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CONTRA (m_X, m_Y)-SEMICONTINUOUS FUNCTIONS IN m-SPACES

Abstract. In this paper, we introduce the notion of contra (m_X, m_Y)-semicontinuous functions between m-spaces. We obtain many characterizations of these functions and deal with decompositions of the functions and other related functions.

Key words: contra (m_X, m_Y)-semicontinuity, m_X-semi-closed set, m_X-semi-open set, minimal structure, minimal space.

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1. Introduction

Generalizations of open sets in a topological space: α-sets [8], preopen sets [3], semi-open sets [1] and β-open sets etc are very important for generalizing continuity in topological spaces. Various generalizations of continuity are defined and investigated by many authors. As a generalization of the topology, Maki [2] define the notion of minimal structures. A subfamily m of the power set $P(X)$ on a nonemty set X is called a minimal structure [2] if $\emptyset \in m$ and $X \in m$. The pair (X, m) is called a minimal space. The elements of m are said to be m-open. Recently, several generalizations of m-open sets have been defined and investigated in [4, 5, 6] and [15]. Quite recently, Sengul and Rosas [14] introduced the notion of contra (m_X, m_Y)-continuity between m-spaces.

The purpose of the present paper is to introduce and study the notion of contra (m_X, m_Y)-semicontinuous functions between m-spaces. In Section 3, we obtain many characterizations of contra (m_X, m_Y)-semicontinuity. In Section 4, we deal with decompositions of contra (m_X, m_Y)-semicontinuity and other related functions. The last section gives some properties of strongly $S - m_X$-closed spaces.

2. Preliminaries

Definition 1 ([2, 11]). *A subfamily* m_X *of the power set* $P(X)$ *of a nonempty set* X *is called a minimal structure (briefly, m-structure) on* X *if* $\emptyset \in m_X$ *and* $X \in m_X$. *The pair* (X, m_X) *is called a minimal space (briefly, m-space). A member of* m_X *is said to be* m_X*-open and the complement of an* m_X*-open set is said to be* m_X*-closed.*

Definition 2 ([2, 11]). *Let* (X, m_X) *be a minimal space. For a subset* A *of* X, *the* m_X*-closure of* A *and the* m_X*-interior of* A *are defined as follows:*
(1) $m_X - Cl(V) = \bigcap\{F : A \subseteq F, X - F \in m_X\}$.
(2) $m_X - Int(V) = \bigcup\{U : U \subseteq A, U \in m_X\}$.

Lemma 1 ([2, 11]). *Let* (X, m_X) *be a minimal space and* $A, B \subseteq X$. *Then the followings hold:*
(1) $m_X - Cl(\emptyset) = \emptyset$, $m_X - Cl(X) = X$.
(2) $m_X - Int(\emptyset) = \emptyset$, $m_X - Int(X) = X$.
(3) *If* $X - A \in m_X$, *then* $m_X - Cl(A) = A$.
(4) *If* $A \in m_X$, *then* $m_X - Int(A) = A$.
(5) $A \subseteq m_X - Cl(A)$, $m_X - Int(A) \subseteq A$.
(6) $m_X - Cl(X - A) = X - (m_X - Int(A))$.
(7) $m_X - Int(X - A) = X - (m_X - Cl(A))$.
(8) $m_X - Cl(m_X - Cl(A)) = m_X - Cl(A)$.
(9) $m_X - Int(m_X - Int(A)) = m_X - Int(A)$.
(10) *If* $A \subseteq B$, *then* $m_X - Cl(A) \subseteq m_X - Cl(B)$.
(11) *If* $A \subseteq B$, *then* $m_X - Int(A) \subseteq m_X - Int(B)$.

Definition 3 ([2]). *Let* (X, m_X) *be a minimal space. The m-structure* m_X *is said to have property* $\mathcal{B}$ *if the union of any family of subsets belonging to* m_X *belongs to* m_X.

Lemma 2 ([11]). *Let* (X, m_X) *be a minimal space and* m_X *satisfy property of* $\mathcal{B}$. *For* $A \subseteq X$, *the followings hold:*
(1) $A \in m_X$ *if and only if* $m_X - Int(A) = A$.
(2) A *is* m_X*-closed if and only if* $m_X - Cl(A) = A$.
(3) $m_X - Int(A) \in m_X$.
(4) $m_X - Cl(A)$ *is* m_X*-closed.*

Lemma 3 ([11]). *Let* (X, m_X) *be a minimal space and* $A \subseteq X$. *Then* $x \in m_X - Cl(A)$ *if and only if* $U \cap A \neq \emptyset$ *for every* $U \in m_X$ *such that* $x \in U$.

Definition 4. *Let* (X, m_X) *be a minimal space. A subset* A *of* X *is said to be* m_X*-clopen if it is* m_X*-open and* m_X*-closed.*

Definition 5. *Let* (X, m_X) *be a minimal space. A subset* A *of* X *is called*

(1) *an αm_X - open set [6] if* $A \subseteq m_X - Int(m_X - Cl(m_X - Int(A)))$.
(2) *an m_X-preopen set [4, 13] if* $A \subseteq m_X - Int(m_X - Cl(A))$.
(3) *a β - m_X-open set [7, 15] if* $A \subseteq m_X - Cl(m_X - Int(m_X - Cl(A)))$.

Definition 6 ([5]). *Let (X, m_X) be a minimal space. A subset A of X is called an m_X-semiopen set if $A \subseteq m_X - Cl(m_X - Int(A))$. The complement of an m_X-semiopen set is called an m_X-semiclosed set. The family of all m_X-semiopen sets in X is denoted by $MSO(X)$.*

Lemma 4 ([5]). *Let (X, m_X) be a minimal space and $A \subseteq X$. Then*
(1) *A is an m_X-semiclosed set if and only if $m_X - Int(m_X - Cl(A)) \subseteq A$.*
(2) *$MSO(X)$ is a minimal structure with property $\mathcal{B}$.*

Definition 7 ([5]). *Let (X, m_X) be a minimal space and $A \subseteq X$. The m_X-semi-closure of A and the m_X-semi-interior of A are defined as follows:*
(1) $m_X sCl(A) = \bigcap\{F : A \subseteq F, F$ *is m_X-semiclosed in X*$\}$.
(2) $m_X sInt(A) = \bigcup\{U : U \subseteq A, U$ *is m_X-semiopen in X*$\}$.

Lemma 5. *Let (X, m_X) be a minimal space. For a subset of A of X, the following hold:*
(1) *A is m_X-semiopen if and only if $m_X sInt(A) = A$.*
(2) *A is m_X-semiclosed if and only if $m_X sCl(A) = A$.*

Proof. This follows easily from Lemmas 2 and 4. ■

Definition 8 ([13]). *Let (X, m_X) be a minimal space. Then a subset A of X is said to be m_X-gs-closed if $m_X sCl(A) \subseteq U$ whenever $A \subseteq U$ and $U \in m_X$.*

Definition 9 ([13]). *Let (X, m_X) be a minimal space. Then $A \subseteq X$ is called an m_X-regular open set if $A = m_X - Int(m_X - Cl(A))$. Also $A \subseteq X$ is called an m_X-regular closed set if $X - A$ is m_X-regular open.*

If A is m_X-closed, then $m_X - cl(A) = A$ but the converse is not always true. Therefore, m_X-regular open (resp. m_X - regular closed) is not always m_X-open (resp. m_X-closed).

Definition 10 ([12]). *A subset U of a nonempty set X with a minimal structure m_X is said to be m_X-compact relative to (X, m_X) if any cover of U by m_X-open sets has a finite subcover.*

Definition 11 ([14]). *Let (X, m_X) and (Y, m_Y) be two minimal spaces. Then a function $f : (X, m_X) \to (Y, m_Y)$ is said to be contra (m_X, m_Y)-continuous if $f^{-1}(V) = m_X - Cl(f^{-1}(V))$ for every m_Y-open set V of Y.*

3. Contra (m_X, m_Y)-semi continuous functions

In this section, we introduce the concept of a contra (m_X, m_Y)-semi continuous function between m-spaces and investigate some characterizations of this continuity.

Definition 12. *Let (X, m_X) and (Y, m_Y) be two minimal spaces. Then a function $f : (X, m_X) \to (Y, m_Y)$ is said to be contra (m_X, m_Y)-semi continuous if $f^{-1}(V)$ is m_X-semiclosed in X for every m_Y-open set V of Y.*

Lemma 6. *Every contra (m_X, m_Y)-continuous function is contra (m_X, m_Y)-semi continuous.*

Proof. Let $f : (X, m_X) \to (Y, m_Y)$ be a contra (m_X, m_Y)-continuous function and V be any m_Y-open set of Y. Then $m_X - Cl(f^{-1}(V)) = f^{-1}(V)$ and $m_X - Int(m_X - Cl(f^{-1}(V))) = m_X - Intf^{-1}(V) \subseteq f^{-1}(V)$. Therefore, Lemma 4, $f^{-1}(V)$ is m_X-semiclosed and f is contra (m_X, m_Y)-semi continuous. ■

Remark 1. The converse of Lemma 6 is not always true as the following example shows.

Example 1. Let $X = \{a, b, c\}$ and m_{X_1}, m_{X_2} be two minimal structures on X as follows:

$$m_{X_1} = \{\emptyset, X, \{b\}, \{c\}, \{b, c\}\},$$
$$m_{X_2} = \{\emptyset, X, \{c\}\}.$$

Define a function $f : (X, m_{X_1}) \to (X, m_{X_2})$ as follows:

$$f(a) = b, \quad f(b) = c, \quad f(c) = a.$$

Then f is contra (m_X, m_Y)-semi continuous, but it is not contra (m_X, m_Y) -continuous.

Theorem 1. *A function $f : (X, m_X) \to (Y, m_Y)$ is contra (m_X, m_Y)-semi continuous if and only if $f : (X, MSO(X)) \to (Y, m_Y)$ is contra (m_X, m_Y)-continuous.*

Proof. *Necessity.* Let $f : (X, m_X) \to (Y, m_Y)$ be contra (m_X, m_Y)-semi continuous and V be any m_Y-open set of Y. Then, by hypothesis $f^{-1}(V)$ is m_X-semiclosed in X and, by Lemma 5, $f^{-1}(V) = m_X sCl(f^{-1}(V))$. Therefore, $f : (X, MSO(X)) \to (Y, m_Y)$ is contra (m_X, m_Y)-continuous.

Sufficieny. Let V be any m_Y-open set of Y. By hypothesis, $f^{-1}(V) = m_X sCl(f^{-1}(V))$ and, by Lemma 5, $f^{-1}(V)$ is m_X-semi-closed. Therefore, $f : (X, m_X) \to (Y, m_Y)$ is contra (m_X, m_Y)-semi continuous. ■

Definition 13. *Let (X, m_X) and (Y, m_Y) be two minimal spaces. Then a function $f : (X, m_X) \to (Y, m_Y)$ is said to be contra (m_X, m_Y)-semicontinuous at $x \in X$ if for each m_Y-closed V of Y containing $f(x)$, there exists an m_X-semiopen set U of X containing x such that $f(U) \subseteq V$.*

Theorem 2. *Let (X, m_X), (Y, m_Y) be two minimal spaces. A function $f : (X, m_X) \to (Y, m_Y)$ is contra (m_X, m_Y)-semi continuous if and only if f is contra (m_X, m_Y)-semicontinuous at each point $x \in X$.*

Proof. *Necessity.* Let $x \in X$ and V be any m_Y-closed set of Y containing $f(x)$. Then $Y - V$ is m_Y-open. By hypothesis, $f^{-1}(Y - V)$ is an m_X-semiclosed subset of X. Thus $f^{-1}(V)$ is m_Y-semiopen. Put $U = f^{-1}(V)$. Then $x \in U$ and $f(U) \subseteq V$. This shows that f is contra (m_X, m_Y)-semicontinuous at each point $x \in X$.

Sufficiency. Let V be any m_Y-open set of Y and $x \in f^{-1}(Y - V)$. Then $f(x) \in Y - V$ and $Y - V$ is m_Y-closed. By hypothesis, there exists an m_X-semiopen set U_x containing x such that $f(U_X) \subseteq Y - V$; hence $x \in U_x \subseteq f^{-1}(Y - V)$. Therefore, we have $\cup\{U_x : x \in f^{-1}(Y - V)\} = f^{-1}(Y - V)$. Since $MSO(X)$ satisfies property $\mathcal{B}$, $f^{-1}(Y - V)$ is m_X-semiopen and $f^{-1}(V)$ is m_X-semiclosed in X. This shows that f contra (m_X, m_Y)-semi continuous. ■

Theorem 3. *Let (X, m_X) and (Y, m_Y) be two minimal spaces. For a function $f : (X, m_X) \to (Y, m_Y)$, the following statements are equivalent:*

(1) *f is contra (m_X, m_Y)-semi continuous;*

(2) *$f^{-1}(V)$ is m_X-semiopen in X for every m_Y-closed subset V of Y;*

(3) *$m_X{-}Int(m_X{-}Cl(f^{-1}(V))) = m_X{-}Int(f^{-1}(V))$ for every m_Y-open subset V of Y;*

(4) *$m_X{-}Cl(m_X{-}Int(f^{-1}(V))) = m_X{-}Cl(f^{-1}(V))$ for every m_Y-closed subset V of Y.*

Proof. (1) $\Rightarrow$ (2). Let V be any m_Y-closed set of Y. Then $Y - V$ is m_Y-open. Using the hypothesis, $f^{-1}(Y - V) = X - f^{-1}(V)$ is m_X-semiclosed in X. As a consequence, $f^{-1}(V)$ is m_X-semiopen in X.

(2) $\Rightarrow$ (3). Let V be any m_Y-open set of Y. Then $Y - V$ is m_Y-closed. By (2), $f^{-1}(Y - V)$ is m_X - semiopen and $f^{-1}(V)$ is m_X - semiclosed in X. By Lemma 4, $m_X - Int(m_X - Cl(f^{-1}(V))) \subseteq f^{-1}(V)$ and hence by Lemma 1 $m_X - Int(m_X - Cl(f^{-1}(V))) \subseteq m_X - Int(f^{-1}(V)) \subseteq m_X - Int(m_X - Cl(f^{-1}(V)))$. Therefore, we obtain (3).

(3) $\Rightarrow$ (4). It is clear from the complement of (3).

(4) $\Rightarrow$ (1). Let V be any m_Y-open subset of Y. Then $Y - V$ is m_Y-closed. By hypothesis,

$$m_X - Cl(m_X - Int(f^{-1}(Y - V))) = m_X - Cl(f^{-1}(Y - V)).$$

Then we obtain that

$$m_X - Int(m_X - Cl(f^{-1}(V))) = m_X - Int(f^{-1}(V)) \subseteq f^{-1}(V).$$

By Lemma 4, $f^{-1}(V)$ is m_X-semiclosed in X. ■

Theorem 4. *Let (X, m_X), (Y, m_Y) be two minimal spaces and m_Y satisfy property $\mathcal{B}$. For a function $f : (X, m_X) \to (Y, m_Y)$, the following statements are equivalent:*

(1) *f is contra (m_X, m_Y)-semi continuous;*
(2) *$f^{-1}(B)$ is m_X-semiopen in X for every m_Y-closed set B in Y;*
(3) *$f^{-1}(B) \subseteq m_X - Cl(m_X - Int(f^{-1}(m_Y - Cl(B))))$ for every subset B in Y;*
(4) *$m_X - Int(m_X - Cl(f^{-1}(m_Y - Int(B)))) \subseteq f^{-1}(B)$ for every subset B in Y;*
(5) *$A \subseteq m_X - Cl(m_X - Int(f^{-1}(m_Y - Cl(f(A)))))$ for every subset A in X.*

Proof. (1) $\Leftrightarrow$ (2). It is obvious from Theorem 3.

(2) $\Rightarrow$ (3). Let $B \subseteq Y$. Then $m_Y - Cl(B)$ is an m_Y-closed set in Y since m_Y satisfies property $\mathcal{B}$. By (2), $f^{-1}(m_Y - Cl(B))$ is m_X-semiopen in X. Therefore, $f^{-1}(m_Y - Cl(B)) \subseteq m_X - Cl(m_X - Int(f^{-1}(m_Y - Cl(B))))$. As a consequence, $f^{-1}(B) \subseteq f^{-1}(m_Y - Cl(B)) \subseteq m_X - Cl(m_X - Int(f^{-1}(m_Y - Cl(B))))$.

(3) $\Leftrightarrow$ (4). It is clear from the complement.

(4) $\Rightarrow$ (5). Let $A \subseteq X$. Then $f(A) \subseteq Y$. By (3), $A \subseteq f^{-1}(f(A)) \subseteq m_X - Cl(m_X - Int(f^{-1}(m_Y - Cl(f(A)))))$.

(5) $\Rightarrow$ (2). Let B be any m_Y - closed set in Y. Then $f^{-1}(B) \subseteq X$. By (5), $f^{-1}(B) \subseteq m_X - Cl(m_X - Int(f^{-1}(m_Y - Cl(f(f^{-1}(B)))))) \subseteq m_X - Cl(m_X - Int(f^{-1}(m_Y - Cl(B))))$. Then we obtain

$$f^{-1}(B) \subseteq m_X - Cl(m_X - Int(f^{-1}(B)))$$

since B is m_Y-closed in Y. As a consequence, $f^{-1}(B)$ is m_X-semiopen in X. ■

Theorem 5. *Let (X, m_X), (Y, m_Y) be two minimal spaces and m_X, m_Y satisfy property $\mathcal{B}$. For a function $f : (X, m_X) \to (Y, m_Y)$, the following statements are equivalent:*

(1) *f is contra (m_X, m_Y)-semi continuous;*
(2) *$f^{-1}(V)$ is m_X-semiopen in X for every m_Y-closed subset V of Y;*
(3) *There exists an m_X-semiclosed set U such that $x \notin U$ and $f^{-1}(V) \subseteq U$ for each $x \in X$ and each m_Y-open V with $f(x) \notin V$;*

(4) $f^{-1}(F) \subseteq m_X sInt(f^{-1}(F))$ *for any* m_Y*-closed set* F *in* Y*;*
(5) $m_X sCl(f^{-1}(F)) \subseteq f^{-1}(F)$ *for any* m_Y*-open set* F *in* Y*;*
(6) $m_X sCl(f^{-1}(m_Y - Int(F))) \subseteq f^{-1}(m_Y - Int(F))$ *for any subset* $F \subseteq Y$*;*
(7) $f^{-1}(m_Y - Cl(F)) \subseteq m_X sInt(f^{-1}(m_Y - Cl(F)))$ *for any subset* $F \subseteq Y$*.*

Proof. (1) $\Leftrightarrow$ (2) is already shown in Theorem 3.

(1) $\Rightarrow$ (3). Let $x \in X$ and V be any m_Y-open subset of Y with $f(x) \notin V$. Then $f^{-1}(V)$ is m_X-semiclosed. Put $U = f^{-1}(V)$. Then $f^{-1}(V) \subseteq U$ and $x \notin U$.

(3) $\Rightarrow$ (1). Let V be any m_Y-open subset of Y. For each $x \in f^{-1}(Y-V)$, $f(x) \notin V$. By hypothesis, there exists an m_X-semiclosed set U_x such that $x \notin U_x$ and $f^{-1}(V) \subseteq U_x$. Then $x \in X - U_x \subseteq X - f^{-1}(V) = f^{-1}(Y-V)$. We obtain

$$\bigcup_{x \in f^{-1}(Y-V)} \{x\} \subseteq \bigcup_{x \in f^{-1}(Y-V)} (X - U_x) \subseteq f^{-1}(Y-V).$$

Hence $f^{-1}(Y-V) = \bigcup_{x \in f^{-1}(Y-V)} (X - U_x)$ is m_X-semiopen. Thus $f^{-1}(V)$ is m_X-semiclosed. As a consequence, f is contra (m_X, m_Y)-semi continuous.

(1) $\Rightarrow$ (4). Let F be any m_Y-closed subset of Y. For each $x \in f^{-1}(F)$, $f(x) \in F$. By Theorem 2, there exists an m_X-semiopen set U such that $x \in U$ and $f(U) \subseteq F$. Since $x \in U \subseteq f^{-1}(F)$, we obtain $x \in m_X sInt(f^{-1}(F))$. As a consequence, $f^{-1}(F) \subseteq m_X sInt(f^{-1}(F))$.

(4) $\Rightarrow$ (5). It is obvious from taking the complement of (4).

(5) $\Rightarrow$ (6). Let F be any subset of Y. Since m_Y satisfies property $\mathcal{B}$, $m_Y - Int(F)$ is an m_Y-open subset of Y and by (5), we obtain

$$m_X sCl(f^{-1}(m_Y - Int(F))) \subseteq f^{-1}(m_Y - Int(F)).$$

(6) $\Rightarrow$ (7). It is clear from the complement of (6).

(7) $\Rightarrow$ (1). Let V be any m_Y-open subset of Y. Then $Y - V$ is m_Y-closed. By (7), $X - f^{-1}(V) = f^{-1}(Y-V) = f^{-1}(m_Y - Cl(Y-V)) \subseteq m_X sInt(f^{-1}(m_Y - Cl(Y-V))) = m_X sInt(f^{-1}(Y-V)) = X - m_X sCl(f^{-1}(V))$. Therefore, $m_X - sCl(f^{-1}(V)) \subseteq f^{-1}(V)$ and hence $m_X - sCl(f^{-1}(V)) = f^{-1}(V)$. Since m_X satisfies property $\mathcal{B}$, $f^{-1}(V)$ is m_X-semiclosed in X. As a consequence, f is contra (m_X, m_Y)-semi continuous. ■

4. Decompositions of contra (m_X, m_Y)-semicontinuity

In this section, we obtain decompositions of contra (m_X, m_Y)-semicontinuous functions and other related functions.

Definition 14. *Let (X, m_X) be a minimal space. A subset A of X is called*

(1) *an m_X-semi-regular set if A is both m_X-semiopen and m_X-semiclosed.*
(2) *an m_X-B-set if $A = U \cap V$, where $U \in m_X$ and V is m_X-semiclosed.*

Lemma 7. *Let (X, m_X) be a minimal space and $A \subseteq X$. Then the following conditions are equivalent:*

(1) *A is m_X-semi-regular;*
(2) *A is both βm_X-open and m_X-semiclosed.*

Proof. It is obvious by Lemma 4. ∎

Remark 2. A βm_X-open set and an m_X-semiclosed set are independent of each other as the following examples show.

Example 2. Let $X = \{a, b, c\}$ and $m_X = \{\emptyset, X, \{a\}, \{b\}, \{a, b\}\}$. Then $A = \{a, b\}$ is an m_X-open set and hence βm_X-open, but it is not an m_X-semiclosed set.

Example 3. Let $X = \{a, b, c\}$ and $m_X = \{\emptyset, X, \{a\}, \{c\}, \{b, c\}\}$. Then $A = \{a, b\}$ is an m_X-closed set and hence m_X-semiclosed set, but it is not a βm_X-open set.

Lemma 8. *Let (X, m_X) be a minimal space and m_X satisfy property $\mathcal{B}$. Then for a subset A of X, the following conditions are equivalent:*

(1) *A is both m_X-open and m_X-semiclosed;*
(2) *A is both αm_X-open and m_X-semiclosed;*
(3) *A is both m_X-preopen and m_X-semiclosed.*

Proof. It is clear. ∎

Remark 3. An m_X-preopen set and an m_X-semiclosed set are independent of each other as the following example shows.

Example 4. Consider Example 2, then the set $A = \{a, b\}$ is an m_X-preopen set, but it is not m_X-semiclosed. Also in Example 3, the set A is an m_X-semiclosed set, but it is not an m_X-preopen set.

Lemma 9. *Let (X, m_X) be a minimal space and $A \subseteq X$. If A is both βm_X-open and m_X-closed, then it is m_X-regular closed.*

Proof. It is an immediate result. ∎

Remark 4. A βm_X-open set and an m_X-closed set are independent of each other as the following example shows.

Example 5. Consider Example 2, then the set $A = \{a, b\}$ is a βm_X-open set, but it is not an m_X-closed set. Also in Example 3, the set A is an m_X-closed set, but it is not a βm_X-open set.

Definition 15. *Let (X, m_X) and (Y, m_Y) be two minimal spaces. Then a function $f : (X, m_X) \to (Y, m_Y)$ is said to be*
(1) *(m_X, m_Y)-perfectly continuous if $f^{-1}(V)$ is m_X-clopen in X for every m_Y-open set V of Y,*
(2) *(m_X, m_Y)-completely continuous if $f^{-1}(V)$ is m_X-regular open in X for every m_Y-open set V of Y,*
(3) *(m_X, m_Y)-semi-regular continuous (briefly, $(m_X, m_Y) - SR$-continuous) if $f^{-1}(V)$ is m_X-semi-regular open in X for every m_Y-open set V of Y,*
(4) *(m_X, m_Y)-regular closed continuous (briefly, $(m_X, m_Y) - RC$-continuous) if $f^{-1}(V)$ is m_X-regular closed in X for every m_Y-open set V of Y,*
(5) *(m_X, m_Y) - B-continuous if $f^{-1}(V)$ is an m_X - B-set in X for every m_Y-open set V of Y.*

Definition 16 ([7]). *Let m_X, m_Y be two minimal structures. A function $f : (X, m_X) \to (Y, m_Y)$ is said to be $M - \beta$-continuous if $f^{-1}(V)$ is βm_X-open in X for every m_Y-open set V of Y.*

Theorem 6. *For a function $f : (X, m_X) \to (Y, m_Y)$, the following statements are equivalent:*
(1) *f is (m_X, m_Y) - SR-continuous;*
(2) *f is M - β-continuous and contra (m_X, m_Y)-semi continuous.*

Proof. It is an immediate result of Lemma 7. ■

Definition 17 ([4]). *Let m_X, m_Y be two minimal structures. A function $f : (X, m_X) \to (Y, m_Y)$ is said to be M-pre continuous if $f^{-1}(V)$ is m_X-preopen in X for every m_Y-open set V of Y.*

Theorem 7. *If a function $f : (X, m_X) \to (Y, m_Y)$ is M-pre continuous and contra (m_X, m_Y)-semi continuous, it is (m_X, m_Y)-completely continuous.*

Proof. It is clear from the fact that every m_X-preopen and m_X-semiclosed set is m_X-regular open. ■

Theorem 8. *If a function $f : (X, m_X) \to (Y, m_Y)$ is M - β-continuous and contra (m_X, m_Y)-continuous, it is (m_X, m_Y) - RC-continuous.*

Proof. It is obvious from Lemma 9. ■

Definition 18. *A function $f : (X, m_X) \to (Y, m_Y)$ is said to be contra (m_X, m_Y) - gs-continuous if $f^{-1}(V)$ is m_X - gs-closed in X for every m_Y-open set V of Y.*

Theorem 9. *For a function $f : (X, m_X) \to (Y, m_Y)$, the following statements are equivalent:*
(1) *f is contra (m_X, m_Y)-semi continuous;*
(2) *f is (m_X, m_Y) - B-continuous and contra (m_X, m_Y) - gs-continuous.*

Proof. (1) $\Rightarrow$ (2). It is clear.
(2) $\Rightarrow$ (1). Let V be any m_Y-open set of Y. Since f is (m_X, m_Y) - B-continuous, $f^{-1}(V) = U \cap F$, where $U \in m_X$ and F is m_X-semiclosed in X. Then $f^{-1}(V) \subseteq U$ and $U \in m_X$. $f^{-1}(V)$ is m_X - gs-closed and since f is contra (m_X, m_Y) - gs-continuous, $m_X sCl(f^{-1}(V)) \subseteq U$. Since $MSO(X)$ satisfies property $\mathcal{B}$, $m_X sCl(f^{-1}(V))$ is m_X - semiclosed and by Lemma 4 $m_X - Int(m_X - Cl(f^{-1}(V))) \subseteq m_X - Int(m_X - Cl(m_X sCl(f^{-1}(V)))) \subseteq m_X sCl(f^{-1}(V)) \subseteq U$. On the other hand, F is m_X - semiclosed and by Lemma 4 $m_X - Int(m_X - Cl(f^{-1}(V))) \subseteq m_X - Int(m_X - Cl(F)) \subseteq F$. Therefore, we obtain $m_X - Int(m_X - Cl(f^{-1}(V))) \subseteq U \cap F = f^{-1}(V)$. As a consequence, $f^{-1}(V)$ is m_X-semiclosed. ■

Remark 5. The notions of (m_X, m_Y) - B-continuity and contra (m_X, m_Y) - gs-continuity are independent of each other as shown by the following example.

Example 6. Let $X = \{1, 2\}$, $Y = \{a, b\}$, $m_X = \{\emptyset, X, \{2\}\}$ and $m_Y = \{\emptyset, Y\}$. Let $f : (X, m_X) \to (X, m_X)$ be the identity function. Then f is (m_X, m_Y)-B-continuous but it is not contra (m_X, m_Y) - gs-continuous. Also, let $g : (Y, m_Y) \to (X, m_X)$ be a function defined as follows:

$$g(a) = 1, \quad g(b) = 2.$$

Then g is contra (m_X, m_Y)-gs-continuous, but it is not (m_X, m_Y)-B-continuous.

Corollary 1. *For a function $f : (X, m_X) \to (Y, m_Y)$, the following statements are equivalent:*
(1) *f is (m_X, m_Y)-SR-continuous;*
(2) *f is M-β-continuous, (m_X, m_Y) - B-continuous and contra (m_X, m_Y)-gs-continuous.*

Proof. It is obvious from Theorems 6 and 9. ■

Remark 6. The function $f : (X, m_X) \to (X, m_X)$ in Example 6 is (m_X, m_Y)-pre continuous, but it is not contra (m_X, m_Y)-gs-continuous. Also, the function $g : (Y, m_Y) \to (X, m_X)$ in Example 6 is (m_X, m_Y)-pre continuous, but it is not (m_X, m_Y)-B-continuous.

Remark 7. We obtain the following diagram which shows the relationships between contra (m_X, m_Y)-semicontinuous functions and other related functions.

DIAGRAM

m-regular closed C
$\downarrow$
m-complete C $\rightarrow$ M-pre C $\rightarrow M-\beta-C$ contra-$m-C$
$\uparrow$ $\uparrow$ $\downarrow$
m-perfect C $\rightarrow$ m-semi-regular C $\rightarrow$ contra m-semi C
$\swarrow$ $\searrow$
m-B-C contra-mgs-C

In the diagram, C denotes continuity and m means (m_X, m_Y).

5. Strongly $S-m_X$-closed spaces

Definition 19. *A minimal space (X, m_X) is said to be*
(1) *m_X-semi-compact if there exists a finite subset J of I such that $X = \cup\{U_i : i \in J\}$ for every m_X-semiopen cover $\{U_i : i \in I\}$ of X,*
(2) *m_X-s-closed if there exists a finite subset J of I such that $X = \cup\{m_X sCl(U_i) : i \in J\}$ for every m_X-semiopen cover $\{U_i : i \in I\}$ of X,*
(3) *m_X - S-closed if there exists a finite subset J of I such that $X = \{m_X - Cl(U_i) : i \in J\}$ for every m_X-semiopen cover $\{U_i : i \in I\}$ of X,*
(4) *[14] m_X-nearly compact if there exists a finite subset J of I such that $X = \cup\{m_X - Int(m_X - Cl(U_i)) : i \in J\}$ for every m_X-open cover $\{U_i : i \in I\}$ of X,*
(5) *[9] m_X-closed if there exists a finite subset J of I such that $X = \cup\{m_X - Cl(U_i) : i \in J\}$ for every m_X-open cover $\{U_i : i \in I\}$ of X,*
(6) *[10] strongly S-m_X-closed if every m_X-closed cover of X has a finite subcover,*
(7) *m_X-mildly compact if every m_X-clopen cover of X has a finite subcover.*

We obtain the following diagram:

DIAGRAM

m_X-semi-compact $\rightarrow$ m_X-s-closed $\rightarrow$ m_X-S-closed strongly S-m_X-closed
$\downarrow$ $\downarrow$ $\downarrow$
m_X-compact $\rightarrow$ m_X-nearly compact $\rightarrow$ m_X-closed $\rightarrow$ m_X-mildly compact

Theorem 10. *Let (X, m_X), (Y, m_Y) be two minimal spaces and a function $f : (X, m_X) \to (Y, m_Y)$ be a surjection. If one of the following statements holds, then (Y, m_Y) is strongly S-m_Y-closed.*

(1) *f is contra (m_X, m_Y)-semi continuous and (X, m_X) is m_X-semi-compact,*

(2) *f is (m_X, m_Y)-perfectly continuous and (X, m_X) is m_X-mildly compact.*

Proof. Suppose (2) holds: Let $\{U_i : i \in I\}$ be an m_Y-closed cover of Y. $\{f^{-1}(U_i) : i \in I\}$ is an m_X-clopen cover of X since f is (m_X, m_Y)-perfectly continuous. Then there exists a finite $J \subseteq I$ such that $X = \bigcup_{i\in J} f^{-1}(U_i)$ as (X, m_X) is m_X-mildly compact. Hence $Y = \bigcup_{i\in J} U_i$. As a consequence, (Y, m_Y) is strongly S-m_Y-closed. ■

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