DE GRUYTER OPEN
FASCICULI MATHEMATICI
Nr 57 2016
DOI:10.1515/fascmath-2016-0018

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ON WIJSMAN $\mathcal{I}_2$-LACUNARY STATISTICAL CONVERGENCE FOR DOUBLE SET SEQUENCES

Abstract. The aim of present work is to present some inclusion relations between the concepts of Wijsman $\mathcal{I}_2-$lacunary statistical convergence and Wijsman strongly $\mathcal{I}_2-$lacunary convergence for double sequences of sets. Also we study the concepts of Wijsman $\mathcal{I}_2-$statistical convergence, Wijsman $\mathcal{I}_2-$ lacunary statistical convergence double sequences of sets and investigate the relationship among them.

Key words: $\mathcal{I}-$convergence, lacunary, double sequences.

AMS Mathematics Subject Classification: 40A05, 40A35.

1. Introduction

Hill [9] was the first who applied methods of functional analysis to double sequence. A lot of useful developments of double sequences in summability methods can be found in Limayea and Zeltser [15], Altay and Başar [1].

Convergence of a sequence of real numbers has been extended to statistical convergence independently by Fast [7] and Schoenberg [24]. This notion was extended to the double sequences by Mursaleen and Edely [16].

Lacunary statistical convergence was defined by Fridy and Orhan [8]. Also, Fridy and Orhan gave the relationships between the lacunary statistical convergence and the Cesàro summability. This notion was extended to the double sequences by Savaş, Patterson [23].

P. Kostyrko et al. [13] introduced the concept of $\mathcal{I}-$convergence of sequences in a metric space and studied some properties of this convergence. The notion of lacunary ideal convergence of real sequences was introduced in [26, 27]. Das, Kostyrko, Wilczyski and Malik [6] defined the notion of $\mathcal{I}-$convergence of double sequences in a metric space and studied some properties of this convergence.

Convergence of numbers has been extended by several authors to convergence of sequences of sets (see Baronti and Papini [2]; Beer [3], [4]; Nuray and Rhoades [17]; Wijsman [32], [33]; Nuray and Kişi [10], [11], [12]). Nuray and Rhoades [17] introduced the notion of statistical convergence of sequences of sets. Ulusu and Nuray [28] defined the Wijsman lacunary statistical convergence of sequence of sets, and considered its relation with Wijsman statistical convergence, which was defined by Nuray and Rhoades [17]. Ulusu and Nuray [29] introduced the notion of Wijsman strongly lacunary summability for sequences of sets and discussed its relation with Wijsman strongly Cesàro summability.

Das et al. [5] introduced new notions of convergence, namely $\mathcal{I}-$statistical convergence and $\mathcal{I}-$lacunary statistical convergence by using ideal approach. $\mathcal{I}-$ convergence of real sequences was extended to the sequences of sets by Kişi and Nuray [10]. Kişi et al. [11] defined Wijsman $\mathcal{I}-$statistical convergence and Wijsman $\mathcal{I}-$ lacunary statistical convergence of sequences of sets. Sever et al. [25] investigated the ideas of Wijsman strongly $\mathcal{I}-$lacunary convergence, Wijsman strongly $\mathcal{I}^*-$lacunary convergence and Wijsman strongly $\mathcal{I}-$lacunary Cauchy sequences of sets. The notions of convergence, statistical convergence and ideal convergence of double sequences of sets were studied by Nuray et. al $[18, 19, 20, 21, 22]$.

Nuray et al. [20] studied Wijsman Cesàro summability and Wijsman lacunary convergence of double sequences of sets.

In this paper, we investigate the relationship between Wijsman $\mathcal{I}_2-$statistical convergence, Wijsman $\mathcal{I}_2-$lacunary statistical convergence and Wijsman strongly $\mathcal{I}_2$lacunary convergence of double sequences of sets.

2. Definitions and notations

In this section, we recall some definitions and notations, which form the base for the present study.

Definition 1 ([13]). *A family of sets $\mathcal{I} \subseteq 2^{\mathbb{N}}$ is called an ideal if and only if*

(i) $\emptyset \in \mathcal{I}$,
(ii) for each $A, B \in \mathcal{I}$ we have $A \cup B \in \mathcal{I}$,
(iii) for each $A \in \mathcal{I}$ and each $B \subseteq A$ we have $B \in \mathcal{I}$.

Definition 2 ([13]). *A family of sets $\mathcal{F} \subseteq 2^{\mathbb{N}}$ is a filter in $\mathbb{N}$ if and only if*
(i) $\emptyset \notin \mathcal{F}$,
(ii) *for each $A, B \in \mathcal{F}$ we have $A \cap B \in \mathcal{F}$,*
(iii) *for each $A \in \mathcal{F}$ and each $B \supseteq A$ we have $B \in \mathcal{F}$.*

If $\mathcal{I}$ is proper ideal of $\mathbb{N}$ (i.e., $\mathbb{N} \notin \mathcal{I}$), then the family of sets

$$\mathcal{F}(\mathcal{I}) = \{M \subset \mathbb{N} : \exists A \in \mathcal{I} : M = \mathbb{N} \setminus A\}$$

is a filter of $\mathbb{N}$ and it is called the filter associated with the ideal $\mathcal{I}$.

An ideal is called non-trivial if $\mathbb{N} \notin \mathcal{I}$ and non-trivial ideal is called admissible if $\{n\} \in \mathcal{I}$ for each $n \in \mathbb{N}$.

Definition 3 ([2]). *Let (X, d) be a metric space. For any non-empty closed subsets $A, A_k \subseteq X$, we say that the sequence $\{A_k\}$ is Wijsman convergent to A if*

$$\lim_{k\to\infty} d(x, A_k) = d(x, A)$$

for each $x \in X$. In this case we write $W - \lim_{k\to\infty} A_k = A$.

As an example, consider the following sequence of circles in the (x, y)-plane: $A_k = \{(x, y) : x^2 + y^2 + 2kx = 0\}$. As $k \to \infty$, the sequence A_k is Wijsman convergent to the y-axis $A = \{(x, y) : x = 0\}$.

Definition 4 ([17]). *Let (X, d) be a metric space. For any non-empty closed subsets $A, A_k \subseteq X$, we say that the sequence $\{A_k\}$ is Wijsman statistically convergent to A if $\{d(x, A_k)\}$ is statistically convergent to $d(x, A)$; i.e., for each $\varepsilon > 0$ and for each $x \in X$,*

$$\lim_{n\to\infty} \frac{1}{n} |\{k \leq n : |d(x, A_k) - d(x, A)| \geq \varepsilon\}| = 0,$$

i.e.,

$$|d(x, A_k) - d(x, A)| < \varepsilon \quad a.a.k. \, .$$

In this case we write $st - \lim_W A_k = A$.

Also the concept of bounded sequence for sequences of sets was given by Nuray and Rhoades [17]. Let (X, d) be a metric space. For any non-empty closed subsets A_k of X, we say that the sequence $\{A_k\}$ is bounded if $\sup_k d(x, A_k) < \infty$ for each $x \in X$.

By a lacunary sequence we mean an increasing integer sequence $\theta = \{k_r\}$ such that $k_0 = 0$ and $h_r = k_r - k_{r-1} \to \infty$, $r \to \infty$. Throughout this paper the intervals determined by θ will be denoted by $I_r = (k_{r-1}, k_r]$, and ratio $\frac{k_r}{k_{r-1}}$ will be abbreviated by q_r.

Definition 5 ([28]). *Let (X, d) be a metric space and $\theta = \{k_r\}$ be a lacunary sequence. For any non-empty closed subsets $A, A_k \subseteq X$, we say that the sequence $\{A_k\}$ is Wijsman lacunary statistical convergent to A if $\{d(x, A_k)\}$ is lacunary statistically convergent to $d(x, A)$; i.e., for $\varepsilon > 0$ and for each $x \in X$,*

$$\lim_{r\to\infty} \frac{1}{h_r} |k \in I_r : |d(x, A_k) - d(x, A)| \geq \varepsilon| = 0.$$

In this case we write $S_\theta - \lim_W A_k = A$ or $A_k \to A(WS_\theta)$.

Definition 6 ([29])**.** *Let* (X,d) *be a metric space and* $\theta=\{k_r\}$ *be a lacunary sequence. For any non-empty closed subsets* A, $A_k\subseteq X$, *we say that* $\{A_k\}$ *is Wijsman lacunary summable to* A *if* $\{d(x,A_k)\}$ *is lacunary summable to* $d(x,A)$; *i.e., for each* $x\in X$,

$$\lim_{r\to\infty}\frac{1}{h_r}\sum_{I_r}d(x,A_k)=d(x,A).$$

In this case we write $A_k\to A(WN_\theta)$.

Definition 7 ([29])**.** *Let* (X,d) *be a metric space and* $\theta=\{k_r\}$ *be a lacunary sequence. For any non-empty closed subsets* A, $A_k\subseteq X$, *we say that* $\{A_k\}$ *is Wijsman strongly lacunary summable to* A *if* $\{d(x,A_k)\}$ *is strongly lacunary summable to* $d(x,A)$; *i.e., for each* $x\in X$,

$$\lim_{r\to\infty}\frac{1}{h_r}\sum_{I_r}|d(x,A_k)-d(x,A)|=0.$$

In this case we write $A_k\to A([WN_\theta])$.

Definition 8 ([10])**.** *Let* (X,d) *be a metric space and* $\mathcal{I}\subseteq 2^{\mathbb{N}}$ *be a proper ideal in* $\mathbb{N}$. *For any non-empty closed subsets* $A,A_k\subset X$, *we say that the sequence* $\{A_k\}$ *is Wijsman* $\mathcal{I}-$*convergent to* A, *if for each* $\varepsilon>0$ *and for each* $x\in X$, *the set,*

$$A\left(x,\varepsilon\right)=\left\{k\in\mathbb{N}:\left|d\left(x,A_k\right)-d\left(x,A\right)\right|\geq\varepsilon\right\}$$

belongs to $\mathcal{I}$. *In this case we write* $\mathcal{I}_W-\lim A_k=A$ *or* $A_k\to A(\mathcal{I}_W)$, *and the set of Wijsman* $\mathcal{I}-$*convergent sequences of sets will be denoted by*

$$\mathcal{I}_W=\left\{\{A_k\}:\left\{k\in\mathbb{N}:\left|d\left(x,A_k\right)-d\left(x,A\right)\right|\geq\varepsilon\right\}\in\mathcal{I}\right\}.$$

A double sequence $x=(x_{k,l})$ has a Pringsheim limit L (denoted by $P-\lim x=L$) provided that for given $\varepsilon>0$, there exists $n\in\mathbb{N}$ such that $|x_{k,l}-L|<\varepsilon$, whenever $k,l>n$. We describe such a double sequence $x=(x_{k,l})$ more briefly as "P-convergent".

The double sequence $(x_{k,l})$ is bounded if there exists a positive integer M such that $|x_{k,l}|<M$ for all k and l. We denote the space of all bounded double sequences by l^2_∞.

Throught the paper, A, $A_{k,l}$ be any non-empty closed subsets of X.

Definition 9 ([18])**.** *The double sequence* $\{A_{k,l}\}$ *is Wijsman convergent to* A, *if for each* $x\in X$

$$P-\lim_{k,l\to\infty}d\left(x,A_{k,l}\right)=d\left(x,A\right)\quad or\quad\lim_{k,l\to\infty}d\left(x,A_{k,l}\right)=d\left(x,A\right).$$

In this case we write $W_2 - \lim A_{k,l} = A$.

Definition 10 ([18])**.** *The double sequence $\{A_{k,l}\}$ is Wijsman statistically convergent to A, if for each $x \in X$ and for every $\varepsilon > 0$,*

$$\lim_{m,n\to\infty} \frac{1}{mn} |\{ \ k \leq m, l \leq n : |d(x, A_{k,l}) - d(x, A) \ \ | \geq \varepsilon \ \}| = 0,$$

that is,

$$|d(x, A_{k,l}) - d(x, A) \ \ | < \varepsilon, \ a.a. \ \ (k, l).$$

In this case we write $st_2 - \lim_W A_{k,l} = A$.

The set of Wijsman statistically convergent double sequences will be denoted by

$$W_2S := \left\{ \{A_{k,l}\} : st_2 - \lim_W A_{k,l} = A \right\}.$$

By $\mathcal{I}_2$ we will denote the admissible ideal of $\mathbb{N} \times \mathbb{N}$ and by $\theta_{r,s} = \{(k_r, l_s)\}$ a double lacunary sequence of positive real numbers, respectively, unless otherwise stated.

A double sequence $\overline{\theta} = \theta_{r,s} = \{(k_r, l_s)\}$ is called double lacunary sequence if there exist two increasing sequences of integers (k_r) and (l_s) such that

$$k_0 = 0, \ \ h_r = k_r - k_{r-1} \to \infty, \ \ r \to \infty$$

and

$$l_0 = 0, \ \ \overline{h}_s = l_s - l_{s-1} \to \infty, \ \ s \to \infty.$$

We will use the following notation $k_{r,s} := k_r l_s$, $h_{r,s} := h_r \overline{h}_s$ and $\theta_{r,s}$ is determined by

$$J_{r,s} := \{(k, l) : k_{r-1} < k \leq k_r \text{ and } l_{s-1} < l \leq l_s\},$$

$$q_r := \frac{k_r}{k_{r-1}}, \ \ \overline{q}_s := \frac{l_s}{l_{s-1}} \ \text{ and } \ q_{r,s} := q_r \overline{q}_s.$$

For details on double lacunary sequence we refer to [22].

Definition 11 ([22])**.** *The double sequence $\{A_{k,l}\}$ is Wijsman lacunary statistically convergent to A, if for each $x \in X$ and for every $\varepsilon > 0$,*

$$\lim_{r,s\to\infty} \frac{1}{h_{r,s}} |\{(k, l) \in J_{r,s} : |d(x, A_{k,l}) - d(x, A)| \geq \varepsilon\} = 0.$$

In this case we write $st_2 - \lim_{W_\theta} A_{k,l} = A$.

Definition 12 ([18])**.** *Let $\mathcal{I}_2$ be an admissible ideal of $\mathbb{N}\times\mathbb{N}$. We say that the double sequence $\{A_{k,l}\}$ is Wijsman $\mathcal{I}_2-$convergent to A, if for each $\varepsilon>0$ and for each $x\in X$, the set,*

$$A(x,\varepsilon)=\{(k,l)\in\mathbb{N}\times\mathbb{N}:|d(x,A_{k,l})-d(x,A)|\geq\varepsilon\}$$

belongs to $\mathcal{I}_2$. In this case we write $\mathcal{I}_2-\lim A_{k,l}=A$ or $A_{k,l}\to A(\mathcal{I}_2)$.

Definition 13 ([18])**.** *Let $\mathcal{I}_2$ be an admissible ideal of $\mathbb{N}\times\mathbb{N}$. We say that the double sequence $\{A_{k,l}\}$ is Wijsman $\mathcal{I}_2-$statistically convergent to A or $S(\mathcal{I}_2)$-convergent to A if for each $\varepsilon>0$, for each $x\in X$ and $\delta>0$,*

$$\left\{(m,n)\in\mathbb{N}\times\mathbb{N}:\frac{1}{mn}|\{k\leq m,l\leq n:|d(x,A_{k,l})-d(x,A)|\geq\varepsilon\}|\geq\delta\right\}\in\mathcal{I}_2.$$

In this case, we write $A_{k,l}\to A(S(\mathcal{I}_2))$. The class of all Wijsman $\mathcal{I}_2-$statistically convergent double set sequences will be denoted by $S(\mathcal{I}_2)$.

Definition 14 ([31])**.** *Let $\theta_{r,s}=(k_{r,s})$ be a double lacunary sequence and $\mathcal{I}_2$ be an admissible ideal of $\mathbb{N}\times\mathbb{N}$. The double set sequence $\{A_{k,l}\}$ is said to be Wijsman $\mathcal{I}_2-$lacunary statistically convergent to A or $S_{\theta_{r,s}}(\mathcal{I}_2)$-convergent to A if for each $\varepsilon>0$, for each $x\in X$ and $\delta>0$,*

$$\left\{(r,\ s)\in\mathbb{N}\times\mathbb{N}:\frac{1}{h_{r,s}}|\{(k,l)\in J_{r,s}:|d(x,A_{k,l})-d(x,A)|\geq\varepsilon\}|\geq\delta\right\}\in\mathcal{I}_2.$$

In this case, we write $A_{k,l}\to A\left(S_{\theta_{r,s}}(\mathcal{I}_2)\right)$. The class of all Wijsman $\mathcal{I}_2-$lacunary statistically convergent sequences will be denoted by $S_{\theta_{r,s}}(\mathcal{I}_2)$.

Definition 15 ([31])**.** *Let $\theta_{r,s}=(k_{r,s})$ be a double lacunary sequence. Then a double set sequence $\{A_{k,l}\}$ is said to be Wijsman strongly $\mathcal{I}_2-$lacunary convergent to A or $N_{\theta_{r,s}}(\mathcal{I}_2)-$convergent to A, is for every $\varepsilon>0$, and for every $x\in X$,*

$$\left\{(r,\ s)\in\mathbb{N}\times\mathbb{N}:\frac{1}{h_{r,s}}\sum_{(k,l)\in J_{r,s}}|d(x,A_{k,l})-d(x,A)|\geq\varepsilon\right\}\in\mathcal{I}_2.$$

In this case, we write $A_{k,l}\to A\left(N_{\theta_{r,s}}(\mathcal{I}_2)\right)$.

3. Main results

In this section, we investigate the relationship between Wijsman $\mathcal{I}_2-$statistical convergence, Wijsman $\mathcal{I}_2-$lacunary statistical convergence and Wijsman strongly $\mathcal{I}_2-$lacunary convergence of double sequences of sets.

The following theorem is a 2-dimensional analogue of Ulusu and Dündar's theorem presented in [30], and Wijsman type of result presented in [14].

Theorem 1. *If $\mathcal{I}_2$ is an admissible ideal of $\mathbb{N}\times\mathbb{N}$, $\theta_{r,s}=(k_{r,s})$ is a double lacunary sequence and A_{kl}, B_{kl} are non-empty closed subsets of X, then*

(i) (a) If $A_{k,l}\to A\left(N_{\theta_{r,s}}(\mathcal{I}_2)\right)$ then $A_{k,l}\to A\left(S_{\theta_{r,s}}(\mathcal{I}_2)\right)$;

(b) $N_{\theta_{r,s}}(\mathcal{I}_2)$ is a proper subset of $S_{\theta_{r,s}}(\mathcal{I}_2)$;

(ii) If $A_{k,l}\to A\left(S_{\theta_{r,s}}(\mathcal{I}_2)\right)$ and $\{A_{k,l}\}\in l_\infty^2$ then $A_{k,l}\to A\left(N_{\theta_{r,s}}(\mathcal{I}_2)\right)$.

Proof. $(i)-(a)$. Let $\varepsilon>0$ and $A_{k,l}\to A\left(N_{\theta_{r,s}}(\mathcal{I}_2)\right)$. Then we can write

$$\sum_{(k,l)\in J_{r,s}}|d(x,A_{k,l})-d(x,A)|\geq\sum_{\substack{(k,l)\in J_{r,s}\\ \left|d\left(x,A_{k,l}\right)-d(x,A)\right|\geq\varepsilon}}|d(x,A_{k,l})-d(x,A)|$$
$$\geq\varepsilon\left|\{(k,l)\in J_{r,s}:|d(x,A_{k,l})-d(x,A)|\geq\varepsilon\}\right|,$$

and so

$$\frac{1}{\varepsilon h_{r,s}}\sum_{(k,l)\in J_{r,s}}|d(x,A_{k,l})-d(x,A)|$$
$$\geq\frac{1}{h_{r,s}}\left|\{(k,l)\in J_{r,s}:|d(x,A_{k,l})-d(x,A)|\geq\varepsilon\}\right|.$$

Then, for each $x\in X$ and for any $\delta>0$, we have the containment

$$\left\{(r,\,s)\in\mathbb{N}\times\mathbb{N}:\tfrac{1}{h_{r,s}}\left|\{(k,l)\in J_{r,s}:|d(x,A_{k,l})-d(x,A)|\geq\varepsilon\}\right|\geq\delta\right\}$$
$$\subseteq\left\{(r,\,s)\in\mathbb{N}\times\mathbb{N}:\tfrac{1}{h_{r,s}}\sum_{(k,l)\in J_{r,s}}|d(x,A_{k,l})-d(x,A)|\geq\varepsilon\delta\right\}.$$

Since $A_{k,l}\to A\left(N_{\theta_{r,s}}(\mathcal{I}_2)\right)$, so that

$$\left\{(r,\,s)\in\mathbb{N}\times\mathbb{N}:\frac{1}{h_{r,s}}\sum_{(k,l)\in J_{r,s}}|d(x,A_{k,l})-d(x,A)|\geq\varepsilon\delta\right\}\in\mathcal{I}_2,$$

which implies that

$$\left\{(r,\,s)\in\mathbb{N}\times\mathbb{N}:\frac{1}{h_{r,s}}\left|\{(k,l)\in J_{r,s}:|d(x,A_{k,l})-d(x,A)|\geq\varepsilon\}\right|\geq\delta\right\}\in\mathcal{I}_2.$$

Hence we have $A_{k,l}\to A\left(S_{\theta_{r,s}}(\mathcal{I}_2)\right)$.

$(i)-(b)$. Let $\theta_{r,s}=(k_{r,s})$ be given and let us define a set sequence $\{A_{k,l}\}$ as follows:

$$\{A_{k,l}\}=\left(\begin{array}{ccccccc}
\{1\} & \{2\} & \{3\} & \ldots & \left\{\left[\sqrt[3]{h_{r,s}}\right]\right\} & \{0\} & \ldots \\
\{2\} & \{2\} & \{3\} & \ldots & \left\{\left[\sqrt[3]{h_{r,s}}\right]\right\} & \{0\} & \ldots \\
. & . & . & . & . & . & . \\
. & . & . & . & . & . & . \\
. & . & . & . & . & . & . \\
\{2\} & \left\{\left[\sqrt[3]{h_{r,s}}\right]\right\} & \ldots & \ldots & \left\{\left[\sqrt[3]{h_{r,s}}\right]\right\} & \{0\} & \ldots \\
\{0\} & \{0\} & \{0\} & \{0\} & \{0\} & \{0\} & \vdots \\
. & . & . & . & . & . & . \\
. & . & . & . & . & . & . \\
. & . & . & . & . & . & .
\end{array}\right)$$

where $[.]$ denotes the greatest integer function.

It is clear that $\{A_{k,l}\}$ is an unbounded double set sequence. Morever, for each $\varepsilon>0$ and for each $x\in X$ we have

$$\frac{1}{h_{r,s}}\left|\{(k,l)\in J_{r,s}:|d(x,A_{k,l})-d(x,\{0\})|\geq\varepsilon\}\right|\leq\frac{\left[\sqrt[3]{h_{r,s}}\right]}{h_{r,s}}.$$

Then for any $\delta>0$ we get

$$\left\{(r,\ s)\in\mathbb{N}\times\mathbb{N}:\frac{1}{h_{r,s}}\left|\{(k,l)\in J_{r,s}:|d(x,A_{k,l})-d(x,\{0\})|\geq\varepsilon\}\right|\geq\delta\right\}$$
$$\subseteq\left\{(r,\ s)\in\mathbb{N}\times\mathbb{N}:\frac{\left[\sqrt[3]{h_{r,s}}\right]}{h_{r,s}}\geq\delta\right\}.$$

Since $P-\lim_{r,s\to\infty}\frac{\left[\sqrt[3]{h_{r,s}}\right]}{h_{r,s}}=0$, it follows that the set on the right side is finite and therefore belongs to $\mathcal{I}_2$. This shows that

$$\left\{(r,\ s)\in\mathbb{N}\times\mathbb{N}:\frac{1}{h_{r,s}}\left|\{(k,l)\in J_{r,s}:|d(x,A_{k,l})-d(x,\{0\})|\geq\varepsilon\}\right|\geq\delta\right\}\in\mathcal{I}_2$$

and therefore we have $\{A_{k,l}\}\to\{(0,0)\}\left(S_{\theta_{r,s}}(\mathcal{I}_2)\right)$. On the other hand for some fixed $x\in X$,

$$\frac{1}{h_{r,s}}\sum_{(k,l)\in J_{r,s}}|d(x,A_{k,l})-d(x,\{0\})|=\frac{\left[\sqrt[3]{h_{r,s}}\right]\left[\sqrt[3]{h_{r,s}}\right]\left(\left[\sqrt[3]{h_{r,s}}\right]+1\right)}{2h_{r,s}}\to\frac{1}{2},$$

implies that the sequence $\frac{[\sqrt[3]{h_{r,s}}][\sqrt[3]{h_{r,s}}]([\sqrt[3]{h_{r,s}}]+1)}{h_{r,s}} \to 1, \quad r,s \to \infty$, which gives for $\varepsilon = \frac{1}{4}$

$$\left\{(r,\, s) \in \mathbb{N}\times\mathbb{N} : \frac{1}{h_{r,s}} \sum_{(k,l)\in J_{r,s}} |d\,(x, A_{k,l}) - d\,(x, \{0\})| \geq \frac{1}{4}\right\}$$
$$= \left\{(r,\, s) \in \mathbb{N}\times\mathbb{N} : \frac{\left[\sqrt[3]{h_{r,s}}\right]\left[\sqrt[3]{h_{r,s}}\right]\left(\left[\sqrt[3]{h_{r,s}}\right]+1\right)}{h_{r,s}} \geq \frac{1}{2}\right\} \in \mathcal{F}\,(\mathcal{I}_2)\,.$$

This shows that $A_{k,l} \to \{(0,0)\}\left(N_{\theta_{r,s}}\,(\mathcal{I}_2)\right)$ does not hold.

(ii) Suppose that $A_{k,l} \to A\left(S_{\theta_{r,s}}\,(\mathcal{I}_2)\right)$ and $\{A_{k,l}\} \in l^2_\infty$. Then there exists a $M > 0$ such that

$$|d\,(x, A_{k,l}) - d\,(x, A)| \leq M$$

for all $(k,l) \in \mathbb{N}\times\mathbb{N}$. Given $\varepsilon > 0$, for each $x \in X$ we have

$$\frac{1}{h_{r,s}} \sum_{(k,l)\in J_{r,s}} |d\,(x, A_{k,l}) - d\,(x, A)|$$
$$= \frac{1}{h_{r,s}} \sum_{\substack{(k,l)\in J_{r,s} \\ \left|d\left(x, A_{k,l}\right) - d(x,A)\right| \geq \frac{\varepsilon}{2}}} |d\,(x, A_{k,l}) - d\,(x, A)|$$
$$+ \frac{1}{h_{r,s}} \sum_{\substack{(k,l)\in J_{r,s} \\ \left|d\left(x, A_{k,l}\right) - d(x,A)\right| < \frac{\varepsilon}{2}}} |d\,(x, A_{k,l}) - d\,(x, A)|$$
$$\leq \frac{M}{h_{r,s}} \left|\left\{(k,l) \in J_{r,s} : |d\,(x, A_{k,l}) - d\,(x, A)| \geq \frac{\varepsilon}{2}\right\}\right| + \frac{\varepsilon}{2}.$$

Hence, for each $x \in X$ we have

$$\left\{(r,\, s) \in \mathbb{N}\times\mathbb{N} : \frac{1}{h_{r,s}} \sum_{(k,l)\in J_{r,s}} |d\,(x, A_{k,l}) - d\,(x, A)| \geq \varepsilon\right\}$$
$$\subseteq \left\{(r,\, s) \in \mathbb{N}\times\mathbb{N} : \frac{1}{h_{r,s}} \left|\left\{(k,l) \in J_{r,s} : |d\,(x, A_{k,l}) - d\,(x, A)| \geq \frac{\varepsilon}{2}\right\}\right| \geq \frac{\varepsilon}{2M}\right\} \in \mathcal{I}_2.$$

Therefore $A_{k,l} \to A\left(N_{\theta_{r,s}}\,(\mathcal{I}_2)\right)$. This completes the proof. ■

Theorem 2. *For any double lacunary sequence* $\theta_{r,s} = (k_{r,s})$, *if* $\liminf_{r,s} q_{r,s} > 1$, *then*

$$A_{k,l} \to A\left(S\left(\mathcal{I}_2\right)\right) \Rightarrow A_{k,l} \to A\left(S_{\theta_{r,s}}\left(\mathcal{I}_2\right)\right).$$

Proof. Suppose first that $\liminf_{r,s} q_{r,s} > 1$, then there exists $\delta > 0$ such that $q_{r,s} \geq 1 + \delta$ for sufficiently large r,s. Then we have

$$\frac{h_{r,s}}{k_r l_s} \geq \frac{\delta}{(1+\delta)}.$$

If $A_{k,l} \to A\left(S\left(\mathcal{I}_2\right)\right)$, then for every $\varepsilon > 0$ and for sufficiently large r, s, we have

$$\begin{aligned} &\frac{1}{k_r l_s}\left|\left\{k \leq k_r, l \leq l_s : \left|d\left(x, A_{k,l}\right) - d\left(x, A\right)\right| \geq \varepsilon\right\}\right| \\ &\quad \geq \frac{1}{k_r l_s}\left|\left\{(k,l) \in J_{r,s} : \left|d\left(x, A_{k,l}\right) - d\left(x, A\right)\right| \geq \varepsilon\right\}\right| \\ &\quad \geq \frac{\delta}{(1+\delta)} \frac{1}{h_{rs}}\left|\left\{(k,l) \in J_{r,s} : \left|d\left(x, A_{k,l}\right) - d\left(x, A\right)\right| \geq \varepsilon\right\}\right|. \end{aligned}$$

Then for any $\mu > 0$, we get

$$\begin{aligned} &\left\{(r,\ s) \in \mathbb{N} \times \mathbb{N} : \frac{1}{h_{rs}}\left|\left\{(k,l) \in J_{r,s} : \left|d\left(x, A_{k,l}\right) - d\left(x, A\right)\right| \geq \varepsilon\right\}\right| \geq \mu\right\} \\ &\subseteq \left\{(r,\ s) \in \mathbb{N} \times \mathbb{N} : \frac{1}{k_r l_s}\left|\left\{k \leq k_r, l \leq l_s : \left|d\left(x, A_{k,l}\right) - d\left(x, A\right)\right| \geq \varepsilon\right\}\right| \right. \\ &\quad \left. \geq \frac{\delta\mu}{(1+\delta)}\right\} \in \mathcal{I}_2. \end{aligned}$$

This completes the proof. ■

Theorem 3. *If* $\mathcal{I}_2 = \mathcal{I}_{2(fin)} = \{A \subset \mathbb{N} \times \mathbb{N} : A \text{ is a finite set}\}$ *is a non-trivial ideal, and* $\theta_{r,s} = (k_{r,s})$ *is a double lacunary sequence with* $\limsup_{r,s} q_{r,s} < \infty$, *then we have*

$$\{A_{k,l}\} \to A\left(S_{\theta_{r,s}}\left(\mathcal{I}_2\right)\right) \Rightarrow \{A_{k,l}\} \to A\left(S\left(\mathcal{I}_2\right)\right).$$

Proof. If $\limsup_{r,s} q_{r,s} < \infty$, then there exists a $K > 0$ such that $q_{r,s} < K$ for all $r, s \geq 1$. Suppose that $\{A_{k,l}\} \to A\left(S_{\theta_{r,s}}\left(\mathcal{I}_2\right)\right)$ and let

$$M_{r,s} = \left|\left\{(k,l) \in J_{r,s} : \left|d\left(x, A_{k,l}\right) - d\left(x, A\right)\right| \geq \varepsilon\right\}\right|.$$

Since $\{A_{k,l}\} \to A\left(S_{\theta_{r,s}}\left(\mathcal{I}_2\right)\right)$, then for every $\varepsilon > 0$ and $\delta > 0$, we have

$$\begin{aligned} &\left\{(r,\ s) \in \mathbb{N} \times \mathbb{N} : \frac{1}{h_{r,s}}\left|\left\{(k,l) \in J_{r,s} : \left|d\left(x, A_{k,l}\right) - d\left(x, A\right)\right| \geq \varepsilon\right\}\right| \geq \delta\right\} \\ &= \left\{(r, s) \in \mathbb{N} \times \mathbb{N} : \frac{M_{r,s}}{h_{r,s}} \geq \delta\right\} \in \mathcal{I}_2, \end{aligned}$$

and therefore, it is a finite set. We choose integers $r_0, s_0 \in \mathbb{N}$ such that

$$\frac{M_{r,s}}{h_{r,s}} < \delta \text{ for all } r > r_0, \quad s > s_0.$$

Let $M = \max\{M_{r,s} : 1 \leq r \leq r_0, 1 \leq s \leq s_0\}$ and m, n are two integers satisfying $k_{r-1} < m \leq k_r$, $l_{s-1} < n \leq l_s$. Then we have

$$\begin{aligned}
&\frac{1}{mn} \left|\{k \leq m,\, l \leq n : |d(x, A_{k,l}) - d(x, A)| \geq \varepsilon\}\right| \\
&\leq \frac{1}{k_{r-1}l_{s-1}} \left|\{k \leq k_r,\, l \leq l_s : |d(x, A_{k,l}) - d(x, A)| \geq \varepsilon\}\right| \\
&= \frac{1}{k_{r-1}l_{s-1}} \{M_{1,1} + M_{2,2} + ... + M_{r_0,s_0} + M_{r_0+1,s_0+1} + ... + M_{r,s}\} \\
&\leq \frac{M}{k_{r-1}l_{s-1}} r_0 s_0 + \frac{1}{k_{r-1}l_{s-1}} \left\{ h_{r_0+1,s_0+1} \left(\frac{M_{r_0+1,s_0+1}}{h_{r_0+1,s_0+1}} \right) + ... + h_{r,s} \frac{M_{r,s}}{h_{r,s}} \right\} \\
&\leq \frac{M}{k_{r-1}l_{s-1}} r_0 s_0 + \frac{1}{k_{r-1}l_{s-1}} \left(\sup_{r>r_0, s>s_0} \frac{M_{r,s}}{h_{r,s}} \right) (h_{r_0+1,s_0+1} + ... + h_{r,s}) \\
&\leq \frac{M}{k_{r-1}l_{s-1}} r_0 s_0 + \delta \left(\frac{k_r l_s - k_{r_0} l_{s_0}}{k_{r-1}l_{s-1}} \right) \\
&\leq \frac{M}{k_{r-1}l_{s-1}} r_0 s_0 + \delta q_{r,s} \leq \frac{M}{k_{r-1}l_{s-1}} r_0 s_0 + \delta K.
\end{aligned}$$

This completes the proof of the theorem. ■

Definition 16. *We say that the sequence $\{A_{k,l}\}$ is Wijsman $\mathcal{I}_2$-Cesàro summable to $\{A\}$ if for each $\varepsilon > 0$ and for each $x \in X$,*

$$\left\{ (m,n) \in \mathbb{N} \times \mathbb{N} : \left| \frac{1}{mn} \sum_{k,l=1}^{m,n} (d(x, A_{k,l}) - d(x, A)) \right| \geq \varepsilon \right\} \in \mathcal{I}_2.$$

In this case, we write $\{A_{k,l}\} \overset{C_1(\mathcal{I}_2)}{\rightarrow} \{A\}$.

Definition 17. *We say that the sequence $\{A_{k,l}\}$ is Wijsman strongly $\mathcal{I}_2$-Cesàro summable to $\{A\}$ if for each $\varepsilon > 0$ and for each $x \in X$,*

$$\left\{ (m,n) \in \mathbb{N} \times \mathbb{N} : \frac{1}{mn} \sum_{k,l=1}^{m,n} |d(x, A_{k,l}) - d(x, A)| \geq \varepsilon \right\} \in \mathcal{I}_2.$$

In this case, we write $\{A_{k,l}\} \overset{C_1[\mathcal{I}_2]}{\rightarrow} \{A\}$.

Theorem 4. *Let $\mathcal{I}_2$ be an admissible ideal of $\mathbb{N}\times\mathbb{N}$, $\theta_{r,s}=(k_{r,s})$ be a double lacunary sequence. If $\{A_{kl}\}\in l_\infty^2$ and $A_{k,l}\to A\,(S\,(\mathcal{I}_2))$, then $\{A_{k,l}\}\overset{C_1(\mathcal{I}_2)}{\to}\{A\}$.*

Proof. Suppose that $\{A_{kl}\}\in l_\infty^2$ and $A_{k,l}\to A\,(S\,(\mathcal{I}_2))$. Then we can assume that

$$|d\,(x,A_{k,l})-d\,(x,A)|\leq M$$

for all $(k,l)\in\mathbb{N}\times\mathbb{N}$. Also for each $\varepsilon>0$, we can write

$$\begin{aligned}\left|\frac{1}{mn}\sum_{k,l=1}^{m,n}(d(x,A_{k,l})-d(x,A))\right| &\leq \frac{1}{mn}\sum_{k,l=1}^{m,n}|d(x,A_{k,l})-d(x,A)|\\ &\leq \frac{1}{mn}\sum_{\substack{k,l=1\\ \left|d\left(x,A_{k,l}\right)-d(x,A)\right|\geq\frac{\varepsilon}{2}}}^{m,n}|d\,(x,A_{k,l})-d\,(x,A)|\\ &\quad+\frac{1}{mn}\sum_{\substack{k,l=1\\ \left|d\left(x,A_{k,l}\right)-d(x,A)\right|<\frac{\varepsilon}{2}}}^{m,n}|d\,(x,A_{k,l})-d\,(x,A)|\\ &\leq M\frac{1}{mn}|\{k\leq m,l\leq n:|d\,(x,A_{k,l})-d\,(x,A)|\geq\frac{\varepsilon}{2}\}|+\frac{1}{mn}mn\frac{\varepsilon}{2}.\end{aligned}$$

Consequently, if $\delta>\frac{\varepsilon}{2}>0$, δ and ε are independent, put $\delta_1=\delta-\frac{\varepsilon}{2}>0$, we have

$$\begin{aligned}&\left\{(m,n)\in\mathbb{N}\times\mathbb{N}:\left|\frac{1}{mn}\sum_{k,l=1}^{m,n}(d(x,A_{k,l})-d(x,A))\right|\geq\delta\right\}\\ &\subseteq\Bigg\{(m,n)\in\mathbb{N}\times\mathbb{N}:\frac{1}{mn}|\{k\leq m,l\leq n:|d\,(x,A_{k,l})-d\,(x,A)|\geq\frac{\varepsilon}{2}\}|\\ &\qquad\geq\frac{\delta_1}{M}\Bigg\}\in\mathcal{I}_2.\end{aligned}$$

This shows that $\{A_{k,l}\}\overset{C_1(\mathcal{I}_2)}{\to}\{A\}$. ■

Acknowledgement. The author would like to thank the referee for his/her much encouragment, support, constructive criticism, careful reading and making a useful comment which imporved the presentation and the readability of the paper.

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Received on 25.06.2015 and, in revised form, on 23.08.2016.