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# ON THE NUMBER OF ZEROS OF A POLYNOMIAL IN A REGION

Abstract. In this paper, we impose restrictions on the complex coefficients of a polynomial in order to give bounds concerning the number of zeros in a specific region of the complex plane. Our results generalize and refine a good number of results in this area of research.

Key words: polynomial, zeros of polynomials, monotonicity, disk.

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## 1. Introduction and statement of results

The study of polynomials and their zeros is well known to have numerous applications in many areas of scientific discipline such as control theory, signal processing, communication theory, coding theory, cryptography, combinatorics and mathematical biology. Due to this fact, several authors have studied extensively problems involving polynomials and their properties in general, and locations of their zeros. In most cases, especially in practice, the roots of polynomials may not be easily obtainable. Therefore, there is need to put some restrictions on the coefficients of polynomials. This would assist us to determine the bounds on the number of zeros of some polynomials in a certain region, thereby reducing the effort of locating these zeros. In this paper, we present some interesting results involving the bounds on the number of zeros of polynomials and in the process, generalize these results.

The earliest record of the Eneström-Kakeya Theorem dates back to 1893 and is stated thus:

Let $p(z) = \sum_{j=0}^{n} a_j z^j$ be a polynomial of degree $n$ such that $0 < a_0 < a_1 < a_2 < ... < a_{n-1} < a_n$, then $p(z)$ has all its zeros in $|z| < 1$.

Since the establishment of this result, several works have been done to improve or extend the result.

One result of fundamental importance pertaining to the number of zeros of polynomials in a given disk is the result of Titchmarsh [15] and this is found in *The Theory of Functions* (second edition, page 171). He proved the following.

**Theorem A.** *Let $F(z)$ be analytic in $|z| \leq R$. Let $|F(z)| \leq M$ in the disk $|z| \leq R$ and suppose $F(0) \neq 0$. Then for $0 < \delta < 1$, the number of zeros of $F(z)$ in the disk $|z| \leq \delta R$ is less than*

$$\frac{1}{\log 1/\delta} \log \frac{M}{|F(0)|}.$$

*By applying Theorem A particularly to polynomial functions and introducing restrictions like that of Eneström-Kakeya's result, Mohammad [12] obtained the bound for number of zeros of polynomials for a particular case of $\delta = 1/2$.*

Dewan [2] weakened Mohammad's hypotheses and proved the following result involving the complex coefficients of the polynomial.

**Theorem B.** *Let $p(z) = \sum_{j=0}^{n} a_j z^j$ such that $|\arg a_j - \beta| \leq \alpha \leq \frac{\pi}{2}$ for all $1 \leq j \leq n$ and some real $\alpha$ and $\beta$, and $0 < |a_0| \leq |a_1| \leq |a_2| \leq \cdots \leq |a_{n-1}| \leq |a_n|$. Then the number of zeros of $p(z)$ in $|z| \leq 1/2$ does not exceed*

$$\frac{1}{\log 2} \log \frac{|a_n|(\cos\alpha + \sin\alpha + 1) + 2\sin\alpha \sum_{j=0}^{n-1} |a_j|}{|a_0|}.$$

Dewan [2] also considered the monotonicity condition for the real parts of the coefficients of a given polynomial.

Pukhta [13] generalized Theorem B by finding the number of zeros in $|z| \leq \delta$ for $0 < \delta < 1$.

The following Theorem deals with the monotonicity condition on the moduli of the coefficients.

**Theorem C.** *Let $p(z) = \sum_{j=0}^{n} a_j z^j$ such that $|\arg a_j - \beta| \leq \alpha \leq \frac{\pi}{2}$ for all $1 \leq j \leq n$ and some real $\alpha$ and $\beta$, and $0 < |a_0| \leq |a_1| \leq |a_2| \leq \cdots \leq |a_{n-1}| \leq |a_n|$. Then the number of zeros of $p$ in $|z| \leq \delta$, $0 < \delta < 1$ does not exceed*

$$\frac{1}{\log 1/\delta} \log \frac{|a_n|(\cos\alpha + \sin\alpha + 1) + 2\sin\alpha \sum_{j=0}^{n-1} |a_j|}{|a_0|}.$$

Pukhta [13] also obtained a similar result dealing with a monotonicity condition on the real part of the complex polynomial.

Gardner and Shields [5] used some "monotonicity flip" conditions like those of Aziz and Mohammad [1] to prove the following result.

**Theorem D.** *Let* $p(z) = \sum_{j=0}^{n} a_j z^j$ *where for some* $R > 0$ *and some* $0 \le k \le n$,

$$\begin{aligned} 0 < |a_0| \le R|a_1| \le R^2|a_2| \le \cdots \le R^{k-1}|a_{k-1}| \le R^k|a_k| \\ \ge R^{k+1}|a_{k+1}| \ge R^{k+2}|a_{k+2}| \ge \cdots \ge R^{n-1}|a_{n-1}| \ge R^n|a_n| \end{aligned}$$

*and* $|\arg a_j - \beta| \le \alpha \le \frac{\pi}{2}$ *for* $1 \le j \le n$ *and for some real* $\alpha$ *and* $\beta$. *Then for* $0 < \delta < 1$ *the number of zeros of* $p(z)$ *in the disk* $|z| \le \delta R$ *is less than*

$$\frac{1}{\log 1/\delta} \log \frac{M}{|a_0|}$$

*where*

$$\begin{aligned} M =& |a_0|R(1 - \cos\alpha - \sin\alpha) + 2|a_k|R^{k+1}\cos\alpha \\ &+ |a_n|R^{n+1}(1 - \cos\alpha + \sin\alpha) + 2\sin\alpha \sum_{j=0}^{n-1} |a_j|R^{j+1}. \end{aligned}$$

Also, Gardner and Shields [5] used similar conditions on the real parts and then on both the real and imaginary parts of the complex polynomial to have the following results:

**Theorem E.** *Let* $p(z) = \sum_{j=0}^{n} a_j z^j$ *where* $Re\ a_j = \alpha_j$ *and* $Im\ a_j = \beta_j$ *for* $0 \le j \le n$. *Suppose that for some* $R > 0$ *and some* $0 \le k \le n$, *we have*

$$\begin{aligned} 0 \ne \alpha_0 \le R\alpha_1 \le R^2\alpha_2 \le \cdots \le R^{k-1}\alpha_{k-1} \le R^k\alpha_k \\ \ge R^{k+1}\alpha_{k+1} \ge \cdots \ge R^{n-1}\alpha_{n-1} \ge R^n\alpha_n. \end{aligned}$$

*Then, for* $0 < \delta < 1$ *the number of zeros of* $p(z)$ *in the disk* $|z| \le \delta R$ *is less than*

$$\frac{1}{\log 1/\delta} \log \frac{M}{|a_0|}$$

*where*

$$M = (|\alpha_0| - \alpha_0)R + 2\alpha_k R^{k+1} + (|\alpha_n| - \alpha_n)R^{n+1} + 2\sum_{j=0}^{n} |\beta_j| R^{j+1}.$$

**Theorem F.** *Let* $p(z) = \sum_{j=0}^{n} a_j z^j$ *where* $Re\ a_j = \alpha_j$ *and* $Im\ a_j = \beta_j$ *for* $0 \le j \le n$. *Suppose that for some* $R > 0$ *and some* $0 \le k \le n$, *we have*

$$\begin{aligned} 0 \ne \alpha_0 \le R\alpha_1 \le R^2\alpha_2 \le \cdots \le R^{k-1}\alpha_{k-1} \le R^k\alpha_k \\ \ge R^{k+1}\alpha_{k+1} \ge \cdots \ge R^{n-1}\alpha_{n-1} \ge R^n\alpha_n \end{aligned}$$

*and for some* $0 \leq l \leq n$ *we have*

$$\beta_0 \leq R\beta_1 \leq R^2\beta_2 \leq \cdots \leq R^{l-1}\beta_{l-1} \leq R^l\beta_l \\ \geq R^{l+1}\beta_{l+1} \geq \cdots \geq R^{n-1}\beta_{n-1} \geq R^n\beta_n.$$

*Then for* $0 < \delta < 1$ *the number of zeros of* $p(z)$ *in the disk* $|z| \leq \delta R$ *is less than*

$$\frac{1}{\log 1/\delta} \log \frac{M}{|a_0|}$$

*where*

$$M = (|\alpha_0| - \alpha_0)R + 2\alpha_k R^{k+1} + (|\alpha_n| - \alpha_n)R^{n+1} \\ + (|\beta_0| - \beta_0)R + 2\beta_l R^{l+1} + (|\beta_n| - \beta_n)R^{n+1}.$$

In this paper, we further weaken the hypotheses of the above results in order to obtain results that include several results in this area.

**Theorem 1.** *Let* $p(z) = \sum_{j=0}^{n} a_j z^j$ *where for some* $R > 0$, $0 \leq \mu < 1$, $0 < \rho \leq 1$ *and some* $0 \leq k \leq n$,

$$0 < \rho|a_0| \leq R|a_1| \leq R^2|a_2| \leq \cdots \leq R^{k-1}|a_{k-1}| \leq R^k|a_k| \\ \geq R^{k+1}|a_{k+1}| \geq \cdots \geq R^{n-1}|a_{n-1}| \geq (R-\mu)R^{n-1}|a_n|$$

*and* $|\arg a_j - \beta| \leq \alpha \leq \frac{\pi}{2}$ *for* $1 \leq j \leq n$ *and for some real* $\alpha$ *and* $\beta$. *Then for* $0 < \delta < 1$ *the number of zeros of* $p(z)$ *in the disk* $|z| \leq \delta R$ *is less than*

$$\frac{1}{\log 1/\delta} \log \frac{M}{|a_0|}$$

*where*

$$M = |a_0|R[\frac{1}{\rho} + \frac{\mu}{\rho|a_0|} - \cos\alpha - \sin\alpha] + 2|a_k|R^{k+1}\cos\alpha \\ + |a_n|R^{n+1}[1 + \frac{\mu}{|a_n|} - \cos\alpha + \sin\alpha] + 2\sin\alpha \sum_{j=0}^{n-1} |a_j|R^{j+1}.$$

Notice that when $R = 1$ in Theorem 1, we get the following:

**Corollary 1.** *Let* $p(z) = \sum_{j=0}^{n} a_j z^j$ *where for some* $0 \leq \mu < 1$, $0 < \rho \leq 1$ *and some* $0 \leq k \leq n$,

$$0 < \rho|a_0| \leq |a_1| \leq |a_2| \leq \cdots \leq |a_{k-1}| \leq |a_k| \geq |a_{k+1}| \geq \cdots \\ \geq |a_{n-1}| \geq (1-\mu)|a_n|$$

*and* $|\arg a_j - \beta| \leq \alpha \leq \frac{\pi}{2}$ *for* $1 \leq j \leq n$ *and for some real* $\alpha$ *and* $\beta$*. Then for* $0 < \delta < 1$ *the number of zeros of* $p(z)$ *in the disk* $|z| \leq \delta$ *is less than*

$$\frac{1}{\log 1/\delta} \log \frac{M}{|a_0|}$$

*where*

$$M = |a_0| \left( \frac{1}{\rho} + \frac{\mu}{\rho |a_0|} - \cos\alpha - \sin\alpha \right) + 2|a_k| \cos\alpha + |a_n| \left( 1 + \frac{\mu}{|a_n|} - \cos\alpha + \sin\alpha \right) + 2 \sin\alpha \sum_{j=0}^{n-1} |a_j|.$$

It is also important to see that when $\mu = 0$ and $\rho = 1$, we recapture the result of Theorem D of Gardner and Shields [5], which in turn, is a generalization of several other theorems.

**Theorem 2.** *Let* $p(z) = \sum_{j=0}^{n} a_j z^j$ *where* $Re\ a_j = \alpha_j$ *and* $Im\ a_j = \beta_j$ *for* $0 \leq j \leq n$*. Suppose that for some* $R > 0$*,* $0 \leq \mu < 1$*,* $0 < \rho \leq 1$ *and some* $0 \leq k \leq n$*, we have*

$$0 \neq \rho\alpha_0 \leq R\alpha_1 \leq R^2\alpha_2 \leq \cdots \leq R^{k-1}\alpha_{k-1} \leq R^k\alpha_k \geq R^{k+1}\alpha_{k+1} \geq \cdots \geq R^{n-1}\alpha_{n-1} \geq (R-\mu)R^{n-1}\alpha_n.$$

*Then for* $0 < \delta < 1$ *the number of zeros of* $p(z)$ *in the disk* $|z| \leq \delta R$ *is less than*

$$\frac{1}{\log 1/\delta} \log \frac{M}{|a_0|}$$

*where*

$$M = \left( \frac{1}{\rho} |\alpha_0 - \mu| - \alpha_0 \right) R + (|\alpha_n - \mu| - \alpha_n) R^{n+1} + \mu R(1 + R^n) + 2\alpha_k R^{k+1} + 2 \sum_{j=0}^{n} |\beta_j| R^{j+1}.$$

Observe that when $R = 1$ in Theorem 2, we have the following.

**Corollary 2.** *Let* $p(z) = \sum_{j=0}^{n} a_j z^j$ *where* $Re\ a_j = \alpha_j$ *and* $Im\ a_j = \beta_j$ *for* $0 \leq j \leq n$*. Suppose that for some* $0 \leq \mu < R$*,* $0 < \rho \leq 1$ *and some* $0 \leq k \leq n$*, we have*

$$0 \neq \rho\alpha_0 \leq \alpha_1 \leq \alpha_2 \leq \cdots \leq \alpha_{k-1} \leq \alpha_k \geq \alpha_{k+1} \geq \cdots \geq \alpha_{n-1} \geq (1-\mu)\alpha_n.$$

*Then for* $0 < \delta < 1$ *the number of zeros of* $p(z)$ *in the disk* $|z| \leq \delta$ *is less than*

$$\frac{1}{\log 1/\delta} \log \frac{M}{|a_0|}$$

*where*

$$M = \left(\frac{1}{\rho}|\alpha_0 - \mu| - \alpha_0\right) + (|\alpha_n - \mu| - \alpha_n) + 2(\mu + \alpha_k) + 2\sum_{j=0}^{n} |\beta_j|.$$

**Theorem 3.** *Let* $p(z) = \sum_{j=0}^{n} a_j z^j$ *where* $Re\ a_j = \alpha_j$ *and* $Im\ a_j = \beta_j$ *for* $0 \leq j \leq n$. *Suppose that for some* $R > 0$, $0 < \rho_1 \leq 1$, $0 < \rho_2 \leq 1$ $0 \leq \mu < 1$, $0 \leq \lambda < 1$ *and some* $0 \leq k \leq n$, *we have*

$$\begin{aligned} 0 \neq \rho_1\alpha_0 &\leq R\alpha_1 \leq R^2\alpha_2 \leq \cdots \leq R^{k-1}\alpha_{k-1} \leq R^k\alpha_k \\ &\geq R^{k+1}\alpha_{k+1} \geq \cdots \geq R^{n-1}\alpha_{n-1} \geq (R-\mu)R^{n-1}\alpha_n \end{aligned}$$

*and for some* $0 \leq l \leq n$ *we have*

$$\begin{aligned} \rho_2\beta_0 &\leq R\beta_1 \leq R^2\beta_2 \leq \cdots \leq R^{l-1}\beta_{l-1} \leq R^l\beta_l \\ &\geq R^{l+1}\beta_{l+1} \geq \cdots \geq R^{n-1}\beta_{n-1} \geq (R-\lambda)R^{n-1}\beta_n. \end{aligned}$$

*Then for* $0 < \delta < 1$ *the number of zeros of* $p(z)$ *in the disk* $|z| \leq \delta R$ *is less than*

$$\frac{1}{\log 1/\delta} \log \frac{M}{|a_0|}$$

*where*

$$\begin{aligned} M &= (\frac{1}{\rho_1}|\alpha_0 - \mu| - \alpha_0)R + (\frac{1}{\rho_2}|\beta_0 - \lambda| - \beta_0)R \\ &+ \ (\mu + \lambda)R(1 + R^n) + 2\alpha_k R^{k+1} + (|\alpha_n - \mu| - \alpha_n)R^{n+1} \\ &+ \ (|\beta_n - \lambda| - \beta_n)R^{n+1} + 2\beta_l R^{l+1}. \end{aligned}$$

Notice that if we set $R = 1$, $\lambda = \mu$, $\rho_1 = \rho_2 = \rho$ and $l = k$, we obtain the following corollary:

**Corollary 3.** *Let* $p(z) = \sum_{j=0}^{n} a_j z^j$ *where* $Re\ a_j = \alpha_j$ *and* $Im\ a_j = \beta_j$ *for* $0 \leq j \leq n$. *Suppose that for some* $0 < \rho \leq 1$, $0 \leq \mu < 1$, *and some* $0 \leq k \leq n$, *we have*

$$0 \neq \rho\alpha_0 \leq \alpha_1 \leq \alpha_2 \leq \cdots \leq \alpha_{k-1} \leq \alpha_k \geq \alpha_{k+1} \geq \cdots \geq \alpha_{n-1} \geq (1-\mu)\alpha_n$$

*and*

$$\rho\beta_0 \leq \beta_1 \leq \beta_2 \leq \cdots \leq \beta_{k-1} \leq \beta_k \geq \beta_{k+1} \geq \cdots \geq \beta_{n-1} \geq (1-\mu)\beta_n.$$

*Then for* $0 < \delta < 1$ *the number of zeros of* $p(z)$ *in the disk* $|z| \leq \delta R$ *is less than*

$$\frac{1}{\log 1/\delta} \log \frac{M}{|a_0|}$$

*where*

$$\begin{aligned} M &= (\frac{1}{\rho}|\alpha_0 - \mu| - \alpha_0) + (\frac{1}{\rho}|\beta_0 - \mu| - \beta_0) + 4\mu \\ &+ \ 2(\alpha_k + \beta_k) + (|\alpha_n - \mu| - \alpha_n) + (|\beta_n - \mu| - \beta_n) \end{aligned}$$

**Example.** Consider the polynomial $p(z) = (z + 1.1)(z^2 + 0.1) = 0.11 + 0.1z + 1.1z^2 + z^3$. We notice that none of the restrictions of the earlier existing Theorems A through F can count the number of zeros, given the coefficients of $p$. However, using corollary 2 with suitable $\rho$ and $\mu$, say, $\rho = 10/11$ and $\mu = 0$, we find out that the number of zeros in the disk $|z| \leq 32/100$ is less than 2.634. However, it is easy to see that exactly two roots, which are $i/\sqrt{10}$ and $-i/\sqrt{10}$, lie in the given disk. Thus, it agrees with our Theorem.

## 2. Proofs of the main theorems

First, we consider the following lemma which is due to Govil and Rahman [9].

**Lemma 1.** *Let* $z, z' \in \mathbb{C}$ *with* $|z| \geq |z'|$. *Suppose* $|\arg z^\star - \beta| \leq \alpha \leq \frac{\pi}{2}$ *for* $z^\star \in \{z, z'\}$ *and for some real* $\alpha$ *and* $\beta$. *Then*

$$|z - z'| \leq (|z| - |z'|) \cos\alpha + (|z| + |z'|) \sin\alpha.$$

**Proof of Theorem 1.** Consider

$$\begin{aligned} F(z) = (R - z)p(z) = (R - z) \sum_{j=0}^{n} a_j z^j &= \sum_{j=0}^{n} (a_j R z^j - a_j z^{j+1}) \\ = a_0 R + \sum_{j=1}^{n} R a_j z^j - \sum_{j=1}^{n} a_{j-1} z^j - a_n z^{n+1} \\ = a_0 R + \sum_{j=1}^{n} (a_j R - a_{j-1}) z^j - a_n z^{n+1}. \end{aligned}$$

For $|z| = R$ we have

$$\begin{aligned} |F(z)| &\leq |a_0| R + \sum_{j=1}^{n} |a_j R - a_{j-1}| R^j + |a_n| R^{n+1} \\ &= |a_0| R + \sum_{j=1}^{k} |a_j R - a_{j-1}| R^j + \sum_{j=k+1}^{n} |a_{j-1} - a_j R| R^j + |a_n| R^{n+1}. \end{aligned}$$

Applying Lemma 1 with $z = a_jR$, $z' = a_{j-1}$ when $1 \le j \le k$ and $z = a_{j-1}$, $z' = a_jR$ when $k+1 \le j \le n$, we have

$$\begin{aligned}
|F(z)| \le{}& |a_0|R + \sum_{j=1}^{k}[(|a_j|R - |a_{j-1}|)\cos\alpha + (|a_{j-1}| + |a_j|R)\sin\alpha]R^j \\
&+ \sum_{j=k+1}^{n}\left[(|a_{j-1}| - |a_j|R)\cos\alpha + (|a_j|R + |a_{j-1}|)\sin\alpha\right]R^j \\
&+ |a_n|R^{n+1} \\
={}& |a_0|R + \sum_{j=1}^{k}|a_j|R^{j+1}\cos\alpha - \sum_{j=1}^{k}|a_{j-1}|R^j\cos\alpha + \sum_{j=1}^{k}|a_{j-1}|R^j\sin\alpha \\
&+ \sum_{j=1}^{k}|a_j|R^{j+1}\sin\alpha + \sum_{j=k+1}^{n}|a_{j-1}|R^j\cos\alpha - \sum_{j=k+1}^{n}|a_j|R^{j+1}\cos\alpha \\
&+ \sum_{j=k+1}^{n}|a_j|R^{j+1}\sin\alpha + \sum_{j=k+1}^{n}|a_{j-1}|R^j\sin\alpha + |a_n|R^{n+1} \\
\le{}& \frac{1}{\rho}|a_0|R + |a_k|R^{k+1}\cos\alpha + \sum_{j=1}^{k-1}|a_j|R^{j+1}\cos\alpha - |a_0|R\cos\alpha \\
&- \sum_{j=1}^{k-1}|a_j|R^{j+1}\cos\alpha + |a_0|R\sin\alpha + \sum_{j=1}^{k-1}|a_j|R^{j+1}\sin\alpha \\
&+ |a_k|R^{k+1}\sin\alpha + \sum_{j=1}^{k-1}|a_j|R^{j+1}\sin\alpha + |a_k|R^{k+1}\cos\alpha \\
&+ \sum_{j=k+1}^{n-1}|a_j|R^{j+1}\cos\alpha - |a_n|R^{n+1}\cos\alpha - \sum_{j=k+1}^{n-1}|a_j|R^{j+1}\cos\alpha \\
&+ |a_n|R^{n+1}\sin\alpha + \sum_{j=k+1}^{n-1}|a_j|R^{j+1}\sin\alpha + |a_k|R^{k+1}\sin\alpha \\
&+ \sum_{j=k+1}^{n-1}|a_j|R^{j+1}\sin\alpha + |a_n|R^{n+1} \\
\le{}& \frac{1}{\rho}(|a_0| + \mu)R + |a_k|R^{k+1}\cos\alpha - |a_0|R\cos\alpha + |a_0|R\sin\alpha \\
&+ |a_k|R^{k+1}\sin\alpha + 2\sum_{j=1}^{k-1}|a_j|R^{j+1}\sin\alpha + |a_k|R^{k+1}\cos\alpha \\
&- |a_n|R^{n+1}\cos\alpha + |a_n|R^{n+1}\sin\alpha
\end{aligned}$$

$$
\begin{aligned}
&+ 2 \sum_{j=k+1}^{n-1} |a_j| R^{j+1} \sin\alpha + |a_k| R^{k+1} \sin\alpha + (|a_n| + \mu) R^{n+1} \\
&= \frac{1}{\rho}|a_0|R + \frac{\mu}{\rho}R - |a_0|R\cos\alpha + |a_0|R\sin\alpha + 2|a_k|R^{k+1}\cos\alpha \\
&\quad + 2|a_k|R^{k+1}\sin\alpha + |a_n|R^{n+1} + \mu R^{n+1} - |a_n|R^{n+1}\cos\alpha \\
&\quad + |a_n|R^{n+1}\sin\alpha + 2\left(\sum_{j=1}^{k-1}|a_j|R^{j+1}\sin\alpha + \sum_{j=k+1}^{n-1}|a_j|R^{j+1}\sin\alpha\right) \\
&= |a_0|R\left(\frac{1}{\rho} + \frac{\mu}{\rho|a_0|} - \cos\alpha + \sin\alpha\right) + 2|a_k|R^{k+1}\cos\alpha \\
&\quad + 2|a_k|R^{k+1}\sin\alpha + |a_n|R^{n+1}\left(1 + \frac{\mu}{|a_n|} - \cos\alpha + \sin\alpha\right) \\
&\quad + 2\sum_{j=0}^{n-1}|a_j|R^{j+1}\sin\alpha - 2|a_k|R^{k+1}\sin\alpha - 2|a_0|R\sin\alpha \\
&= |a_0|R\left(\frac{1}{\rho} + \frac{\mu}{\rho|a_0|} - \cos\alpha - \sin\alpha\right) + 2|a_k|R^{k+1}\cos\alpha \\
&\quad + |a_n|R^{n+1}\left(1 + \frac{\mu}{|a_n|} - \cos\alpha + \sin\alpha\right) + 2\sum_{j=0}^{n-1}|a_j|R^{j+1}\sin\alpha = M.
\end{aligned}
$$

Now, $F(z)$ is analytic in $|z| \le R$ and $|F(z)| \le M$ for $|z| = R$. So by Theorem A and the Maximum Modulus Theorem, the number of zeros of $F(z)$ (and hence of $p(z)$) in $|z| \le \delta R$ is less than or equal to

$$\frac{1}{\log 1/\delta} \log \frac{M}{|a_0|}.$$

This completes the proof. ■

**Proof of Theorem 2.** As in the proof of Theorem 1,

$$F(z) = (R - z)P(z) = a_0 R + \sum_{j=1}^{n}(a_j R - a_{j-1})z^j - a_n z^{n+1}.$$

Notice that $a_j = \alpha_j + i\beta_j$, thus

$$
\begin{aligned}
F(z) \;&=\; (\alpha_0 + i\beta_0)R + \sum_{j=1}^{n}((\alpha_j + i\beta_j)R \\
&\qquad - (\alpha_{j-1} + i\beta_{j-1}))z^j - (\alpha_n + i\beta_n)z^{n+1}
\end{aligned}
$$

$$= (\alpha_0 + i\beta_0)R + \sum_{j=1}^{n}(\alpha_j R - \alpha_{j-1})z^j$$

$$+ i\sum_{j=1}^{n}(\beta_j R - \beta_{j-1})z^j - (\alpha_n + i\beta_n)z^{n+1}.$$

For $|z| = R$, we have

$$\begin{aligned}
|F(z)| \leq& (|\alpha_0| + |\beta_0|)R + \sum_{j=1}^{n}|\alpha_j R - \alpha_{j-1}|R^j \\
&+ \sum_{j=1}^{n}(|\beta_j|R - |\beta_{j-1}|)R^j + (|\alpha_n| + |\beta_n|)R^{n+1} \\
=&(|\alpha_0| + |\beta_0|)R + \sum_{j=1}^{k}(\alpha_j R - \alpha_{j-1})R^j \\
&+ \sum_{j=k+1}^{n}(\alpha_{j-1} - \alpha_j R)R^j + \sum_{j=1}^{n-1}(|\beta_j|R^{j+1} + |\beta_n|R^{n+1} \\
&+ |\beta_0|R + \sum_{j=1}^{n-1}|\beta_j|R^{j+1} + (|\alpha_n| + |\beta_n|)R^{n+1} \\
\leq& (|\alpha_0 - \mu| + \mu + |\beta_0|)R + \sum_{j=1}^{k-1}\alpha_j R^{j+1} + \alpha_k R^{k+1} \\
&- \alpha_0 R - \sum_{j=1}^{k-1}\alpha_j R^{j+1} + \sum_{j=k+1}^{n-1}\alpha_j R^{j+1} \\
&+ \alpha_k R^{k+1} - \sum_{j=k+1}^{n-1}\alpha_j R^{j+1} - \alpha_n R^{n+1} + 2\sum_{j=1}^{n-1}|\beta_j|R^{j+1} \\
&+ |\beta_0|R + |\beta_n|R^{n+1} + (|\alpha_n - \mu| + \mu + |\beta_n|)R^{n+1} \\
=& (|\alpha_0 - \mu| + |\beta_0|)R + \mu R(1 + R^n) + 2\alpha_k R^{k+1} - \alpha_0 R \\
&- \alpha_n R^{n+1} + (|\alpha_n - \mu| + |\beta_n|)R^{n+1} + |\beta_0|R \\
&+ |\beta_n|R^{n+1} + 2\sum_{j=1}^{n-1}|\beta_j|R^{j+1} \\
\leq& (\frac{1}{\rho}|\alpha_0 - \mu| + |\beta_0|)R + \mu R(1 + R^n) + 2\alpha_k R^{k+1} - \alpha_0 R - \alpha_n R^{n+1} \\
&+ (|\alpha_n - \mu| + |\beta_n|)R^{n+1} + |\beta_0|R + |\beta_n|R^{n+1} + 2\sum_{j=1}^{n-1}|\beta_j|R^{j+1}
\end{aligned}$$

$$= \left(\frac{1}{\rho}|\alpha_0 - \mu| - \alpha_0\right) R + (|\alpha_n - \mu| - \alpha_n)R^{n+1} + \mu R(1 + R^n)$$

$$+ 2\alpha_k R^{k+1} + 2|\beta_0|R + 2|\beta_n|R^{n+1} + 2\sum_{j=1}^{n-1} |\beta_j|R^{j+1}$$

$$= \left(\frac{1}{\rho}|\alpha_0 - \mu| - \alpha_0\right) R + (|\alpha_n - \mu| - \alpha_n)R^{n+1} + \mu R(1 + R^n)$$

$$+ 2\alpha_k R^{k+1} + 2\sum_{j=0}^{n} |\beta_j|R^{j+1} = M.$$

The result now follows as in the proof of Theorem 1. ■

**Proof of Theorem 3.** As in the proof of Theorem 2,

$$F(z) = (\alpha_0 + i\beta_0)R + \sum_{j=1}^{n}(\alpha_j R - \alpha_{j-1})z^j$$

$$+ i\sum_{j=1}^{n}(\beta_j R - \beta_{j-1})z^j - (\alpha_n + i\beta_n)z^{n+1}.$$

For $|z| = R$ we have

$$|F(z)| \leq (|\alpha_0| + |\beta_0|)R + \sum_{j=1}^{n} |\alpha_j R - \alpha_{j-1}|R^j$$

$$+ \sum_{j=1}^{n} |\beta_j R - \beta_{j-1}|R^j + (|\alpha_n| + |\beta_n|)R^{n+1}$$

$$= (|\alpha_0| + |\beta_0|)R + \sum_{j=1}^{k} |\alpha_j R - \alpha_{j-1}|R^j$$

$$+ \sum_{j=k+1}^{n} |\alpha_j R - \alpha_{j-1}|R^j + \sum_{j=1}^{l} |\beta_j R - \beta_{j-1}|R^j$$

$$+ \sum_{j=l+1}^{n} |\beta_j R - \beta_{j-1}|R^j + (|\alpha_n| + |\beta_n|)R^{n+1}$$

$$= (|\alpha_0| + |\beta_0|)R + \sum_{j=1}^{k}(\alpha_j R - \alpha_{j-1})R^j$$

$$+ \sum_{j=k+1}^{n} (\alpha_{j-1} - \alpha_j R)R^j + \sum_{j=1}^{l}(\beta_j R - \beta_{j-1})R^j$$

$$
\begin{aligned}
&+\sum_{j=l+1}^{n}(\beta_{j-1}-\beta_j R)R^j+(|\alpha_n|+|\beta_n|)R^{n+1}\\
&=(|\alpha_0|+|\beta_0|)R+\sum_{j=1}^{k-1}\alpha_j R^{j+1}+\alpha_k R^{k+1}-\alpha_0 R-\sum_{j=1}^{k-1}\alpha_j R^{j+1}\\
&\quad+\alpha_k R^{k+1}+\sum_{j=k+1}^{n-1}\alpha_j R^{j+1}-\sum_{j=k+1}^{n-1}\alpha_j R^{j+1}-\alpha_n R^{n+1}\\
&\quad+\sum_{j=1}^{l-1}\beta_j R^{j+1}+\beta_l R^{l+1}-\beta_0 R-\sum_{j=1}^{l-1}\beta_j R^{j+1}+\beta_l R^{l+1}\\
&\quad+\sum_{j=l+1}^{n-1}\beta_j R^{j+1}-\sum_{j=l+1}^{n-1}\beta_j R^{j+1}-\beta_n R^{n+1}+(|\alpha_n|+|\beta_n|)R^{n+1}\\
&=(|\alpha_0|+|\beta_0|)R-\alpha_0 R+2\alpha_k R^{k+1}-\alpha_n R^{n+1}+2\beta_l R^{l+1}\\
&\quad-\beta_0 R-\beta_n R^{n+1}+(|\alpha_n|+|\beta_n|)R^{n+1}\\
&\leq(|\alpha_0-\mu|+\mu+|\beta_0-\lambda|+\lambda)R-\alpha_0 R+2\alpha_k R^{k+1}+2\beta_l R^{l+1}\\
&\quad-\beta_0 R-\alpha_n R^{n+1}-\beta_n R^{n+1}+(|\alpha_n-\mu|+\mu+|\beta_n-\lambda|+\lambda)R^{n+1}\\
&=(|\alpha_0-\mu|+|\beta_0-\lambda|)R+(\mu+\lambda)R-\alpha_0 R-\beta_0 R+2\alpha_k R^{k+1}+2\beta_l R^{l+1}\\
&\quad+(|\alpha_n-\mu|+|\beta_n-\lambda|)R^{n+1}+(\mu+\lambda)R^{n+1}-\alpha_n R^{n+1}-\beta_n R^{n+1}\\
&\leq\frac{1}{\rho_1}|\alpha_0-\mu|R+\frac{1}{\rho_2}|\beta_0-\lambda|R+(\mu+\lambda)R(1+R^n)-\alpha_0 R-\beta_0 R\\
&\quad+2\alpha_k R^{k+1}+2\beta_l R^{l+1}+(|\alpha_n-\mu|+|\beta_n-\lambda|)R^{n+1}-\alpha_n R^{n+1}-\beta_n R^{n+1}\\
&=(\frac{1}{\rho_1}|\alpha_0-\mu|-\alpha_0)R+(\frac{1}{\rho_2}|\beta_0-\lambda|-\beta_0)R+(\mu+\lambda)R(1+R^n)\\
&\quad+2\alpha_k R^{k+1}+(|\alpha_n-\mu|-\alpha_n)R^{n+1}+(|\beta_n-\lambda|-\beta_n)R^{n+1}+2\beta_l R^{l+1}\\
&=M.
\end{aligned}
$$

The result now follows as in the proof of Theorem 1. ■

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