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**UNIQUENESS OF DIFFERENTIAL POLYNOMIALS
 OF MEROMORPHIC FUNCTIONS SHARING
 A SMALL FUNCTION WITHOUT
 COUNTING MULTIPLICITY**

ABSTRACT. The paper concerns interesting problems related to the field of Complex Analysis, in particular Nevanlinna theory of meromorphic functions. The author have studied certain uniqueness problem on differential polynomials of meromorphic functions sharing a small function without counting multiplicity. The results of this paper are extension of some problems studied by K. Boussaf et. al. in [2] and generalization of some results of S.S. Bhoosnurmath et. al. in [4].

KEY WORDS: uniqueness theorem, differential polynomials, meromorphic function, Nevanlinna theory.

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1. Introduction and main results

A meromorphic function means meromorphic in the whole complex plane. We assume that the reader is used to the standard notations and fundamental results of Nevanlinna theory. Let f, g be two meromorphic function in \mathbb{C} and $a \in \mathbb{C} \cup \{\infty\}$. We say that f and g share $a - CM$ if $f - a$ and $g - a$ have the same zeros with multiplicities.

Let m and p be a positive integer. We denote by $\overline{N}_{(m)}(r, a; f)$ (or $\overline{N}_{(m)}(r, \frac{1}{f-a})$) the reduced counting function of a -point of f whose multiplicities are not less than m and $\overline{N}_m(r, a; f)$ (or $\overline{N}_m(r, \frac{1}{f-a})$) the reduced counting function of a -point of f whose multiplicities are at most m . The counting

functions $N_p(r, \frac{1}{f-a})$ and $N_2(r, f)$ are defined by

$$\begin{aligned} N_p(r, \frac{1}{f-a}) &= \overline{N}(r, \frac{1}{f-a}) + \overline{N}_{(2)}(r, \frac{1}{f-a}) + \dots \\ &\quad + \overline{N}_{(p)}(r, \frac{1}{f-a}), \quad a \in \mathbb{C}; \\ N_2(r, f) &= \overline{N}(r, f) + \overline{N}_{(2)}(r, f), \quad \text{respectively.} \end{aligned}$$

First, we remind the definition of uniqueness polynomial. Recall that a polynomial $P \in \mathbb{C}[x]$ is called a *polynomial of uniqueness* for a class of functions \mathcal{F} if for any two functions $f, g \in \mathcal{F}$ the property $P(f) = P(g)$ implies $f = g$.

The definition of polynomials of uniqueness was introduced in [11] by Li and Yang. Since then, many authors have been studying this problem for complex functions ([1], [10], [9]) or p -adic functions ([5], [6]).

Let $\mathcal{M}(\mathbb{C})$ ($\mathcal{A}(\mathbb{C})$) be the set of meromorphic (entire) functions on complex plane. Given $f \in \mathcal{M}(\mathbb{C})$, we denote by $\mathcal{M}_f(\mathbb{C})$ the field of small functions with respect to f . Given $f \in \mathcal{A}(\mathbb{C})$, we denote by $\mathcal{A}_f(\mathbb{C})$ the ring of small entire functions with respect to f .

Given $\alpha \in \mathcal{M}(\mathbb{C})$, we say that $f, g \in \mathcal{M}(\mathbb{C})$ share a function α CM if $f - \alpha$ and $g - \alpha$ have the same zeros with multiplicities. Furthermore, if $f - \alpha$ and $g - \alpha$ have the same zeros without multiplicity, we said that $f, g \in \mathcal{M}(\mathbb{C})$ share a function α IM.

Definition 1 ([7]). *Let k be a nonnegative integer or ∞ , and let $a(z)$ be a small function of both f and g . Denote by $E_k(a; f)$ the set of all zeros of $f - a$, where a zero of $f - a$ with multiplicity m is counted m times if $m \leq k$ and $k + 1$ times if $m > k$. If $E_k(a; f) = E_k(a; g)$, then we say that f and g share a with weight k .*

By Definition 1, if f and g share a with weight k , then z_0 is a zero of $f - a$ with multiplicity $m(\leq k)$ if and only if it is a zero of $g - a$ with multiplicity $m(\leq k)$ and z_0 is a zero of $f - a$ with multiplicity $m(> k)$ if and only if it is a zero of $g - a$ with multiplicity $n(> k)$, where m is not necessarily equal to n .

We write that f and g share (a, k) to mean that f and g share a with weight k . Clearly, if f and g share (a, k) , then f and g share (a, p) for all integer p , $0 \leq p \leq k$. Also we note that f and g share $a - IM$ (resp. $a - CM$) if and only if f and g share $(a, 0)$ (resp. (a, ∞)).

In 2013, K. Boussaf, A. Escassut and J. Ojeda [2] proved

Theorem A. *Let P be a polynomial of uniqueness for $\mathcal{M}(\mathbb{C})$, let $P' = b(x - a_1)^n \prod_{i=2}^l (x - a_i)^{k_i}$ with $b \in \mathbb{C}^*$, and $l \geq 2$, $k_i \geq k_{i+1}$, $2 \leq i \leq l - 1$*

when $l > 2$ and let $k = \sum_{i=2}^l k_i$. Suppose that P satisfies the following conditions:

-) $n \geq 10 + \sum_{i=3}^l \max(0, 4 - k_i) + \max(0, 5 - k_2)$,
-) $n \geq k + 3$,
-) if $l = 2$, then $n \neq 2k, 2k + 1, 3k + 1$,
-) if $l = 3$, then $n \neq 2k + 1, 3k_i - k, i = 2, 3$.

Let $f, g \in \mathcal{M}(\mathbb{C})$ be transcendental and let $\alpha \in \mathcal{M}_f(\mathbb{C}) \cap \mathcal{M}_g(\mathbb{C})$ be non-identically zero. If $f'P'(f)$ and $g'P'(g)$ share α CM, then $f \equiv g$.

Theorem B. Let P be a polynomial of uniqueness for $\mathcal{A}(\mathbb{C})$, let $P' = b(x - a_1)^n \prod_{i=2}^l (x - a_i)^{k_i}$ with $b \in \mathbb{C}^*$, and $l \geq 2$, $k_i \geq k_{i+1}$, $2 \leq i \leq l - 1$ when $l > 2$ and let $k = \sum_{i=2}^l k_i$. Suppose that P satisfies the following conditions:

-) $n \geq 5 + \sum_{i=3}^l \max(0, 4 - k_i) + \max(0, 5 - k_2)$,
-) $n \geq k + 2$.

Let $f, g \in \mathcal{A}(\mathbb{C})$ be transcendental and let $\alpha \in \mathcal{A}_f(\mathbb{C}) \cap \mathcal{A}_g(\mathbb{C})$ be non-identically zero. If $f'P'(f)$ and $g'P'(g)$ share α CM, then $f \equiv g$.

Theorem C. Let $f, g \in \mathcal{M}(\mathbb{C})$ be transcendental and let $\alpha \in \mathcal{M}_f(\mathbb{C}) \cap \mathcal{M}_g(\mathbb{C})$ be non-identically zero. Let $a \in \mathbb{C}^*$. If $f'f^n(f - a)$ and $g'g^n(g - a)$ share the function α CM and if $n \geq 12$ then either $f \equiv g$ or there exists $h \in \mathcal{M}(\mathbb{C})$ such that

$$f = \frac{a(n+2)(1-h^{n+1})}{(n+1)(1-h^{n+2})}h, \quad g = \frac{a(n+2)(1-h^{n+1})}{(n+1)(1-h^{n+2})}.$$

In 2015, S.S. Bhoosnurmath et. al. [4] gave the following result.

Theorem D. Let f and g be two non-constant meromorphic functions, whose zeros and poles are of multiplicities at least s , where s is a positive integer. Let n, m be positive integers satisfying $(n - m - 7)s \geq 19$. Suppose that $f^n(f - 1)^m f'$ and $g^n(g - 1)^m g'$ share $(1, 0)$, then

(i) for $m = 1$, either $f \equiv g$ or

$$g = \frac{(n+2)(1-h^{n+1})}{(n+1)(1-h^{n+2})}, \quad f = \frac{(n+2)h(1-h^{n+1})}{(n+1)(1-h^{n+2})};$$

(ii) for $m = 2$, $f \equiv g$.

Moreover, they prove that Theorem D for the case $f^n(f - 1)^m f'$ and $g^n(g - 1)^m g'$ share $(1, l)$, $l = 1, 2, \infty$ and $m \geq 3$.

According to the theorems in this direction, we will extend the result's K. Boussaf et. al. [2] for condition $f'P'(f)$ and $g'P'(g)$ sharing $(\alpha(z), 0)$. Furthermore, we will generalize the result's S. S. Bhoosnurmath et. al. [4] for a larger class of differential polynomials. Note that, by using our technique, we are easy to get the results for the cases of sharing a small function with weight $1, 2, \infty$. We are leading this work for reader. Namely, we prove

Theorem 1. *Let P be a polynomial of uniqueness for $\mathcal{M}(\mathbb{C})$ and s, p be positive integers (or ∞). Let $P' = b(x - a_1)^n \prod_{i=2}^l (x - a_i)^{k_i}$ with $b \in \mathbb{C}^*$, and $l \geq 2, k_i \geq k_{i+1}, 2 \leq i \leq l - 1$ when $l > 2$ and let $k = \sum_{i=2}^l k_i$. Suppose that P satisfies the following conditions:*

-) $n \geq k + 3$;
-) $n > \frac{7}{s} + \frac{10}{p} + \max(0, 10 - k_2) + \sum_{i=3}^l \max(0, 7 - k_i)$;
-) if $l = 2$, then $n \neq 2k, 2k + 1, 3k + 1$,
-) if $l = 3$, then $n \neq 2k + 1, 3k_i - k, i = 2, 3$.

Let $f, g \in \mathcal{M}(\mathbb{C})$ be transcendental whose a_1 - point and poles are of multiplicities at least s, p respectively and let $\alpha \in \mathcal{M}_f(\mathbb{C}) \cap \mathcal{M}_g(\mathbb{C})$ be non-identically zero. If $f'P'(f)$ and $g'P'(g)$ share $(\alpha, 0)$, then $f \equiv g$.

Theorem 2. *Let P be a polynomial of uniqueness for $\mathcal{A}(\mathbb{C})$ and s be a positive integer (or ∞). Let $P' = b(x - a_1)^n \prod_{i=2}^l (x - a_i)^{k_i}$ with $b \in \mathbb{C}^*$, and $l \geq 2, k_i \geq k_{i+1}, 2 \leq i \leq l - 1$ when $l > 2$ and let $k = \sum_{i=2}^l k_i$. Suppose that P satisfies the following conditions:*

-) $n \geq k + 2$;
-) $n > \frac{7}{s} + \max(0, 10 - k_2) + \sum_{i=3}^l \max(0, 7 - k_i)$.

Let $f, g \in \mathcal{A}(\mathbb{C})$ be transcendental whose a_1 - point is of multiplicities at least s , and let $\alpha \in \mathcal{A}_f(\mathbb{C}) \cap \mathcal{A}_g(\mathbb{C})$ be non-identically zero. If $f'P'(f)$ and $g'P'(g)$ share $(\alpha, 0)$, then $f \equiv g$.

Remark 1. In Theorem 1 and Theorem 2, if a_1 is an exceptional value of f and g then $s = +\infty$. Futhermore, if a_1 is an exceptional value of f or g then s is the smallest number in the set of multiples zeros of $g - a_1$ or $f - a_1$, respectively. In the general case, for many pair of transcendental meromorphic functions f and g , we may choose $s = p = 1$.

Theorem 3. *Let $f, g \in \mathcal{M}(\mathbb{C})$ be transcendental meromorphic functions whose zeros and poles are of multiplicities at least s and p respectively, where s, p are positive integers (or ∞) and let $\alpha \in \mathcal{M}_f(\mathbb{C}) \cap \mathcal{M}_g(\mathbb{C})$ be non-identically zero. Let $a \in \mathbb{C}^*$. If $f'f^n(f - a)$ and $g'g^n(g - a)$ share the function $(\alpha, 0)$ and if $n > 7 + \frac{7}{s} + \frac{10}{p}$, then either $f \equiv g$ or there exists $h \in \mathcal{M}(\mathbb{C})$ such that*

$$f = \frac{a(n + 2)(1 - h^{n+1})}{(n + 1)(1 - h^{n+2})}h, \quad g = \frac{a(n + 2)(1 - h^{n+1})}{(n + 1)(1 - h^{n+2})}.$$

Remark 2. In Theorem 3, take $s = p, \alpha(z) \equiv 1$, we need $n \geq 8 + \frac{17}{s}$. This is an improvement of the result due to S. S. Bhoosnurmath et. al. [4].

2. Some lemmas

Lemma 1 ([8, 16]). *Let f and g be two non-constant meromorphic functions, and let $a(z)$ ($a \neq 0, \infty$) be a small function of both f and g . If f and g share $(a(z), 0)$, one of the following three cases holds:*

$$(i) \quad T(r, f) \leq N_2(r, f) + N_2(r, \frac{1}{f}) + N_2(r, g) + N_2(r, \frac{1}{g}) + 2(\overline{N}(r, \frac{1}{f}) + \overline{N}(r, f)) + (\overline{N}(r, \frac{1}{g}) + \overline{N}(r, g)) + S(r, f) + S(r, g),$$

and the similar inequality holds for $T(r, g)$;

$$(ii) \quad f \equiv g;$$

$$(iii) \quad fg \equiv a^2.$$

Lemma 2 ([2]). *Let $P'(x) = b(x - a_1)^n \prod_{i=2}^l (x - a_i)^{k_i} \in \mathbb{C}[x]$ ($a_i \neq a_j, \forall i \neq j$) with $l \geq 2$ and $n \geq \max\{k_2, \dots, k_l\}$ and let $k = \sum_{i=2}^l k_i$. Let $f, g \in \mathcal{M}(\mathbb{C})$ be transcendental and let $\theta = P'(f)f'P'(g)g'$. If θ belongs to $\mathcal{M}_f(\mathbb{C}) \cap \mathcal{M}_g(\mathbb{C})$, then we have the following:*

$$\text{--} \text{) if } l = 2 \text{ then } n \text{ belongs to } \{k, k + 1, 2k, 2k + 1, 3k + 1\},$$

$$\text{--} \text{) if } l = 3 \text{ then } n \text{ belongs to } \{\frac{k}{2}, k + 1, 2k + 1, 3k_2 - k, 3k_3 - k\},$$

$$\text{--} \text{) if } l \geq 4 \text{ then } n = k + 1.$$

Moreover, if f, g belong to $\mathcal{A}(\mathbb{C})$, then θ does not belong to $\mathcal{A}_f(\mathbb{C})$.

Lemma 3 ([12]). *Let $f \in \mathcal{M}(\mathbb{C})$ be non-constant meromorphic function, then*

$$T(r, f) - N(r, \frac{1}{f}) \leq T(r, f') - N(r, \frac{1}{f'}) + S(r, f).$$

3. Proofs of theorems

Proofs of Theorem 1. Without loss of generality, we can assume that $a_1 = 0$. Let $F = P(f)$, $G = P(g)$, by hypothesis of Theorem 1, we get that F' and G' share $(\alpha(z), 0)$. By Lemma 1, we suppose that the case (i) is true. Then

$$(1) \quad T(r, F') \leq N_2(r, F') + N_2(r, \frac{1}{F'}) + N_2(r, G') + N_2(r, \frac{1}{G'}) + 2(\overline{N}(r, \frac{1}{F'}) + \overline{N}(r, F')) + (\overline{N}(r, \frac{1}{G'}) + \overline{N}(r, G')) + S(r, f) + S(r, g);$$

$$(2) \quad T(r, G') \leq N_2(r, F') + N_2(r, \frac{1}{F'}) + N_2(r, G') + N_2(r, \frac{1}{G'}) + 2(\overline{N}(r, \frac{1}{G'}) + \overline{N}(r, G')) + (\overline{N}(r, \frac{1}{F'}) + \overline{N}(r, F')) + S(r, f) + S(r, g).$$

From the definition of F and G , we have

$$\begin{aligned}
& N_2(r, F') + N_2(r, \frac{1}{F'}) + N_2(r, G') + N_2(r, \frac{1}{G'}) \\
& \quad + 2(\overline{N}(r, \frac{1}{F'}) + \overline{N}(r, F')) + (\overline{N}(r, \frac{1}{G'}) + \overline{N}(r, G')) \\
& \leq 2\overline{N}(r, f) + 2\overline{N}(r, \frac{1}{f}) + 2 \sum_{i=2}^l \overline{N}(r, \frac{1}{f-a_i}) + N(r, \frac{1}{f'}) \\
& \quad + 2\overline{N}(r, f) + 2\overline{N}(r, \frac{1}{f}) + 2 \sum_{i=2}^l \overline{N}(r, \frac{1}{f-a_i}) + 2\overline{N}(r, \frac{1}{f'}) \\
& \quad + 2\overline{N}(r, g) + 2\overline{N}(r, \frac{1}{g}) + 2 \sum_{i=2}^l \overline{N}(r, \frac{1}{g-a_i}) + N(r, \frac{1}{g'}) \\
& \quad + (\overline{N}(r, g) + \overline{N}(r, \frac{1}{g}) + \sum_{i=2}^l \overline{N}(r, \frac{1}{g-a_i}) + \overline{N}(r, \frac{1}{g'})).
\end{aligned}$$

This implies

$$\begin{aligned}
(3) \quad & N_2(r, F') + N_2(r, \frac{1}{F'}) + N_2(r, G') + N_2(r, \frac{1}{G'}) \\
& \quad + 2(\overline{N}(r, \frac{1}{F'}) + \overline{N}(r, F')) + (\overline{N}(r, \frac{1}{G'}) + \overline{N}(r, G')) \\
& \leq \frac{4}{p}N(r, f) + \frac{4}{s}N(r, \frac{1}{f}) + 4 \sum_{i=2}^l \overline{N}(r, \frac{1}{f-a_i}) \\
& \quad + N(r, \frac{1}{f'}) + 2\overline{N}(r, \frac{1}{f'}) + \frac{3}{p}N(r, g) + \frac{3}{s}N(r, \frac{1}{g}) \\
& \quad + 3 \sum_{i=2}^l N(r, \frac{1}{g-a_i}) + N(r, \frac{1}{g'}) + \overline{N}(r, \frac{1}{g'})
\end{aligned}$$

and similarly,

$$\begin{aligned}
(4) \quad & N_2(r, G') + N_2(r, \frac{1}{G'}) + N_2(r, F') + N_2(r, \frac{1}{F'}) \\
& \quad + 2(\overline{N}(r, \frac{1}{G'}) + \overline{N}(r, G')) + (\overline{N}(r, \frac{1}{F'}) + \overline{N}(r, F')) \\
& \leq 2\overline{N}(r, g) + 2\overline{N}(r, \frac{1}{g}) + 2 \sum_{i=2}^l \overline{N}(r, \frac{1}{g-a_i}) + N(r, \frac{1}{g'}) \\
& \quad + 2\overline{N}(r, g) + 2\overline{N}(r, \frac{1}{g}) + 2 \sum_{i=2}^l \overline{N}(r, \frac{1}{g-a_i}) + 2\overline{N}(r, \frac{1}{g'})
\end{aligned}$$

$$\begin{aligned}
& + 2\bar{N}(r, f) + 2\bar{N}(r, \frac{1}{f}) + 2 \sum_{i=2}^l \bar{N}(r, \frac{1}{f-a_i}) + N(r, \frac{1}{f'}) \\
& + (\bar{N}(r, f) + \bar{N}(r, \frac{1}{f}) + \sum_{i=2}^l \bar{N}(r, \frac{1}{f-a_i}) + \bar{N}(r, \frac{1}{f'})) \\
& \leq \frac{4}{p}N(r, g) + \frac{4}{s}N(r, \frac{1}{g}) + 4 \sum_{i=2}^l \bar{N}(r, \frac{1}{g-a_i}) + N(r, \frac{1}{g'}) \\
& + 2\bar{N}(r, \frac{1}{g'}) + \frac{3}{p}N(r, f) + \frac{3}{s}N(r, \frac{1}{f}) \\
& + 3 \sum_{i=2}^l N(r, \frac{1}{f-a_i}) + N(r, \frac{1}{f'}) + \bar{N}(r, \frac{1}{f'}).
\end{aligned}$$

By Lemma 3, we have

$$(5) \quad T(r, F) \leq T(r, F') + N(r, \frac{1}{F}) - N(r, \frac{1}{F'}) + S(r, f),$$

$$(6) \quad T(r, G) \leq T(r, G') + N(r, \frac{1}{G}) - N(r, \frac{1}{G'}) + S(r, g).$$

We notice that $P(x) = x^{n+1}Q(x)$, where $\deg Q = k$. Therefore, from (5) and (6), we obtain

$$\begin{aligned}
(7) \quad T(r, F) & \leq T(r, F') + (n+1)N(r, \frac{1}{f}) + N(r, \frac{1}{Q(f)}) - nN(r, \frac{1}{f}) \\
& - \sum_{i=2}^l k_i N(r, \frac{1}{f-a_i}) - N(r, \frac{1}{f'}) + S(r, f) \\
& = T(r, F') + N(r, \frac{1}{f}) + N(r, \frac{1}{Q(f)}) \\
& - \sum_{i=2}^l k_i N(r, \frac{1}{f-a_i}) - N(r, \frac{1}{f'}) + S(r, f),
\end{aligned}$$

$$\begin{aligned}
(8) \quad T(r, G) & \leq T(r, G') + (n+1)N(r, \frac{1}{g}) + N(r, \frac{1}{Q(g)}) - nN(r, \frac{1}{g}) \\
& - \sum_{i=2}^l k_i N(r, \frac{1}{g-a_i}) - N(r, \frac{1}{g'}) + S(r, g) \\
& = T(r, G') + N(r, \frac{1}{g}) + N(r, \frac{1}{Q(g)}) \\
& - \sum_{i=2}^l k_i N(r, \frac{1}{g-a_i}) - N(r, \frac{1}{g'}) + S(r, g).
\end{aligned}$$

Combining from (1)-(6), (7) and (8), we have

$$\begin{aligned}
 (9) \quad T(r, F) &\leq N(r, \frac{1}{f}) + N(r, \frac{1}{Q(f)}) - \sum_{i=2}^l k_i N(r, \frac{1}{f - a_i}) \\
 &\quad - N(r, \frac{1}{f'}) + \frac{4}{p} N(r, f) + \frac{4}{s} N(r, \frac{1}{f}) + 4 \sum_{i=2}^l \bar{N}(r, \frac{1}{f - a_i}) \\
 &\quad + N(r, \frac{1}{f'}) + 2\bar{N}(r, \frac{1}{f'}) + \frac{3}{p} N(r, g) + \frac{3}{s} N(r, \frac{1}{g}) \\
 &\quad + 3 \sum_{i=2}^l N(r, \frac{1}{g - a_i}) + \bar{N}(r, \frac{1}{g'}) + S(r, f) + S(r, g).
 \end{aligned}$$

Similarly,

$$\begin{aligned}
 (10) \quad T(r, G) &\leq N(r, \frac{1}{g}) + N(r, \frac{1}{Q(g)}) - \sum_{i=2}^l k_i N(r, \frac{1}{g - a_i}) \\
 &\quad - N(r, \frac{1}{g'}) + \frac{4}{p} N(r, g) + \frac{4}{s} N(r, \frac{1}{g}) + 4 \sum_{i=2}^l \bar{N}(r, \frac{1}{g - a_i}) \\
 &\quad + N(r, \frac{1}{g'}) + 2\bar{N}(r, \frac{1}{g'}) + \frac{3}{p} N(r, f) + \frac{3}{s} N(r, \frac{1}{f}) \\
 &\quad + 3 \sum_{i=2}^l N(r, \frac{1}{f - a_i}) + \bar{N}(r, \frac{1}{f'}) + S(r, f) + S(r, g).
 \end{aligned}$$

Thus, from (9) and (10), we get

$$\begin{aligned}
 (11) \quad T(r, F) + T(r, G) &\leq (1 + \frac{7}{s})(N(r, \frac{1}{f}) + N(r, \frac{1}{g})) \\
 &\quad + \sum_{i=2}^l (7 - k_i)(N(r, \frac{1}{f - a_i}) + N(r, \frac{1}{g - a_i})) \\
 &\quad + \frac{7}{p}(N(r, f) + N(r, g)) + N(r, \frac{1}{Q(f)}) + N(r, \frac{1}{Q(g)}) \\
 &\quad + 3(N(r, \frac{1}{f'}) + N(r, \frac{1}{g'})) + S(r, f) + S(r, g).
 \end{aligned}$$

By Lemma 3, we get

$$(12) \quad N(r, \frac{1}{f'}) \leq T(r, f') - T(r, f) + N(r, \frac{1}{f}) + S(r, f).$$

Clearly

$$\begin{aligned}
 (13) \quad T(r, f') - T(r, f) &= m(r, f') + N(r, f') - T(r, f) \\
 &\leq m(r, \frac{f'}{f}) + m(r, f) + N(r, f) + \overline{N}(r, f) - T(r, f) \\
 &= \overline{N}(r, f) + S(r, f).
 \end{aligned}$$

Combining (12) and (13), we obtain

$$N(r, \frac{1}{f'}) \leq N(r, \frac{1}{f}) + \overline{N}(r, f) + S(r, f).$$

This implies that

$$\begin{aligned}
 N(r, \frac{1}{f'}) + N(r, \frac{1}{g'}) &\leq N(r, \frac{1}{f - a_2}) + N(r, \frac{1}{g - a_2}) \\
 &\quad + \overline{N}(r, f) + \overline{N}(r, g) + S(r, f) + S(r, g) \\
 &\leq N(r, \frac{1}{f - a_2}) + N(r, \frac{1}{g - a_2}) \\
 &\quad + \frac{1}{p}(N(r, f) + N(r, g)) + S(r, f) + S(r, g)
 \end{aligned}$$

and

$$\begin{aligned}
 N(r, \frac{1}{Q(f)}) &\leq kT(r, f) + S(r, f); \\
 N(r, \frac{1}{Q(g)}) &\leq kT(r, g) + S(r, g).
 \end{aligned}$$

Then, from (11), we obtain

$$\begin{aligned}
 (14) \quad T(r, F) + T(r, G) &\leq (1 + \frac{7}{s})(N(r, \frac{1}{f}) + N(r, \frac{1}{g})) \\
 &\quad + \sum_{i=3}^l (7 - k_i)(N(r, \frac{1}{f - a_i}) + N(r, \frac{1}{g - a_i})) \\
 &\quad + (10 - k_2)(N(r, \frac{1}{f - a_2}) + N(r, \frac{1}{g - a_2})) + \frac{10}{p}(N(r, f) \\
 &\quad + N(r, g)) + k(T(r, f) + T(r, g)) + S(r, f) + S(r, g).
 \end{aligned}$$

Furthermore,

$$(15) \quad T(r, F) = (n + k + 1)T(r, f) + S(r, f);$$

$$(16) \quad T(r, G) = (n + k + 1)T(r, g) + S(r, g).$$

Hence, from (14), (15) and (16), we get

$$\begin{aligned} (n - (\frac{7}{s} + \frac{10}{p} + \max(0, 10 - k_2) + \sum_{i=3}^l \max(0, 7 - k_i)))(T(r, f) + T(r, g)) \\ \leq S(r, f) + S(r, g), \end{aligned}$$

which is contradiction with $n > \frac{7}{s} + \frac{10}{p} + \max(0, 10 - k_2) + \sum_{i=3}^l \max(0, 7 - k_i)$.

Hence, by Lemma 1, $F' \equiv G'$ or $F'G' \equiv \alpha^2(z)$.

If $F'G' \equiv \alpha^2(z)$, by Lemma 2, we get a contradiction. If $F' \equiv G'$, then $F = G + c$, where c is a constant. From this equality, we conclude that

$$T(r, f) = T(r, g) + O(1).$$

If $c \neq 0$, by the second main theorem, we have

$$\begin{aligned} T(r, F) &\leq \bar{N}(r, F) + \bar{N}(r, \frac{1}{F}) + \bar{N}(r, \frac{1}{F-c}) + S(r, f) \\ &\leq \bar{N}(r, f) + \bar{N}(r, \frac{1}{f}) + \bar{N}(r, \frac{1}{Q(f)}) \\ &\quad + \bar{N}(r, \frac{1}{g}) + \bar{N}(r, \frac{1}{Q(g)}) + S(r, f) \\ &\leq (2k + 3)T(r, f) + S(r, f). \end{aligned}$$

Which implies that

$$(n - (k + 2))T(r, f) \leq S(r, f),$$

which is a contradiction with $n \geq k + 3$. Therefore $c \equiv 0$, then $P(f) \equiv P(g)$. Since $P(z)$ is polynomial of uniqueness for $\mathcal{M}(\mathbb{C})$ in Theorem 1, then $f \equiv g$. \blacksquare

Proofs of Theorem 2. Regarding Theorem 1, we can easily prove the statement in Theorem 2. Since f and g are entire functions, then $N(r, f) = N(r, g) = 0$. Thus, the inequality (14) can be replaced by

$$\begin{aligned} T(r, F) + T(r, G) &\leq (1 + \frac{7}{s})(N(r, \frac{1}{f}) + N(r, \frac{1}{g})) \\ &\quad + \sum_{i=3}^l (7 - k_i)(N(r, \frac{1}{f - a_i}) + N(r, \frac{1}{g - a_i})) \\ &\quad + (10 - k_2)(N(r, \frac{1}{f - a_2}) + N(r, \frac{1}{g - a_2})) \\ &\quad + k(T(r, f) + T(r, g)) + S(r, f) + S(r, g). \end{aligned}$$

This will lead to

$$\begin{aligned} (n - (\frac{7}{s} + \max(0, 10 - k_2)) + \sum_{i=3}^l \max(0, 7 - k_i))(T(r, f) + T(r, g)) \\ \leq S(r, f) + S(r, g). \end{aligned}$$

Which is contradiction with $n > \frac{7}{s} + \max(0, 10 - k_2) + \sum_{i=3}^l \max(0, 7 - k_i)$.

Hence, by Lemma 1, $F' \equiv G'$ or $F'G' \equiv \alpha^2(z)$.

If $F'G' \equiv \alpha^2(z)$, by Lemma 2, we get a contradiction. If $F' \equiv G'$, then $F = G + c$, where c is a constant. From this equality, we conclude that

$$T(r, f) = T(r, g) + O(1).$$

If $c \neq 0$, by the second main theorem and note that $\bar{N}(r, f) = 0$, we have

$$\begin{aligned} T(r, F) &\leq \bar{N}(r, F) + \bar{N}(r, \frac{1}{F}) + \bar{N}(r, \frac{1}{F-c}) + S(r, f) \\ &\leq \bar{N}(r, f) + \bar{N}(r, \frac{1}{f}) + \bar{N}(r, \frac{1}{Q(f)}) \\ &\quad + \bar{N}(r, \frac{1}{g}) + \bar{N}(r, \frac{1}{Q(g)}) + S(r, f) \\ &\leq (2k + 2)T(r, f) + S(r, f). \end{aligned}$$

Which implies that

$$(n - (k + 1))T(r, f) \leq S(r, f),$$

which is a contradiction with $n \geq k + 2$. Therefore $c \equiv 0$, then $P(f) \equiv P(g)$. Since $P(z)$ is polynomial of uniqueness for $\mathcal{A}(\mathbb{C})$ in Theorem 2, then $f \equiv g$. \blacksquare

Proofs of Theorem 3. In Theorem 3, we have $l = 2, k_2 = 1, a_2 = a$. By an argument as Theorem 1, we see

$$\begin{aligned} N_2(r, F') + N_2(r, \frac{1}{F'}) + N_2(r, G') + N_2(r, \frac{1}{G'}) \\ + 2(\bar{N}(r, \frac{1}{F'}) + \bar{N}(r, F')) + (\bar{N}(r, \frac{1}{G'}) + \bar{N}(r, G')) \\ \leq 2\bar{N}(r, f) + 2\bar{N}(r, \frac{1}{f}) + N(r, \frac{1}{f-a}) + N(r, \frac{1}{f'}) \\ + 2\bar{N}(r, f) + 2\bar{N}(r, \frac{1}{f}) + 2\bar{N}(r, \frac{1}{f-a}) + 2\bar{N}(r, \frac{1}{f'}) \\ + 2\bar{N}(r, g) + 2\bar{N}(r, \frac{1}{g}) + N(r, \frac{1}{g-a}) + N(r, \frac{1}{g'}) \\ + (\bar{N}(r, g) + \bar{N}(r, \frac{1}{g}) + \bar{N}(r, \frac{1}{g-a}) + \bar{N}(r, \frac{1}{g'})). \end{aligned}$$

This implies

$$\begin{aligned}
 (17) \quad & N_2(r, F') + N_2(r, \frac{1}{F'}) + N_2(r, G') + N_2(r, \frac{1}{G'}) \\
 & + 2(\overline{N}(r, \frac{1}{F'}) + \overline{N}(r, F')) + (\overline{N}(r, \frac{1}{G'}) + \overline{N}(r, G')) \\
 & \leq \frac{4}{p}N(r, f) + \frac{4}{s}N(r, \frac{1}{f}) + 3N(r, \frac{1}{f-a}) + N(r, \frac{1}{f'}) \\
 & + 2\overline{N}(r, \frac{1}{f'}) + \frac{3}{p}N(r, g) \\
 & + \frac{3}{s}N(r, \frac{1}{g}) + 2N(r, \frac{1}{g-a}) + N(r, \frac{1}{g'}) + \overline{N}(r, \frac{1}{g'})
 \end{aligned}$$

and similarly,

$$\begin{aligned}
 (18) \quad & N_2(r, G') + N_2(r, \frac{1}{G'}) + N_2(r, F') + N_2(r, \frac{1}{F'}) \\
 & + 2(\overline{N}(r, \frac{1}{G'}) + \overline{N}(r, G')) + (\overline{N}(r, \frac{1}{F'}) + \overline{N}(r, F')) \\
 & \leq 2\overline{N}(r, g) + 2\overline{N}(r, \frac{1}{g}) + N(r, \frac{1}{g-a}) + N(r, \frac{1}{g'}) \\
 & + 2\overline{N}(r, g) + 2\overline{N}(r, \frac{1}{g}) + 2\overline{N}(r, \frac{1}{g-a}) + 2\overline{N}(r, \frac{1}{g'}) \\
 & + 2\overline{N}(r, f) + 2\overline{N}(r, \frac{1}{f}) + N(r, \frac{1}{f-a}) + N(r, \frac{1}{f'}) \\
 & + (\overline{N}(r, f) + \overline{N}(r, \frac{1}{f}) + \overline{N}(r, \frac{1}{f-a}) + \overline{N}(r, \frac{1}{f'})) \\
 & \leq \frac{4}{p}N(r, g) + \frac{4}{s}N(r, \frac{1}{g}) + 3N(r, \frac{1}{g-a}) + N(r, \frac{1}{g'}) + 2\overline{N}(r, \frac{1}{g'}) \\
 & + \frac{3}{p}N(r, f) + \frac{3}{s}N(r, \frac{1}{f}) + 2N(r, \frac{1}{f-a}) + N(r, \frac{1}{f'}) + \overline{N}(r, \frac{1}{f'}).
 \end{aligned}$$

From (1), (2), (5), (6), (17) and (18), we have

$$\begin{aligned}
 (19) \quad & T(r, F) \leq N(r, \frac{1}{f}) + N(r, \frac{1}{Q(f)}) - N(r, \frac{1}{f-a}) - N(r, \frac{1}{f'}) \\
 & + \frac{4}{p}N(r, f) + \frac{4}{s}N(r, \frac{1}{f}) + 3N(r, \frac{1}{f-a}) + N(r, \frac{1}{f'}) \\
 & + 2\overline{N}(r, \frac{1}{f'}) + \frac{3}{p}N(r, g) + \frac{3}{s}N(r, \frac{1}{g}) \\
 & + 2N(r, \frac{1}{g-a}) + \overline{N}(r, \frac{1}{g'}) + S(r, f) + S(r, g).
 \end{aligned}$$

Similarly,

$$\begin{aligned}
 (20) \quad T(r, G) &\leq N(r, \frac{1}{g}) + N(r, \frac{1}{Q(g)}) - N(r, \frac{1}{g-a}) - N(r, \frac{1}{g'}) \\
 &\quad + \frac{4}{p}N(r, g) + \frac{4}{s}N(r, \frac{1}{g}) + 3N(r, \frac{1}{g-a}) + N(r, \frac{1}{g'}) \\
 &\quad + 2\bar{N}(r, \frac{1}{g'}) + \frac{3}{p}N(r, f) + \frac{3}{s}N(r, \frac{1}{f}) \\
 &\quad + 2N(r, \frac{1}{f-a}) + \bar{N}(r, \frac{1}{f'}) + S(r, f) + S(r, g).
 \end{aligned}$$

Thus, from (19) and (20), we get

$$\begin{aligned}
 (21) \quad T(r, F) + T(r, G) &\leq (1 + \frac{7}{s})(N(r, \frac{1}{f}) + N(r, \frac{1}{g})) \\
 &\quad + 4(N(r, \frac{1}{f-a}) + N(r, \frac{1}{g-a})) + \frac{7}{p}(N(r, f) + N(r, g)) \\
 &\quad + N(r, \frac{1}{Q(f)}) + N(r, \frac{1}{Q(g)}) + 3(N(r, \frac{1}{f'}) + N(r, \frac{1}{g'})) \\
 &\quad + S(r, f) + S(r, g).
 \end{aligned}$$

We have

$$\begin{aligned}
 N(r, \frac{1}{f'}) + N(r, \frac{1}{g'}) &\leq N(r, \frac{1}{f-a}) + N(r, \frac{1}{g-a}) \\
 &\quad + \bar{N}(r, f) + \bar{N}(r, g) + S(r, f) + S(r, g) \\
 &\leq N(r, \frac{1}{f-a}) + N(r, \frac{1}{g-a}) + \frac{1}{p}(N(r, f) \\
 &\quad + N(r, g)) + S(r, f) + S(r, g).
 \end{aligned}$$

Since $P(x) = x^{n+1}Q(x)$, $\deg Q = 1$, from (21), we have

$$\begin{aligned}
 T(r, F) + T(r, G) &\leq (1 + \frac{7}{s})(N(r, \frac{1}{f}) + N(r, \frac{1}{g})) \\
 &\quad + 4(N(r, \frac{1}{f-a}) + N(r, \frac{1}{g-a})) \\
 &\quad + 3(N(r, \frac{1}{f-a}) + N(r, \frac{1}{g-a})) + \frac{10}{p}(N(r, f) + N(r, g)) \\
 &\quad + T(r, f) + T(r, g) + S(r, f) + S(r, g).
 \end{aligned}$$

Thus, we obtain

$$(n - (7 + \frac{7}{s} + \frac{10}{p}))(T(r, f) + T(r, g)) \leq S(r, f) + S(r, g),$$

which is contradiction with $n > 7 + \frac{7}{s} + \frac{10}{p}$.

Hence, by Lemma 1, $F' \equiv G'$ or $F'G' \equiv \alpha^2(z)$.

If $F'G' \equiv \alpha^2(z)$, by Lemma 2, we get a contradiction. If $F' \equiv G'$, then $F = G + c$, where c is a constant. From this equality, we conclude that

$$T(r, f) = T(r, g) + O(1).$$

If $c \neq 0$, by the second main theorem, we have

$$\begin{aligned} T(r, F) &\leq \bar{N}(r, F) + \bar{N}(r, \frac{1}{F}) + \bar{N}(r, \frac{1}{F-c}) + S(r, f) \\ &\leq \bar{N}(r, f) + \bar{N}(r, \frac{1}{f}) + \bar{N}(r, \frac{1}{Q(f)}) \\ &\quad + \bar{N}(r, \frac{1}{g}) + \bar{N}(r, \frac{1}{Q(g)}) + S(r, f) \\ &\leq 5T(r, f) + S(r, f). \end{aligned}$$

Which implies that

$$(n-3)T(r, f) \leq S(r, f),$$

which is a contradiction with $n > 7 + \frac{7}{s} + \frac{10}{p}$. Therefore $c \equiv 0$, then $P(f) \equiv P(g)$. Then, we obtain

$$P(f) = \frac{f^{n+2}}{n+2} - a \frac{f^{n+1}}{n+1}, \quad P(g) = \frac{g^{n+2}}{n+2} - a \frac{g^{n+1}}{n+1}.$$

From $P(f) = P(g)$, we have

$$(22) \quad f^{n+1} \left(f - a \frac{n+2}{n+1} \right) = g^{n+1} \left(g - a \frac{n+2}{n+1} \right).$$

Let $h = \frac{f}{g}$. If $h \equiv 1$ then $f \equiv g$. If $h \not\equiv 1$, then $f \not\equiv g$. Hence, from (22), we deduce that

$$f = \frac{a(n+2)(1-h^{n+1})}{(n+1)(1-h^{n+2})}h, \quad g = \frac{a(n+2)(1-h^{n+1})}{(n+1)(1-h^{n+2})}.$$

■

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