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NEW ALGORITHM FOR FIRST ORDER STIFF INITIAL VALUE PROBLEMS

Abstract. In this paper, we consider the development and implementation of algorithms for the solution of stiff first order initial value problems. Method of interpolation and collocation of basis function to give system of nonlinear equations which is solved for the unknown parameters to give a continuous scheme that is evaluated at selected grid points to give discrete methods. The stability properties of the method is verified and numerical experiments show that the new method is efficient in handling stiff problems.

Key words: exponentially-fitted method, interpolation, collocation, stability properties, continuous method, stiff problems.

AMS Mathematics Subject Classification: 65L05, 65L06.

1. Introduction

In this paper, we develop an exponentially fitted two step, one hybrid point numerical integrator for initial value problems (IVPs) of first order differential equations in the form

$$y' = f(x, y), \quad y(x_n) = y_0, \quad x_n \le x \le x_N , \tag{1}$$

where x_n is the initial point, $y : [x_n, x_N] \to \mathbb{R}^m$, $f : [x_n, x_N] \times \mathbb{R} \to \mathbb{R}^m$, $m \geqq 1$ is continuously differentiable, the Jacobian arising from (1) vary slowly and the eigenvalues have negative real part; moreover, the solution is decaying or exhibit a pronounced exponential behavior.

Classical general purpose method developed using finite power series basis function cannot produce satisfactory results due to the special nature of the problems. Such problems are found in the modeling of disease outbreak, war, radioactive decay, diffusion process in biology and chemical reactions. Several scholars have developed exponentially fitted methods, among them are Berghe et al. [7], Abhulimen [1], Fengjian, Xinming and Yiping [14], Simon [16], Ying and Yaacob [20], Carroll [8], Yang et al. [19], Xiao, Zhang and Yi [17].

2. Methodology

We consider the approximate solution

$$y\left(x\right)=\sum_{j=0}^{k}a_jx^j+\sum_{j=1}^{k}b_je^{-x^j}, \tag{2}$$

where a$_j$ and b'$_j$s are constants to be determined. We seek approximation at an equidistant set of points defined by the integration interval $x_n<x_1<\cdots<x_{N-1}<x_N$, $h=\dfrac{x_N-x_n}{N-1}$, N is a positive integer.

Interpolating (2) at $x_{n+i}, i=0,1,\cdots,r$ and collocating (2) at $x_{n+i}, i=0,1,\cdots,s$ give

$$XA=U, \tag{3}$$

where

$$A=\left[\begin{array}{cccccccc} a_0 & a_1 & \cdots & a_{k-1} & a_k & b_1 & \cdots & b_k\end{array}\right]^T,$$

$$U=\left[\begin{array}{cccccccc} y_n & y_{n+1} & \cdots & y_{n+r} & f_n & f_{n+1} & \cdots & f_{n+s}\end{array}\right]^T,$$

$$X=\left[\begin{array}{cccccccc} 1 & x_n & x_n^2 & \cdots & x_n^k & e^{-x_n} & \cdots & e^{-x_n^k}\\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & & \vdots\\ 1 & x_{n+r} & x_{n+r}^2 & \cdots & x_{n+r}^k & e^{-x_{n+r}} & \cdots & e^{-x_{n+r}^k}\\ 0 & 1 & 2x_n & \cdots & kx_n^{k-1} & -e^{-x_n} & \cdots & -kx_n^{k-1}e^{-x_n^k}\\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & & \vdots\\ 0 & 1 & 2x_{n+s} & \cdots & kx_{n+s}^{k-1} & -e^{-x_{n+s}} & \cdots & -kx_{n+s}^{k-1}e^{-x_{n+s}^k}\end{array}\right].$$

We then impose the following conditions on $y\left(x\right)$ in (2)

$$y\left(x_{n+i}\right)=y_{n+i},\ \ i=0,1\cdots,r,\quad y'\left(x_{n+i}\right)=f_{n+i},\ \ i=0,1,\cdots,s, \tag{4}$$

where r and s are the numbers of interpolation and collocation points respectively. Solving (3)using Crammer's rule, substituting the result into (2) and after some algebraic simplifications gives the continuous Linear multistep method (LMM)

$$y_{n+t}=\alpha_0\left(t\right)y_n+\sum_{j=1}^{r}\alpha_j\left(t\right)y_{n+j}+\gamma_0\left(t\right)f_n+\sum_{j=1}^{s}\gamma_j\left(t\right)f_{n+j}\,, \tag{5}$$

where $t = \dfrac{x - x_n}{h}$. For consistency, $\sum_{j=0}^{r} \alpha_j(t) = 1$, $\sum_{j=0}^{s} \gamma_j(t) = ht$.

It should be noted that if α_j and γ_j in (5) are not functions of t or if they are constants, then it is referred to as discrete LMM. Evaluating (5) at the grid points gives a discrete method implemented in block to give

$$\zeta^{(1)} Y_{m+1} = \zeta^{(0)} Y_m + h\left(\eta^{(0)} F_m + \eta^{(1)} F_{m+1}\right), \tag{6}$$

where $\zeta^{(1)}$ being the coefficients of y_{n+t} in matrix is $r \times r$ identity matrix, $\zeta^{(0)} = \eta^{(0)}$ being the coefficients of y_n and f_n respectively are $r \times r$ matrices in the form

$$\begin{bmatrix} 0 & 0 & \cdots & 1 \\ 0 & 0 & \cdots & 1 \\ \vdots & \vdots & \cdots & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$\eta^{(1)}$ being the coefficients of f_{n+j} is $r \times r$ matrix

$$Y_{m+1} = \begin{bmatrix} y_{n+1} & y_{n+2} & \cdots & y_{n+r} \end{bmatrix}^T, \; F_m = \begin{bmatrix} f_{n-1} & f_{n-2} & \cdots & f_n \end{bmatrix}^T,$$

$$F_{m+1} = \begin{bmatrix} f_{n+1} & f_{n+2} & \cdots & f_{n+s} \end{bmatrix}^T, \; Y_m = \begin{bmatrix} y_{n-1} & y_{n-2} & \cdots & y_n \end{bmatrix}^T.$$

Y_{m+1}, F_m, F_{m+1} and Y_m are $r \times 1$ matrices.

2.1. Stability properties

2.1.1. Order of the method

The operator ℓ is associated with the linear method defined by

$$\ell[y(x):h] = y_{n+t} - \alpha_0(t) y_n - \sum_{j=1}^{r} \alpha_j(t) y_{n+j} - \gamma_0(t) f_n - \sum_{j=1}^{s} \gamma_j(t) f_{n+j}, \tag{7}$$

where $y(x)$ is an arbitrary function, continuously differentiable on an interval of integration. Equation (3) can be written in Taylor expansion about the point x to obtain

$$\ell[y(x):h] = c_0 y(x) + c_1 h y'(x) + c_2 h^2 y''(x) + ... + c_p h^p y^{(p)}(x) + \cdots,$$

where

$$c_p = \frac{1}{p!}\left[\sum_{j=1}^{r} j^p \theta_j - \frac{1}{(p-1)!}\sum_{j=1}^{r} j^{p-1} \gamma_j\right]$$

equation (3) is of order p if

$$\ell[y(x):h] = 0(h^{p+1}), \;\; c_0 = c_1 = \cdots = c_p = 0, \;\; c_{p+1} \neq 0.$$

Hence c_{p+1} is called the error constant and $c_{p+1}h^{p+1}y^{(p+1)}(x)$ is called the local truncation error (LTE) [18]

2.1.2. Consistency

A block method (6) is said to be consistent if it has order $p \geq 1$.

2.1.3. Zero stability

A block method (6) is said to be zero stable if the roots $z_s, s = 1, 2, 3, \cdots n$ of the first characteristic polynomial $\rho(z)$, defined by

$$\rho(z) = \det\left[z\zeta_1^{(1)} - \zeta_2^{(0)}\right] = 0$$

satisfies $|z_s| \leq 1$ and every root with $|z_s| \leq 1$ has multiplicity not exceeding the order of the differential equation as $h \to 0$ [6].

2.1.4. Convergence

The necessary and sufficient condition for a method to be convergent is that it must be consistent and zero stable [18].

2.1.5. Linear Stability

The linear stability is derived by applying the test equation $y^{(k)} = \lambda^{(k)} y_n$ to yield $y_{m+1} = \mu\left(z\right) y_m$, $\mu\left(z\right)$ is the amplification equation given by

$$\mu\left(z\right) = -\left(\zeta^{(1)} - z\eta^{(1)}\right)^{-1}\left(\zeta^{(0)} + z\eta^{(0)}\right)$$

the matrix $\mu\left(z\right)$ has eigenvalues $(0, 0, \cdots, \xi_k)$ where ξ_k is called the stability function which is a rational function with real coefficients [6].

2.1.5. Region of Absolute Stability (RAS)

A Region of absolute stability (RAS) of a LMM is the set

$$\begin{aligned} R \; = \; & \left\{\overline{h} : \text{for } \overline{h} \text{ where the root of the stability polynomial} \right. \\ & \left. \quad \text{are absolute less than one } \right\}. \text{ [11]} \end{aligned}$$

We use boundary locus method to get the region of absolute stability.

In this paper, we consider interpolation at $x = x_n$ and collocation at $x = x_{n+i}, i = 0, 1, \frac{3}{2}, 2.$ Solving the resulting systems of equation, (5) reduces to

$$y_{n+t} = y_n + \gamma_0\left(t\right) f_n + \gamma_1\left(t\right) f_{n+1} + \gamma_2\left(t\right) f_{n+\frac{3}{2}} + \gamma_3\left(t\right) f_{n+2}, \tag{8}$$

where

$$\gamma_0=-\frac{\left[\begin{array}{c} e^{-h}+e^{-2h}-2e^{-\frac{3}{2}h}-e^{-h^2t^2}e^{-h}-e^{-h^2t^2}e^{-2h}+2e^{-h^2t^2}e^{-\frac{3}{2}h}\\ -2he^{-h^2}-4he^{-4h^2}+6he^{-\frac{9}{4}h^2}-6h^2te^{-2h-h^2}+12h^2te^{-h-4h^2}\\ +8h^2te^{-\frac{3}{2}h-h^2}-8h^2te^{-\frac{3}{2}h-4h^2}-12h^2te^{-h-\frac{9}{4}h^2}+6h^2te^{-2h-\frac{9}{4}h^2}\\ +2he^{-h^2}e^{-ht}+4he^{-4h^2}e^{-ht}-6he^{-\frac{9}{4}h^2}e^{-ht}+2h^2t^2e^{-2h-h^2}\\ -4h^2t^2e^{-h-4h^2}-2h^2t^2e^{-\frac{3}{2}h-h^2}+4h^2t^2e^{-\frac{3}{2}h-4h^2}\\ +3h^2t^2e^{-h-\frac{9}{4}h^2}-3h^2t^2e^{-2h-\frac{9}{4}h^2} \end{array}\right]}{2h\left[\begin{array}{c} 3e^{-2h-h^2}-6e^{-h-4h^2}-4e^{-\frac{3}{2}h-h^2}+4e^{-\frac{3}{2}h-4h^2}\\ +6e^{-h-\frac{9}{4}h^2}-3e^{-2h-\frac{9}{4}h^2}+e^{-h^2}+2e^{-4h^2}-3e^{-\frac{9}{4}h^2} \end{array}\right]},$$

$$\gamma_1=\frac{\left[\begin{array}{c} -e^{-h^2t^2}+3e^{-2h}-4e^{-\frac{3}{2}h}-3e^{-h^2t^2}e^{-2h}+4e^{-h^2t^2}e^{-\frac{3}{2}h}\\ -12he^{-4h^2}+12he^{-\frac{9}{4}h^2}+12he^{-4h^2}e^{-ht}-12he^{-\frac{9}{4}h^2}e^{-ht}\\ +12h^2te^{-4h^2}-12h^2te^{-\frac{9}{4}h^2}+4h^2t^2e^{-\frac{3}{2}h-4h^2}\\ -3h^2t^2e^{-2h-\frac{9}{4}h^2}-4h^2t^2e^{-4h^2}+3h^2t^2e^{-\frac{9}{4}h^2}+1 \end{array}\right]}{2h\left[\begin{array}{c} 3e^{-2h-h^2}-6e^{-h-4h^2}-4e^{-\frac{3}{2}h-h^2}+4e^{-\frac{3}{2}h-4h^2}\\ +6e^{-h-\frac{9}{4}h^2}-3e^{-2h-\frac{9}{4}h^2}+e^{-h^2}+2e^{-4h^2}-3e^{-\frac{9}{4}h^2} \end{array}\right]},$$

$$\gamma_2=\frac{\left[\begin{array}{c} e^{-h^2t^2}+2e^{-h}-e^{-2h}-2e^{-h^2t^2}e^{-h}+e^{-h^2t^2}e^{-2h}-4he^{-h^2}\\ +4he^{-4h^2}+4he^{-h^2}e^{-ht}-4he^{-4h^2}e^{-ht}+4h^2te^{-h^2}-4h^2te^{-4h^2}\\ +h^2t^2e^{-2h-h^2}-2h^2t^2e^{-h-4h^2}-h^2t^2e^{-h^2}+2h^2t^2e^{-4h^2}-1 \end{array}\right]}{h\left[\begin{array}{c} 3e^{-2h-h^2}-6e^{-h-4h^2}-4e^{-\frac{3}{2}h-h^2}+4e^{-\frac{3}{2}h-4h^2}\\ +6e^{-h-\frac{9}{4}h^2}-3e^{-2h-\frac{9}{4}h^2}+e^{-h^2}+2e^{-4h^2}-3e^{-\frac{9}{4}h^2} \end{array}\right]},$$

$$\gamma_3=-\frac{1}{2}\frac{\left[\begin{array}{c} e^{-h^2t^2}+3e^{-h}-2e^{-\frac{3}{2}h}-3e^{-h^2t^2}e^{-h}+2e^{-h^2t^2}e^{-\frac{3}{2}h}\\ -6he^{-h^2}+6he^{-\frac{9}{4}h^2}+6he^{-h^2}e^{-ht}-6he^{-\frac{9}{4}h^2}e^{-ht}+6h^2te^{-h^2}\\ -6h^2te^{-\frac{9}{4}h^2}+2h^2t^2e^{-\frac{3}{2}h-h^2}-3h^2t^2e^{-h-\frac{9}{4}h^2}\\ -2h^2t^2e^{-h^2}+3h^2t^2e^{-\frac{9}{4}h^2}-1 \end{array}\right]}{h\left[\begin{array}{c} 3e^{-2h-h^2}-6e^{-h-4h^2}-4e^{-\frac{3}{2}h-h^2}+4e^{-\frac{3}{2}h-4h^2}\\ +6e^{-h-\frac{9}{4}h^2}-3e^{-2h-\frac{9}{4}h^2}+e^{-h^2}+2e^{-4h^2}-3e^{-\frac{9}{4}h^2} \end{array}\right]}.$$

The order of (8) is four with $LTE=\frac{1}{69\,120}h^5t^2\left(576t^3-3240t^2+6240t-4320\right)$. Evaluating (8) at $t=1,\frac{3}{2}$ and 2 give the discrete method which is implemented in block.The LTE of the block is $\left[\begin{array}{ccc} -\frac{31}{2880} & -\frac{51}{5120} & -\frac{1}{90} \end{array}\right]$ with stability function $\xi_k=\frac{25z^2-175z+384}{36z^3-156z^2+324z-288}$.

The region of absolute stability of the block is shown in Figure 1.

4. Numerical examples

We considered four problems to test the efficiency of the method and compare the results with results of other methods established in literature. It should be noted that $error = |y(x) - y_n|$ where $y(x)$ is the exact results and y_n is the computed results. $XXe-(xx) = XX * 10^{-xx}$.

Problem 1. We consider the linear system in the range $0 \leq x \leq 1$ solved by Jackson and Kenue [15], Cash [10] and Ehigie, Okunuga and Sofoluwe [11].

$$y' = \begin{pmatrix} -1 & 95 \\ -1 & -97 \end{pmatrix} y, \; y(0) = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \; y(x) = \frac{1}{47}\begin{pmatrix} 95e^{-2x} - 48e^{-95x} \\ 48e^{-96x} - e^{-2x} \end{pmatrix}.$$

The eigenvalues of the Jacobian matrix are $\lambda_1 = -2$, $\lambda_2 = -96$ with the stiffness ratio 1:48.

Table 1. *Comparison of results of Problem 1 with existing methods*

Stepsize	Method	$y(1)$ (|error|)	$y(1)$ (|error|)
0.0625	J-K	$0.2735523\ (3\times10^{-7})$	$-0.002879477\ (4\times10^{-7})$
	Cash4	$0.2735498\ (3\times10^{-7})$	$-0.002879471\ (4\times10^{-7})$
	Cash5	$0.27355005\ (1\times10^{-8})$	$-0.002879474\ (4\times10^{-7})$
	ABOT	$0.27354656\ (3\times10^{-6})$	$-0.002879474\ (4\times10^{-7})$
	SDEBDF	$0.27355004\ (3\times10^{-6})$	$-0.002879471\ (4\times10^{-7})$
	NMTD	$0.27355001\ (3\times10^{-8})$	$-0.002879474\ (4\times10^{-10})$
	Exact values	0.2735500405	−0.002879474114

We solved this problem using $h = 0.0625$ in order for comparison as shown in Table 1. The following notations are used: J-K are the results of Jackson and Kenue [15], Cash 4 and 5 implies results of order 4 and 5 method of Cash [10] and Cash [9] respectively, ABOT is the results of the method of Abhulimen and Otunta [5], SDEBDF is the results of Ehigie, Okunuga and Sofoluwe [11], NMTD implies results of the new method. The results in Table 1 show that the new method compete favorably with the existing methods.

Problem 2. We consider a four dimensional problems by Enright and Pryce [12]

$$\begin{pmatrix} y_1'(x) \\ y_2'(x) \\ y_3'(x) \\ y_4'(x) \end{pmatrix} = \begin{pmatrix} -10^4 y_1(x) + 100 y_2(x) - 10 y_3(x) + y_4(x) \\ -1000 y_2(x) + 10 y_3(x) - 10 y_4(x) \\ -y_3(x) + 10 y_4(x) \\ -0.1 y_4(x) \end{pmatrix},$$

$$\begin{pmatrix} y_1(0) \\ y_2(0) \\ y_3(0) \\ y_4(0) \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$$

within the range $0 \leq x \leq 1$. The eigenvalues of the Jacobian matrix $\lambda_1 = -0.1$, $\lambda_2 = -1.0$, $\lambda_3 = -1000$ and $\lambda_4 = -10000$. The exact solution is given as

$$\begin{aligned}
y_1(x) &= -\frac{89990090}{89990100}e^{-0.1x} + \frac{818090}{89901009}e^{-x} + \frac{9989911}{899010090}e^{-1000x} \\
&\quad + \frac{89071119179}{89990100090}e^{-10000x}, \\
y_2(x) &= \frac{9100}{89991}e^{-0.1x} - \frac{910}{8991}e^{-x} + \frac{9989911}{9989001}e^{-1000x}, \\
y_3(x) &= \frac{100}{9}e^{-0.1x} - \frac{91}{9}e^{-x}, \\
y_4(x) &= e^{-0.1x}.
\end{aligned}$$

Table 2. *Comparison of results of Problem 2 with existing results at* $x = 20$

h	*Method*	$y_1(20)\,(\lvert error\rvert)$	$y_2(20)\,(\lvert error\rvert)$
0.1	*SDEBDF*	-1.35335×10^{-3}	1.368527×10^{-2}
		(2.25×10^{-10})	(2.29×10^{-9})
	NMTD	-1.353352×10^{-3}	1.368526×10^{-2}
		(8.0613×10^{-13})	(8.1517×10^{-12})
	Exact Solution	-1.353352×10^{-3}	1.368526×10^{-2}

h	*Method*	$y_3(20)\,(\lvert error\rvert)$	$y_4(20)\,(\lvert error\rvert)$
0.1	*SDEBDF*	1.50372560	1.3533530×10^{-1}
		(2.50×10^{-7})	(2.06×10^{-8})
	NMTD	1.50372534	1.3533528×10^{-1}
		(8.9570×10^{-10})	(8.0643×10^{-11})
	Exact Solution	1.50372534	1.3533528×10^{-1}

Table 3. *Comparison of resukts of Problem 2 with existing results at* $x = 1$

h	*Method*	$y_1(1)$	$y_2(1)$	$y_3(1)$	$y_4(1)$
0.05	*AB*7	3.2×10^{-2}	3.2×10^{-2}	3.3×10^{-1}	3.7×10^{-5}
	*NM*9	2.2×10^{-3}	3.5×10^{-2}	3.2×10^{-5}	3.2×10^{-6}
	CEGE	3.5×10^{-5}	3.8×10^{-4}	3.5×10^{-7}	3.7×10^{-8}
	NMTD	7.446×10^{-11}	8.576×10^{-10}	8.273×10^{-8}	2.920×10^{-9}
0.1	*AB*7	2.5×10^{-2}	2.1×10^{-1}	2.4×10^{-3}	2.7×10^{-5}
	*NM*9	2.7×10^{-3}	2.4×10^{-3}	2.2×10^{-4}	2.5×10^{-6}
	CEGE	2.9×10^{-5}	2.7×10^{-4}	2.6×10^{-6}	2.6×10^{-8}
	NMTD	1.36×10^{-8}	1.30×10^{-8}	2.12×10^{-9}	2.69×10^{-11}

The following notations are used in Tables 2 and 3. SDEBDF is the method of Ehigie, Okunuga and Sofoluwe [11], AB7 is order seven method of Abhulimen and Otunta [4], CEGE is the method of Abhulimen and Omeike [2] and NMTD is the new method. Results of Tables 2 and 3 show that the new method gives the best approximation.

Problem 3. Consider a system in the range $0 \le t \le 10$ solved by Ezzeddine and Hojjati [13], Akinfenwa and Jator [6]

$$y' = \begin{pmatrix} -1 & -30 \\ 30 & -1 \end{pmatrix} y + \begin{pmatrix} 30e^{-t} \\ -30e^{-t} \end{pmatrix} y,$$

$$y(0) = \begin{pmatrix} 1 \\ 1 \end{pmatrix} y(x) = \left(e^{-t},\ e^{-t}\right)^T$$

the stiffness ratio is 1 : 200

Table 4. *Comparison of results of Problem 3 with existing*

t	y_i	*EBDF*	*HEBDF*	*ECBBDF*	*NMTD*
1	y_1	1.71×10^{-13}	8.15×10^{-15}	1.28×10^{-15}	4.8847×10^{-15}
	y_2	2.60×10^{-12}	8.48×10^{-13}	1.17×10^{-14}	4.9960×10^{-15}
10	y_1	5.03×10^{-17}	9.83×10^{-18}	1.08×10^{-19}	1.8431×10^{-18}
	y_2	3.36×10^{-16}	7.71×10^{-17}	1.62×10^{-18}	6.2541×10^{-19}
20	y_1	1.17×10^{-20}	1.29×10^{-21}	7.24×10^{-23}	9.9261×10^{-24}
	y_2	7.83×10^{-21}	2.79×10^{-21}	5.29×10^{-23}	1.5302×10^{-23}

The following notations are used in Table 4. EBDF and HEBDF are absolute errors in the methods of Ezzeddine and Hojjati [13]. ECBBDF is the absolute error in Akinfenwa and Jator [6] and NMTD is the new method. The results show that the new method give best approximation.

Problem 4. We consider a nonlinear two dimensional Kaps problems within the interval $0 \le x \le 20$

$$\begin{bmatrix} y_1' \\ y_2' \end{bmatrix} = \begin{bmatrix} -1002y_1 + 1000y_2^2 \\ y_1 - y_2(1 + y_2) \end{bmatrix}, \quad \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

with the exact solution

$$\begin{bmatrix} y_1(0) \\ y_2(0) \end{bmatrix} = \begin{bmatrix} e^{-2x} \\ e^{-x} \end{bmatrix}.$$

Table 5 shows the comparison with the existing methods. The following natations are used in Table 5; SDM$_{10}$ and SDM$_{14}$ represent second derivative method of order 10 and 14 of Yakubu and Marcus [18] .

Table 5. *Comparison of results of Problem 4 with existing results*

x	y_i	SDM_{10}	SDM_{14}	*NMTD*
5	y_1	$4.6889e-03$	$5.8258e-02$	$9.7751e-04$
	y_2	$4.8326e-03$	$3.2259e-02$	$1.0556e-06$
50	y_1	$1.4156e-02$	$6.7358e-03$	$2.6559e-05$
	y_2	$1.9419e-02$	$2.6181e-02$	$1.1303e-07$
150	y_1	$6.3883e-04$	$2.4686e-06$	$8.7651e-09$
	y_2	$6.1134e-03$	$5.3608e-04$	$2.5430e-09$
250	y_1	$1.7895e-05$	$8.1636e-10$	$2.8923e-12$
	y_2	$1.2275e-03$	$9.7597e-06$	$6.0129e-11$
500	y_1	$1.6011e-09$	$1.6165e-18$	$5.7208e-21$
	y_2	$1.5267e-05$	$4.3431e-10$	$4.2348e-15$

3. Conclusion

We have discussed the construction of order four exponentially fitted hybrid method for the solution of first order stiff IVPs. The method has good stability properties that is suitable for stiff problems. Results of numerical examples show that the method is efficient and compete favourably with the existing results established in literature.

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