_H} \left|\left(D_{M-}^{\nu} f\right)\right|^q\right)^{\frac{1}{q}}. \tag{69}$$

We need

Definition 6 ([2], p. 345)**.** *Let $\nu > 0$, $n := [\nu]$, $\alpha := \nu - n$, $0 < \alpha < 1$, $f \in C([m, M])$. The right Riemann-Liouville fractional integral operator in given by*

$$\left(J_{M-}^{\nu} f\right)(z) := \frac{1}{\Gamma(\nu)} \int_z^M (J - z)^{\nu - 1} f(J)\, dJ, \tag{70}$$

$\forall\ z \in [m, M]$, $J_{M-}^{0} f := f$.

Define the subspace of functions

$$C_{M-}^{\nu}([m, M]) := \left\{f \in C^n([m, M]) : J_{M-}^{1-\alpha} f^{(n)} \in C^1([m, M])\right\}. \tag{71}$$

Define the right generalized ν-fractional derivative of f over $[m, M]$ as

$$\overline{D}_{M-}^{\nu} f := (-1)^{n-1} \left(J_{M-}^{1-\alpha} f^{(n)}\right)'. \tag{72}$$

Notice that

$$J_{M-}^{1-\alpha} f^{(n)}(z) = \frac{1}{\Gamma(1-\alpha)} \int_z^M (J - z)^{-\alpha} f^{(n)}(J)\, dJ, \tag{73}$$

exists for $f \in C_{M-}^{\nu}([m, M])$, and

$$\left(\overline{D}_{M-}^{\nu} f\right)(z) = \frac{(-1)^{n-1}}{\Gamma(1-\alpha)} \frac{d}{dz} \int_z^M (J - z)^{-\alpha} f^{(n)}(J)\, dJ. \tag{74}$$

That is

$$\left(\overline{D}_{M-}^{\nu} f\right)(z) = \frac{(-1)^{n-1}}{\Gamma(n - \nu + 1)} \frac{d}{dz} \int_z^M (J - z)^{n-\nu} f^{(n)}(J)\, dJ. \tag{75}$$

If $\nu \in \mathbb{N}$, then $\alpha = 0$, $n = \nu$, and

$$\left(\overline{D}_{M-}^{\nu} f\right)(z) = (-1)^n f^{(n)}(z), \tag{76}$$

$\forall\ z \in [m, M]$, *and* $\overline{D}_{M-}^{0} f = f$.

We will use the following fractional Taylor formula:

Theorem 21 ([2], p. 348). *Let* $f \in C^{\nu}_{M-}([m, M])$, $\nu > 0$, $n := [\nu]$. *Then*

1) *for* $\nu \geq 1$, *we get*

$$f(\lambda) = \sum_{k=0}^{n-1} \frac{f^{(k)}(M)}{k!} (\lambda - M)^k + \frac{1}{\Gamma(\nu)} \int_{\lambda}^{M} (J - \lambda)^{\nu-1} \left(\overline{D}^{\nu}_{M-} f\right)(J)\, dJ, \tag{77}$$

$\forall\ \lambda \in [m, M]$.

2) *if* $0 < \nu < 1$, *we obtain*

$$f(\lambda) = \frac{1}{\Gamma(\nu)} \int_{\lambda}^{M} (J - \lambda)^{\nu-1} \left(\overline{D}^{\nu}_{M-} f\right)(J)\, dJ, \tag{78}$$

$\forall\ \lambda \in [m, M]$.

We present the following operator representation formula:

Theorem 22. *Let* A *be a selfadjoint operator in the Hilbert space* H *with the spectrum* $Sp(A) \subseteq [m, M]$ *for some real numbers* $m < M$, $\{E_\lambda\}_\lambda$ *be its spectral family,* I *be a closed subinterval on* $\mathbb{R}$ *with* $[m, M] \subset \overset{\circ}{I}$ (*the interior of* I) *and* $n \in \mathbb{N}$, *with* $n := [\nu]$, $\nu > 0$. *We consider* $f \in C^{\nu}_{M-}([m, M])$, *where* $f : I \to \mathbb{R}$.

Then

i) case of $\nu \geq 1$,

$$f(A) = \sum_{k=0}^{n-1} \frac{f^{(k)}(M)}{k!} (A - M1_H)^k + \frac{1}{\Gamma(\nu)} \int_{m-0}^{M} \left(\int_{\lambda}^{M} (J - \lambda)^{\nu-1} \left(\overline{D}^{\nu}_{M-} f\right)(J)\, dJ \right) dE_\lambda, \tag{79}$$

ii) case of $0 < \nu < 1$,

$$f(A) = \frac{1}{\Gamma(\nu)} \int_{m-0}^{M} \left(\int_{\lambda}^{M} (J - \lambda)^{\nu-1} \left(\overline{D}^{\nu}_{M-} f\right)(J)\, dJ \right) dE_\lambda. \tag{80}$$

Proof. Integrate (77), (78) against E_λ, apply spectral representation theorem. ∎

We have proved

Theorem 23. *All as in Theorem 22. In case of $\nu \geq 1$, we assume further $f^{(k)}(M) = 0$, for $k = 0, 1, ..., n-1$. Let $p, q > 1 : \frac{1}{p} + \frac{1}{q} = 1$, with $\nu > \frac{1}{q}$. Then*

$$|\langle f(A)x, x\rangle| \leq \frac{\left\|\overline{D}_{M-}^{\nu} f\right\|_{q,[m,M]}}{(p(\nu-1)+1)^{\frac{1}{p}}\Gamma(\nu)} \left\langle (M1_H - A)^{\nu-\frac{1}{q}} x, x \right\rangle, \tag{81}$$

$\forall x \in H$.

Inequality (81) means

$$\|f(A)\| \leq \frac{\left\|\overline{D}_{M-}^{\nu} f\right\|_{q,[m,M]}}{(p(\nu-1)+1)^{\frac{1}{p}}\Gamma(\nu)} \left\|(M1_H - A)^{\nu-\frac{1}{q}}\right\|, \tag{82}$$

and in particular,

$$f(A) \leq \left(\frac{\left\|\overline{D}_{M-}^{\nu} f\right\|_{q,[m,M]}}{(p(\nu-1)+1)^{\frac{1}{p}}\Gamma(\nu)}\right) (M1_H - A)^{\nu-\frac{1}{q}}. \tag{83}$$

Proof. Very similar to Theorem 16. ■

We give the following Poincaré type fractional inequality:

Theorem 24. *Let $f \in C_{M-}^{\nu}([m, M])$, $\nu > 0$, $n = [\nu]$. If $\nu \geq 1$, we assume $f^{(k)}(M) = 0$, $k = 0, ..., n-1$. Let $p, q > 1 : \frac{1}{p} + \frac{1}{q} = 1$, $\nu > \frac{1}{q}$. Then*

$$\int_w^M |f(\lambda)|^q d\lambda \leq \frac{(M-w)^{\nu q}}{(p(\nu-1)+1)^{\frac{q}{p}} (\Gamma(\nu))^q \nu q} \int_w^M \left|\left(\overline{D}_{M-}^{\nu} f\right)(\lambda)\right|^q d\lambda, \tag{84}$$

$\forall\ w \in [m, M]$.

Proof. Similar to Theorem 17. ■

We present the following operator Poincaré type inequality:

Theorem 25. *All as in Theorem 24. Then*

$$\int_A^{M1_H} |f|^q \leq \frac{(M1_H - A)^{\nu q}}{(p(\nu-1)+1)^{\frac{q}{p}} (\Gamma(\nu))^q \nu q} \left(\int_A^{M1_H} \left|\overline{D}_{M-}^{\nu} f\right|^q\right). \tag{85}$$

We give the following Sobolev type fractional inequality:

Theorem 26. *All as in Theorem 24, and $r \geq 1$. Then*

$$\|f\|_{r,[w,M]} \leq \frac{(M-w)^{\nu-\frac{1}{q}+\frac{1}{r}}}{\left(\nu r - \frac{r}{p} + 1\right)^{\frac{1}{r}} (p(\nu-1)+1)^{\frac{1}{p}}\Gamma(\nu)} \left\|\overline{D}_{M-}^{\nu} f\right\|_{q,[w,M]}, \tag{86}$$

$\forall\ w \in [m, M]$.

Proof. Similar to Theorem 19. ■

Next we give an operator Sobolev type inequality:

Theorem 27. *All as in Theorem 26. Then*

$$\left(\int_A^{M1_H} |f|^r\right)^{\frac{1}{r}} \leq \frac{(M1_H - A)^{\nu - \frac{1}{q} + \frac{1}{r}}}{\left(\nu r - \frac{r}{p} + 1\right)^{\frac{1}{r}} (p(\nu-1)+1)^{\frac{1}{p}} \Gamma(\nu)} \times \left(\int_A^{M1_H} \left|\overline{D}_{M-}^{\nu} f\right|^q\right)^{\frac{1}{q}}. \tag{87}$$

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