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Luong Quoc Tuyen

CAUCHY SYMMETRIC SPACES WITH POINT-COUNTABLE cs-NETWORKS

Abstract. In this paper, we prove that a Cauchy symmetric space has a point-countable cs-network if and only if it is a 1-sequence-covering compact-covering quotient π, s-image of a metric space; if and only if it is a sequence-covering quotient π, s-image of a metric space.

Key words: Cauchy symmetric, σ-strong network, cs-network, point-countable, 1-sequence-covering map, π-map, s-map.

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1. Introduction and preliminaries

One of the central problems in general topology is to establish relationships between various topological spaces and metric spaces by means of various maps. Some characterizations for certain quotient π-images of metric spaces are obtained by means of σ-strong networks ([8]), and some characterizations around sequence-covering quotient π-images of metric spaces are obtained in terms of symmetric spaces ([13]).

In this paper, we prove that a Cauchy symmetric space has a point-countable cs-network if and only if it is a 1-sequence-covering compact-covering quotient π, s-image of a metric space, if and only if it is a sequence-covering quotient π, s-image of a metric space.

We assume that all spaces are T_1 and regular, all maps are continuous and onto, $\mathbb{N}$ denotes the set of all natural numbers. Let $\mathcal{P}$ and $\mathcal{Q}$ be two families of subsets of X, we denote $(\mathcal{P})_x = \{P \in \mathcal{P} : x \in P\}$, $\mathcal{P} \bigwedge \mathcal{Q} = \{P \cap Q : P \in \mathcal{P}, Q \in \mathcal{Q}\}$ and $\mathtt{St}(x, P) = \{P \in \mathcal{P} : x \in P\}$. For a sequence $\{x_n\}$ converging to x, we say that $\{x_n\}$ is *eventually* in P, if $\{x\} \bigcup \{x_n : n \geq m\} \subset P$ for some $m \in \mathbb{N}$, and $\{x_n\}$ is *frequently* in P, if some subsequence of $\{x_n\}$ is eventually in P.

Definition 1 ([13]). *Let $\mathcal{P}$ be a family of subsets of a space X.*

(a) $\mathcal{P}$ *is* point-countable, *if each point* $x \in X$ *belongs to only countably many members of* $\mathcal{P}$.

(b) $\mathcal{P}$ *is a* network at x *in* X, *if* $x \in P$ *for every* $P \in \mathcal{P}$, *and whenever* $x \in U$ *with* U *is open in* X, *then* $x \in P \subset U$ *for some* $P \in \mathcal{P}$.

(c) $\mathcal{P}$ *is a cs*-network *for* X, *if each sequence* S *converging to a point* $x \in U$ *with* U *open in* X, S *is eventually in* $P \subset U$ *for some* $P \in \mathcal{P}$.

(d) $\mathcal{P}$ *is a cfp-cover for* X, *if whenever* K *is compact subset of* X, *there exists a finite family* $\{K_i : i \leq n\}$ *of closed subsets of* K *and* $\{P_i : i \leq n\} \subset \mathcal{P}$ *such that* $K = \bigcup\{K_i : i \leq n\}$ *and each* $K_i \subset P_i$.

(e) $\mathcal{P}$ *is a cs*-cover *for* X, *if every convergent sequence is eventually in some* $P \in \mathcal{P}$.

(f) $\mathcal{P}$ *is an sn*-cover *for* X, *if for every* $P \in \mathcal{P}$, P *is a sequential neighborhood of some* $x \in X$, *and for every* $x \in X$ *there exists* $P \in \mathcal{P}$ *such that* P *is a sequential neighborhood of* x.

Definition 2 ([2]). *Let* $\mathcal{P} = \bigcup\{\mathcal{P}_x : x \in X\}$ *be a cover of a space* X. *Assume that* $\mathcal{P}$ *satisfies the following (1) and (2) for every* $x \in X$.

(a) $\mathcal{P}_x$ *is a network at* x.

(b) *If* $P_1, P_2 \in \mathcal{P}_x$, *then* $P \subset P_1 \cap P_2$ *for some* $P \in \mathcal{P}_x$.

$\mathcal{P}$ *is a* weak base *for* X, *if for* $G \subset X$, G *is open in* X *if and only if for every* $x \in G$, *there exists* $P \in \mathcal{P}_x$ *such that* $P \subset G$; $\mathcal{P}_x$ *is said to be a* weak neighborhood base *at* x.

Definition 3 ([5, 11]). *Let* d *be a* d*-function on a space* X.

(a) *For each* $x \in X$, $n \in \mathbb{N}$, *let*

$$S_n(x) = \Big\{y \in X : d(x, y) < \frac{1}{n}\Big\}.$$

(b) *For every* $P \subset X$, *put*

$$d(P) = \sup\{d(x, y) : x, y \in P\}.$$

(c) X *is* symmetric, *if* $\{S_n(x) : n \in \mathbb{N}\}$ *is a weak neighborhood base at* x *for each* $x \in X$.

(d) X *is* Cauchy symmetric, *if* X *is symmetric and every convergent sequence is* d*-Cauchy.*

Remark 1 ([11]). X is Cauchy symmetric if and only if for each $x \in X$, $d\big(S_n(x)\big)$ converges to 0.

Definition 4 ([8]). *Let* $\{\mathcal{P}_n : n \in \mathbb{N}\}$ *be a sequence of covers of a space* X *such that* $\mathcal{P}_{n+1}$ *refines* $\mathcal{P}_n$ *for every* $n \in \mathbb{N}$.

(a) $\bigcup\{\mathcal{P}_n : n \in \mathbb{N}\}$ *is a* σ-strong network *for* X, *if* $\{St(x, \mathcal{P}_n) : n \in \mathbb{N}\}$ *is a network at each point* $x \in X$.

(*b*) $\bigcup\{\mathcal{P}_n : n \in \mathbb{N}\}$ *is a* σ-point-countable strong network *for* X*, if it is a* σ*-strong network and each* $\mathcal{P}_n$ *is point-countable.*

(*c*) $\bigcup\{\mathcal{P}_n : n \in \mathbb{N}\}$ *is a* σ-point-countable strong network consisting of *cs*-covers *(resp., sn-*covers*) for* X*, if it is a* σ*-strong network and each* $\mathcal{P}_n$ *is a point-countable cs-cover (resp., sn-cover).*

Definition 5 ([1, 8, 13])**.** *Let* $f : X \to Y$ *be a map.*

(*a*) f *is* weak-open, *if there exists a weak base* $\mathcal{B} = \bigcup\{\mathcal{B}_y : y \in Y\}$ *for* Y*, and for every* $y \in Y$*, there exists* $x \in f^{-1}(y)$ *such that for each open neighborhood* U *of* x*,* $B \subset f(U)$ *for some* $B \in \mathcal{B}_y$*.*

(*b*) f *is* 1-sequence-covering, *if for each* $y \in Y$*, there exists* $x \in f^{-1}(y)$ *such that each sequence converging to* y *is an image of some sequence converging to* x*.*

(*c*) f *is* sequence-covering, *if every convergent sequence of* Y *is the image of some convergent sequence of* X*.*

(*d*) f *is* compact-covering, *if for each compact subset* K *of* Y*, there exists a compact subset* L *of* X *such that* $f(L) = K$*.*

(*e*) f *is* quotient, *if whenever* $U \subset Y$*,* U *open in* Y *if and only if* $f^{-1}(U)$ *open in* X*.*

(*f*) f *is a* π-map, *if for every* $y \in Y$ *and for every neighborhood* U *of* y *in* Y*,* $d\big(f^{-1}(y); X - f^{-1}(U)\big) > 0$*, where* X *is a metric space with a metric* d*.*

(*g*) f *is an* s-map, *if* $f^{-1}(y)$ *is separable in* X *for each* $y \in Y$*.*

Notation. *Let* $\bigcup\{\mathcal{P}_n : n \in \mathbb{N}\}$ *be a* σ*-strong network for a space* X*. For each* $n \in \mathbb{N}$*, put* $\mathcal{P}_n = \{P_\alpha : \alpha \in \Lambda_n\}$ *and endow* Λ_n *with the discrete topology. Then,*

$$M = \Big\{\alpha = (\alpha_n) \in \prod_{n\in\mathbb{N}} \Lambda_n : \{P_{\alpha_n}\} \text{ forms a network at some point } x_\alpha \in X\Big\}$$

is a metric space and the point x_α *is unique in* X *for every* $\alpha \in M$*. Define* $f : M \to X$ *by* $f(\alpha) = x_\alpha$*. Let us call* $(f, M, X, \mathcal{P}_n)$ *a* Ponomarev's system, *following [4].*

For some undefined or related concepts, we refer the reader to [3], [12] and [13].

2. Main results

Theorem 1. *The following are equivalent for a space* X*.*

(*a*) X *is a Cauchy symmetric space has a point-countable cs-network;*

(*b*) X *is a 1-sequence-covering compact-covering quotient* π*, s-image of metric space;*

(*c*) *X is a sequence-covering quotient* π, *s-image of metric space.*

Proof. $(a) \Longrightarrow (b)$. Let X be a Cauchy symmetric and $\mathcal{U}$ be a point-countable cs-network for X. We can assume that $\mathcal{U}$ is closed under finite intersections. Put

$$\mathcal{P}_x = \big\{P \in \mathcal{U} : S_n(x) \subset P \ \text{ for some } \ n \in \mathbb{N}\big\}.$$

Claim. For each U open in X and $x \in U$, there exists $P \in \mathcal{P}_x$ such that $P \subset U$.

In fact, conversely assume that there exist U open in X and $x \in U$ such that $P \not\subset U$ for all $P \in \mathcal{P}_x$. Let

$$\{P \in \mathcal{P}_x : x \in P \subset U\} = \{P_m(x) : m \in \mathbb{N}\}.$$

Then $S_n(x) \not\subset P_m(x)$ for all $n, m \in \mathbb{N}$, so choose $x_{n,m} \in S_n(x) - P_m(x)$. For $n \geq m$, we denote $x_{n,m} = y_k$ with $k = m + n(n-1)/2$. Because $\{S_n(x)\}$ is a decreasing weak neighborhood base at x, the sequence $\{y_k : k \in \mathbb{N}\}$ converges to the point x in X. Thus, there exist $m, i \in \mathbb{N}$ such that

$$\{x\} \bigcup \{y_k : k \geq i\} \subset P_m(x) \subset U.$$

Take $j \geq i$ with $y_j = x_{n,m}$ for some $n \geq m$. Then $x_{n,m} \in P_m(x)$. This is a contradiction.

Then we have

(*a*) $\mathcal{P}_x$ is a network at x in X. Let U be an open subset of X and $x \in U$. Then there exists $P \in \mathcal{P}_x$ such that $P \subset U$ by the Claim.

(*b*) Let $P_1, P_2 \in \mathcal{P}_x$ and $P = P_1 \cap P_2$. Hence, there exist $n, m \in \mathbb{N}$ such that $S_m(x) \subset P_1$ and $S_n(x) \subset P_2$. If put $k = \max\{m, n\}$, then $S_k(x) \subset P \in \mathcal{U}$. Thus, $P \in \mathcal{P}_x$ and $P \subset P_1 \cap P_2$.

(*c*) Let U be an open subset of X. By the Claim, there exists $P \in \mathcal{P}_x$ such that $P \subset G$. Conversely, if $G \subset X$ satisfies that for each $x \in G$, there exists $P \in \mathcal{P}_x$ with $P \subset G$, then for each $x \in G$, there exists $n \in \mathbb{N}$ such that $S_n(x) \subset G$. Because $\{S_n(x)\}$ is a weak neighborhood at x for all $x \in X$, G is open in X.

Therefore, $\mathcal{P} = \bigcup\{\mathcal{P}_x : x \in X\}$ is a weak base for X and $\mathcal{P} \subset \mathcal{U}$.

Since $\mathcal{U}$ is a σ-point-countable cs-network, $\mathcal{P}$ is a σ-point-countable weak base. For each $m, n \in \mathbb{N}$, put

$$\mathcal{Q}_{m,n}(x) = \Big\{P \in \mathcal{P}_x : S_m(x) \subset P \ \text{ and } \ d(P) < \frac{1}{n}\Big\},$$

$$A_{m,n} = \{x \in X : \mathcal{Q}_{m,n}(x) = \emptyset\},$$

$$B_{m,n} = X - A_{m,n}$$

$$\mathcal{Q}_{m,n} = \bigcup\{\mathcal{Q}_{m,n}(x) : x \in B_{m,n}\},$$

$$\mathcal{F}_{m,n} = \mathcal{Q}_{m,n} \bigcup\{A_{m,n}\}$$

Then, each $\mathcal{F}_{m,n}$ is point-countable. Furthermore, we have

(i) *Each* $\mathcal{F}_{m,n}$ *is a cs-cover for* X.

Let $x \in X$ and $S = \{x_i : i \in \mathbb{N}\}$ be a sequence converging to x in X, then

Case 1. If $x \in B_{m,n}$, then there is $P \in \mathcal{Q}_{m,n}(x)$ such that $S_m(x) \subset P$. Hence, S is eventually in $P \in \mathcal{F}_{m,n}$.

Case 2. If $x \notin B_{m,n}$ and $S \cap B_{m,n}$ is finite, then S is eventually in $A_{m,n} \in \mathcal{F}_{m,n}$.

Case 3. If $x \notin B_{m,n}$ and $S \cap B_{m,n}$ is infinite, then we can assume that

$$S \cap B_{m,n} = \{x_{i_k} : k \in \mathbb{N}\}.$$

Since X is Cauchy symmetric and S converges to x, there exists $n_0 \in \mathbb{N}$ such that

$$d(x_i, x_j) < \frac{1}{m} \text{ and } d(x, x_i) < \frac{1}{m} \text{ for every } i, j \geq n_0.$$

Now, we pick $k_0 \in \mathbb{N}$ such that $i_{k_0} \geq n_0$. Because

$$d(x_{i_{k_0}}, x) < \frac{1}{m} \text{ and } d(x_{i_{k_0}}, x_i) < \frac{1}{m} \text{ for every } i \geq n_0,$$

it implies that S is eventually in $S_m(x_{i_{k_0}})$. Furthermore, since $x_{i_{k_0}} \in B_{m,n}$, we get $S_m(x_{i_{k_0}}) \subset P$ for some $P \in Q_{m,n}(x_{i_{k_0}})$. Hence, $P \in \mathcal{F}_{m,n}$ and S is eventually in P.

Therefore, each $\mathcal{F}_{m,n}$ is a *cs*-cover for X.

(ii) $\{\mathtt{St}(x, \mathcal{F}_{m,n}) : m, n \in \mathbb{N}\}$ *is a network at* x.

Let $x \in U$ with U is open in X. Then, $S_n(x) \subset U$ for some $n \in \mathbb{N}$. Since X is Cauchy symmetric, there exists $j \in \mathbb{N}$ such that $d\big(S_j(x)\big) < 1/n$. Furthermore, we have $P \subset S_j(x)$ for some $P \in \mathcal{P}_x$. Indeed, since $\mathcal{P}$ is point-countable, we can put

$$\mathcal{P}_x = \{P_n(x) : n \in \mathbb{N}\}.$$

On the other hand, because $\mathcal{P}$ is a weak base, we can choose sequence $\{n_i : i \in \mathbb{N}\}$ such that $\{P_{n_i}(x) : i \in \mathbb{N}\}$ is a decreasing network at x. Then, there exists $i \in \mathbb{N}$ such that $P_{n_i}(x) \subset S_j(x)$.

Because P is a sequential neighborhood at x, there exists $i \in \mathbb{N}$ such that $S_m(x) \subset P$. If not, for each $n \in \mathbb{N}$, there exists $x_n \in S_n(x) - P$. Hence, $\{x_n\}$ converges to x. Then, there exists $k \in \mathbb{N}$ such that $x_n \in P$ for every $n \geq k$. This is a contradiction.

Then, we have

$$S_m(x) \subset P \in \mathcal{P}_x.$$

Since $d\big(S_j(x)\big) < 1/n$, we get $d(P) < 1/n$. This implies that $P \in \mathcal{F}_{m,n}$. Then, we have

$$\mathtt{St}(x, \mathcal{F}_{m,n}) \subset S_n(x) \subset U.$$

It follows that $\{\mathtt{St}(x, \mathcal{F}_{m,n}) : m, n \in \mathbb{N}\}$ is a network at x.

Next, we write

$$\{\mathcal{F}_{m,n} : m, n \in \mathbb{N}\} = \{\mathcal{H}_n : n \in \mathbb{N}\},$$

and for each $n \in \mathbb{N}$, put

$$\mathcal{G}_n = \bigwedge\{\mathcal{H}_i : i \leq n\}.$$

Then, $\bigcup\{\mathcal{G}_n : n \in \mathbb{N}\}$ is a σ-point-countable strong network consisting of cs-covers for X.

For each $i \in \mathbb{N}$, put

$$\mathcal{L}_i = \big\{P \in \mathcal{G}_i : \text{ there exist } x \in X, n \in \mathbb{N} \text{ such that } S_n(x) \subset P\big\}.$$

Then,

(a) For each $x \in X$, by using the proof of the Claim, there exist $n \in \mathbb{N}$ and $P \in \mathcal{G}_i$ such that $S_n(x) \subset P$, it implies that P is a sequential neighborhood of x and $P \in \mathcal{L}_i$.

(b) For each $P \in \mathcal{L}_i$, there exist $x \in X$ and $n \in \mathbb{N}$ such that $S_n(x) \subset P$. This implies that P is a sequential neighborhood at x.

(c) Since each $\mathcal{L}_i \subset \mathcal{G}_i$, $\bigcup\{\mathcal{P}_n : n \in \mathbb{N}\}$ is a σ-point-countable strong network.

Therefore, $\bigcup\{\mathcal{L}_n : n \in \mathbb{N}\}$ is a σ-point-countable strong network consisting of sn-covers.

Finally, consider the Ponomarev's system $(f, M, X, \mathcal{L}_n)$. Because each $\mathcal{L}_n$ is a point-countable sn-cover, it follow from Theorem 3.7 (1) and Theorem 3.10 in [4] that f is a 1-sequence-covering s-map. On the other hand, since each $\mathcal{L}_n$ is an sn-cover, each $\mathcal{L}_n$ is a cfp-cover by Lemma 3.10 in [13]. Thus, f is compact-covering by Lemma 2.2 in [13]. Furthermore, since X is sequential, it follows from Lemma 3.5 in [10] that f is quotient.

(b) $\Longrightarrow$ (c). It is obvious.

(c) $\Longrightarrow$ (a). Let X be a sequence-covering quotient π, s-image of a metric space. By Theorem 3.11 in [13], X is Cauchy symmetric. Furthermore, it follows from Theorem 1.1 in [6] that X has a point-countable cs-network. ■

By Corollary 2.8 [1] and Theorem 1, we have

Corollary 1. *The following are equivalent for a space X.*
(a) *X is a Cauchy symmetric with a point-countable cs-network;*
(b) *X is a weak-open compact-covering π, s-image of a metric space;*
(c) *X is a weak-open π, s-image of a metric space;*

Example 1 ([7])**.** There exists a Hausdorff space with a countable base, which is not a symmetric space. There also exists a regular Cauchy symmetric space without point-countable *cs*-networks.

Question. *Is a Cauchy symmetric space has a point-countable cs*-network a space with a point-countable cs-network?*

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LUONG QUOC TUYEN
DEPARTMENT OF MATHEMATICS
DA NANG UNIVERSITY
VIETNAM
e-mail: tuyendhdn@gmail.com

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