DE GRUYTER OPEN
F A S C I C U L I   M A T H E M A T I C I
Nr 59 2017
DOI:10.1515/fascmath-2017-0014

G.V. Ravindranadh Babu and M. Dula Tolera

FIXED POINTS OF GENERALIZED (α, ψ, ϕ)-RATIONAL CONTRACTIVE MAPPINGS IN α-COMPLETE METRIC SPACES

Abstract. In this paper, we introduce generalized (α, ψ, ϕ)-rational contractive mappings in α-complete metric spaces and prove some new fixed point results for this class of mappings. We provide examples in support of our results. Our results generalize the fixed point results of Singh, Kamal, Sen and Chugh [22] and Piri and Kumam [18].

Key words: α-complete metric space, α-admissible mapping, generalized (α, ψ, ϕ)-rational contractive mappings, fixed point.

AMS Mathematics Subject Classification: 47H10, 54H25.

1. Introduction and preliminaries

In 1922, Banach proved a remarkable and powerful result on the existence of fixed points in complete metric spaces, called the Banach contraction principle. Because of its fruitful applications, a lot of authors ([3], [5], [6], [12], [15], [16]) established generalizations and extensions of it in many directions.

In 1977, Jaggi [13] introduced a new type of contractions involving rational expressions and proved the existence of fixed points of such mappings. The latest development in this direction are Hieu and Dung [9], Haung, Ansari, Diana Dolicanin-Dekic, and Radenovic [10], and Huang, Deng, Chen, Radenovic [11].

In 1997, Alber and Guerre-Delabriere [2] introduced the notion of weakly contractive maps in Hilbert spaces and proved that any weakly contractive map defined on Hilbert spaces has a unique fixed point. Rhoades [20] reintroduced the notion of weakly contractive maps in the setting of metric spaces and proved that any weakly contractive map defined on complete metric spaces has a unique fixed point. In 2008, Dutta and Choudhury [8] introduced (ψ, φ)-weakly contractive maps and proved the existence of fixed points in complete metric spaces. Interestingly, Doric [7] extended it to a pair of generalized (ψ, φ)-weakly contractive maps. Recently, Singh, Kamal, Sen and Chugh [22] further generalized Doric [7] result in complete spaces.

On the other direction, Wardowski [23] introduced a new contraction called F-contraction and proved a fixed point result as a generalization of the Banach contraction principle. Many authors ([24], [19], [1]) studied fixed point results for F-contraction type maps. Recently Piri and Kumam [18] proved some Wardowski and Suzuki type fixed point results in metric spaces.

Throughout this paper, $\mathbb{N}$ denotes the set of all natural numbers.

Now, we begin with some basic well-known definitions and results which will be used in the rest of this paper.

Samet, Vetro and Vetro [21] introduced the concept of α-admissible mappings in the following.

Definition 1 ([21]). *Let $T : X \to X$ be a mapping and let $\alpha : X \times X \to [0, \infty)$ be a function. We say that T is an α-admissible mapping if $x, y \in X, \alpha(x, y) \geqslant 1 \implies \alpha(Tx, Ty) \geqslant 1$.*

Definition 2 ([14]). *Let $T : X \to X$ and $\alpha : X \times X \to [0, \infty)$. We say that T is a triangular α-admissible mapping if*
(i) T is α-admissible mapping and
(ii) $\alpha(x, y) \geqslant 1$ and $\alpha(y, z) \geqslant 1 \implies \alpha(x, z) \geqslant 1$, $x, y, z \in X$.

Lemma 1 ([14], Lemma 7). *Let T be a triangular α-admissible mapping. Assume that there exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geqslant 1$. We define a sequence $\{x_n\}$ by $x_{n+1} = Tx_n$. Then $\alpha(x_m, x_n) \geqslant 1$ for all $m, n \in \mathbb{N}$ with $m < n$.*

Definition 3 ([12]). *Let (X, d) be a metric space and $\alpha : X \times X \to [0, \infty)$ be a mapping. A metric space X is said to be α-complete if every Cauchy sequence $\{x_n\}$ in X with $\alpha(x_n, x_{n+1}) \geqslant 1$ for all $n \in \mathbb{N}$ converges in X.*

If X is a complete metric space, then X is also an α-complete metric space, but its converse is not true ([17], Example 1.8).

Theorem 1 ([8], Theorem 2.1). *Let (X, d) be a complete metric space and $T : X \to X$ be a selfmap of X. Suppose that*

$$\psi(d(Tx, Ty)) \leqslant \psi(d(x, y)) - \varphi(d(x, y)) \text{ for all } x, y \in X, \tag{1}$$

where $\psi, : [0, \infty) \to [0, \infty)$ is continuous, non-decreasing and $\psi(t) = 0$ if and only if $t = 0$, and $\varphi : [0, \infty) \to [0, \infty)$ is continuous, non-decreasing and $\varphi(t) = 0$ if and only if $t = 0$. Then T has a unique fixed point.

Doric [7] proved the following result as an extension of Theorem 1.

Theorem 2 ([7], Theorem 2.2). *Let (X,d) be a complete metric space and let $T : X \to X$ be a selfmap satisfying the inequality*

$$\psi(d(Tx,Ty)) \leqslant \psi(M(x,y)) - \varphi(M(x,y)), \tag{2}$$

where $M(x,y) = \max\{d(x,y), d(x,Tx), d(y,Ty), \frac{d(x,Ty)+d(y,Tx)}{2}\}$, *$\psi : [0,\infty) \to [0,\infty)$ is continuous, non-decreasing and $\psi(t) = 0$ if and only if $t = 0$, and $\varphi : [0,\infty) \to [0,\infty)$ is lower semi-continuous function with $\varphi(t) = 0$ if and only if $t = 0$. Then T has a unique fixed point.*

Recently, Singh, Kamal, Sen and Chugh [22] obtained the following result as a generalization of Theorem 2.

Theorem 3 ([22], Theorem 2.1). *Let X be a complete metric space and $T : X \to X$ be a selfmap of X such that for every $x, y \in X$,*

$$\begin{aligned} &\frac{1}{2}d(x,Tx) \leqslant d(x,y) \ \textit{implies} \\ &\qquad \psi(d(Tx,Ty)) \leqslant \psi(M(x,y)) - \varphi(M(x,y)) \end{aligned} \tag{3}$$

where $M(x,y) = \max\{d(x,y), d(x,Tx), d(y,Ty), \frac{d(x,Ty)+d(y,Tx)}{2}\}$, *$\psi : [0,\infty) \to [0,\infty)$ is continuous, non-decreasing and $\psi(t) = 0$ if and only if $t = 0$, and $\varphi : [0,\infty) \to [0,\infty)$ is lower semi-continuous function with $\varphi(t) = 0$ if and only if $t = 0$. Then T has a unique fixed point.*

On the other hand, Piri and Kumam [18] proved the following fixed point results in metric spaces.

Theorem 4 ([18], Theorem 2.2). *Let X be a complete metric space and $T : X \to X$ be a selfmap of X. Assume that there exists $\tau > 0$ such that for all $x, y \in X$ with $Tx \neq Ty$,*

$$\frac{1}{2}d(x,Tx) \leqslant d(x,y) \implies \tau + F(d(Tx,Ty)) \leqslant F(d(x,y))$$

where $F : (0,\infty) \to (-\infty,\infty)$ is continuous strictly increasing and $\inf F = -\infty$.

Then T has a unique fixed point $z \in X$ and for every $x \in X$, the sequence $\{T^n x\}$ converges to z.

We denote
$\Psi = \{\psi : [0,\infty) \to [0,\infty) |\ \psi$ is continuous and non-decreasing function$\}$,
$\Phi = \{\phi : [0,\infty) \to [0,\infty) |\ \lim_{n\to\infty} \phi(t_n) = 0 \implies \lim_{n\to\infty} t_n = 0\}$ and
$\Psi_1 = \{\psi_1 : [0,\infty)^6 \to [0,\infty) |\ (i)$ ψ_1 is continuous and non-decreasing in each coordinate, (ii) $\psi_1(t_1,t_2,t_3,t_4,t_5,t_6) = 0$ implies $t_1 = t_2 = t_3 = t_4 = t_5 = t_6 = 0$ and (iii) $\psi_1(t,t,t,t,t,t) \leqslant t$ for all $t > 0\}$.

Remark 1. If $\phi \in \Phi$ then $\phi(t) = 0 \implies t = 0$.

Motivated by the works of Piri and Kumam [18] and Singh, Kamal, Sen and Chugh [22], we now introduce a generalized (α, ψ, ϕ)-rational contractive mappings in metric spaces in the following.

Definition 4. *Let (X, d) be a metric space and $\alpha : X \times X \to [0, \infty)$ be a given map. Let $T : X \to X$ be a selfmap of X. If there exist $\psi \in \Psi$, $\phi \in \Phi$ and $\psi_1 \in \Psi_1$ such that*

$$\text{(4)} \quad \text{for all } x, y \in X \text{ with } \frac{1}{2}d(x, Tx) \leqslant d(x, y) \text{ and } \alpha(x, y) \geqslant 1 \text{ implies}$$
$$\psi(d(Tx, Ty)) \leqslant \psi(M_{\psi_1}(x, y)) - \phi(M_{\psi_1}(x, y)),$$

where

$$\text{(5)} \quad \begin{aligned} M_{\psi_1}(x, y) \ = \ & \psi_1(d(x, y), d(x, Tx), d(y, Ty), \frac{d(x, Ty) + d(y, Tx)}{2}, \\ & \frac{d(y, Ty)[1 + d(x, Tx)]}{1 + d(x, y)}, \frac{d(y, Tx)[1 + d(x, Ty)]}{1 + d(x, y)}), \end{aligned}$$

then we say that T is a generalized (α, ψ, ϕ)-rational contractive mapping.

Example 1. Let $X = [0, 10)$ with the usual metric. We define $T : X \to X$ by

$$Tx = \begin{cases} \frac{3x}{8} & \text{if } x \in [0, 1] \\ x & \text{if } x \in (1, 10) \end{cases}$$

and $\alpha : X \times X \to [0, \infty)$ by

$$\alpha(x, y) = \begin{cases} 2 & \text{if} |; x \in [0, 1] \\ 0 & \text{otherwise.} \end{cases}$$

Since $\alpha(x, y) \geqslant 1$ if and only if $x, y \in [0, 1]$, we verify the inequality (4) for $x, y \in [0, 1]$ with $\frac{1}{2}d(x, Tx) \leqslant d(x, y)$. For this purpose, we choose

$$\psi, \phi : [0, \infty) \to [0, \infty) \text{ by } \psi(t) = \frac{8}{3}t \text{ and } \phi(t) = \begin{cases} 2 & \text{if } t = 0 \\ \frac{5t}{3} & \text{if } t \neq 0 \end{cases}$$

and $\psi_1 : [0, \infty)^6 \to [0, \infty)$ by $\psi_1(t_1, t_2, t_3, t_4, t_5, t_6) = \max\{t_1, t_2, t_3, t_4, t_5.t_6\}$. Let $x, y \in [0, 1]$. We assume without loss of generality that $x \leqslant y$. Let $\frac{1}{2}d(x, Tx) \leqslant d(x, y)$ i.e., $\frac{13}{8}x \leqslant y$. Now,

$$\begin{aligned} \psi(d(Tx, Ty)) \ &= \ \psi(d(\frac{3x}{8}, \frac{3y}{8})) = \psi(|\frac{3y}{8} - \frac{3x}{8}|) = \frac{8}{3}|\frac{3y}{8} - \frac{3x}{8}| = |y - x| \\ &\leqslant \ M_{\psi_1}(x, y) = \frac{8}{3}M_{\psi_1} - \frac{5}{3}M_{\psi_1}(x, y), \\ &= \ \psi(M_{\psi_1}(x, y)) - \phi(M_{\psi_1}(x, y)). \end{aligned}$$

Hence T is a generalized (α, ψ, ϕ)-rational contractive mapping.

In Section 2, we prove our main results in which we study the existence of fixed points of generalized (α, ψ, ϕ)-rational contractive mappings in α-complete metric spaces. We provide corollaries and examples in support of our results in Section 3.

The following lemma is useful in our subsequent discussion.

Lemma 2 ([4], Lemma 1.4). *Suppose (X, d) is a metric space. Let $\{x_n\}$ be a sequence in X such that $d(x_n, x_{n+1}) \to 0$ as $n \to \infty$. If $\{x_n\}$ is not a Cauchy sequence then there exists an $\epsilon > 0$ and sequences of positive integers $\{m_k\}$ and $\{n_k\}$ with $n_k > m_k \geqslant k$ such that $d(m_k, n_k) \geqslant \epsilon$. For each $k > 0$, corresponding to m_k, we can choose n_k to be the smallest positive integer such that $d(x_{m_k}, x_{n_k}) \geqslant \epsilon$, $d(x_{m_k}, x_{n_k-1}) < \epsilon$ and*

(i) $\lim_{k\to\infty} d(x_{n_k}, x_{m_k+1}) = \epsilon$,

(ii) $\lim_{k\to\infty} d(x_{n_k}, x_{m_k}) = \epsilon$,

(iii) $\lim_{k\to\infty} d(x_{m_k-1}, x_{n_k}) = \epsilon$ *and*

(iv) $\lim_{k\to\infty} d(x_{n_k}, x_{m_k+1}) = \epsilon$.

2. Main results

Theorem 5. *Let (X, d) be a metric space and $T : X \to X$ be a generalized (α, ψ, ϕ)-rational contractive mapping. Suppose that the following conditions hold:*

(i) (X, d) is α-complete metric space,

(ii) T is a triangular α-admissible mapping,

(iii) there exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geqslant 1$, and

(iv) T is continuous.

Then T has a fixed point in X.

Proof. By hypotheses (iii), we have $x_0 \in X$ is such that $\alpha(x_0, Tx_0) \geqslant 1$. With this $x_0 \in X$ as an initial point, we define an iterative sequence $\{x_n\}$ by $x_{n+1} = Tx_n$ for $n = 0, 1, 2, \ldots$. If $x_{n_0+1} = x_{n_0}$ for some $n_0 \in \mathbb{N} \cup \{0\}$, we have $Tx_{n_0} = x_{n_0+1} = x_{n_0}$, so that x_{n_0} is a fixed point of T and we are through.

Hence, without loss of generality, we assume that $x_{n+1} \neq x_n$ for all $n \in \mathbb{N} \cup \{0\}$. Since T is α-admissible mapping and $\alpha(x_0, x_1) = \alpha(x_0, Tx_0) \geqslant 1$, we deduce that $\alpha(x_1, x_2) = \alpha(Tx_0, Tx_1) \geqslant 1$. On continuing this process, we get that

$$\alpha(x_n, x_{n+1}) \geqslant 1 \quad \text{for all } n \in \mathbb{N} \cup \{0\}. \tag{6}$$

First we show that $\lim_{n\to\infty} d(x_n, x_{n+1}) = 0$. Since $\alpha(x_n, x_{n+1}) \geqslant 1$ for all $n \in \mathbb{N} \cup \{0\}$, and $\frac{1}{2} d(x_n, Tx_n) \leqslant d(x_n, Tx_n) = d(x_n, x_{n+1})$, and hence by (4), we have

$$\psi(d(Tx_n, Tx_{n+1})) \leqslant \psi(M_{\psi_1}(x_n, x_{n+1})) - \phi(M_{\psi_1}(x_n, x_{n+1})), \tag{7}$$

where

$$\begin{aligned}
M_{\psi_1}(x_n, x_{n+1}) = \psi_1(&d(x_n,x_{n+1}), d(x_n, Tx_n), d(x_{n+1}, Tx_{n+1}), \\
&\frac{d(x_n, Tx_{n+1}) + d(x_{n+1}, Tx_n)}{2}, \\
&\frac{d(x_{n+1}, Tx_{n+1})[1 + d(x_n, Tx_n)]}{1 + d(x,x_{n+1})}, \\
&\frac{d(x_{n+1}, Tx_n)[1 + d(x_n, Tx_{n+1})]}{1 + d(x_n, x_{n+1})}), \\
= \psi_1(&d(x_n,x_{n+1}), d(x_n, x_{n+1}), d(x_{n+1}, x_{n+2}), \\
&\frac{d(x_n, x_{n+2})}{2}, d(x_{n+1}, x_{n+2}), 0).
\end{aligned}$$

Now, if $d(x_n, x_{n+1}) < d(x_{n+1}, x_{n+2})$ for some $n \in \mathbb{N} \cup \{0\}$, from property (i) of ψ_1 we have $M_{\psi_1}(x_n, x_{n+1}) \leqslant d(x_{n+1}, x_{n+2})$, and hence from (7) we have

$$\begin{aligned}
\psi(d(x_{n+1}, x_{n+2})) &= \psi(d(Tx_n, Tx_{n+1})) \\
&\leqslant \psi(M_{\psi_1}(x_n, x_{n+1})) - \phi(M_{\psi_1}(x_n, x_{n+1})), \\
&= \psi(d(x_{n+1}, x_{n+2})) - \phi(M_{\psi_1}(x_n, x_{n+1})),
\end{aligned}$$

which implies that $\phi(M_{\psi_1}(x_n, x_{n+1})) = 0$, which further implies that $M_{\psi_1}(x_n, x_{n+1}) = 0$ i.e., $\psi_1(d(x_n, x_{n+1}), d(x_n, x_{n+1}), d(x_{n+1}, x_{n+2}), \frac{d(x_n,x_{n+2})}{2}, d(x_{n+1}, x_{n+2}), 0) = 0$. Hence from property (ii) of ψ_1 we have $d(x_n, x_{n+1}) = 0$, a contradiction since $x_n \neq x_{n+1}$. Therefore $d(x_n, x_{n+1}) \geqslant d(x_{n+1}, x_{n+2})$ for all $n \in \mathbb{N} \cup \{0\}$. Hence the sequence $\{d(x_n, x_{n+1})\}$ is decreasing and bounded from below. Thus there exists $r \geqslant 0$ such that $\lim_{n\to\infty} d(x_n, x_{n+1}) = r$. Now, from (7) we have

$$\begin{aligned}
\psi(d(x_{n+1}, x_{n+2})) &= \psi(d(Tx_n, Tx_{n+1})) \\
&\leqslant \psi(d(x_n, x_{n+1})) - \phi(M_{\psi_1}(x_n, x_{n+1})).
\end{aligned} \tag{8}$$

Since $d(x_n, x_{n+1})$ and $d(x_{n+1}, x_{n+2})$ are bounded, and $d(x_n, x_{n+2}) \leqslant d(x_n, x_{n+1})+d(x_{n+1}, x_{n+2})$ for all $n \in \mathbb{N}\cup\{0\}$, we have $\{d(x_n, x_{n+2})\}$ is a bounded sequence. Hence there exists a subsequence $\{d(x_{n_k}, x_{n_k+2})\}$ converges to s (say), and $s \geqslant 0$. Corresponding to this subsequence and using (8) we have

$$\begin{aligned}
\psi(d(x_{n_k+1}, x_{n_k+2})) &= \psi(d(Tx_{n_k}, Tx_{n_k+1})) \\
&\leqslant \psi(d(x_{n_k}, x_{n_k+1})) - \phi(M_{\psi_1}(x_{n_k}, x_{n_k+1})) \\
&\leqslant \psi(d(x_{n_k}, x_{n_k+1})).
\end{aligned} \tag{9}$$

On letting $k \to \infty$ in (9) and using the continuity of ψ, we have the first and last terms of (9) converge to the same limit $\psi(r)$, and hence it follows that $\lim_{k\to\infty}(\psi(d(x_{n_k},x_{n_k+1}))-\phi(M_{\psi_1}(x_{n_k},x_{n_k+1})))$ exists and is equal to $\psi(r)$.

Now $\phi(M_{\psi_1}(x_{n_k},x_{n_k+1})) = (\phi(M_{\psi_1}(x_{n_k},x_{n_k+1}))-\psi(d(x_{n_k},x_{n_k+1})))+\psi(d(x_{n_k},x_{n_k+1}))$, and on letting $k\to\infty$, we get

$$\begin{aligned}\lim_{k\to\infty}\phi(M_{\psi_1}(x_{n_k},x_{n_k+1})) &= \lim_{k\to\infty}(\phi(M_{\psi_1}(x_{n_k},x_{n_k+1}))-\psi(d(x_{n_k},x_{n_k+1})))\\ &\quad+\lim_{k\to\infty}\psi(d(x_{n_k},x_{n_k+1}))\\ &= -\psi(r)+\psi(r)=0,\end{aligned}$$

and it implies that $\lim_{k\to\infty}M_{\psi_1}(x_{n_k},x_{n_k+1})=0$. Now by using the continuity property of ψ_1 it follows that $\psi_1(r,r,r,r,r,\frac{s}{2},0)=0$. Hence from property (ii) of ψ_1, we have $r=0$ and $s=0$.

Therefore

$$\lim_{n\to\infty}d(x_n,x_{n+1})=0. \tag{10}$$

We now prove that $\{x_n\}$ is a Cauchy sequence. Suppose that $\{x_n\}$ is not a Cauchy sequence. Then by Lemma 2, there exist $\epsilon>0$ and sequences of positive integers $\{n_k\}$ and $\{m_k\}$ with $n_k>m_k\geqslant k$ satisfying

$$d(x_{m_k},x_{n_k})\geqslant\epsilon. \tag{11}$$

Let us choose the smallest n_k satisfying (11). Then we have $n_k>m_k\geqslant k$ with $d(x_{m_k},x_{n_k})\geqslant\epsilon$, $d(x_{m_k},x_{n_k-1})<\epsilon$ and (i)-(iv) of Lemma 2 hold.

From Lemma 1, we have $\alpha(x_{m_k},x_{n_k})\geqslant 1$ and from (10), we can choose $n_1\in\mathbb{N}\cup\{0\}$ such that $\frac{1}{2}d(x_{m_k},Tx_{m_k})\leqslant\frac{1}{2}\epsilon<d(x_{m_k},x_{n_k})$. Hence from (4) for every $k\geqslant n_1$ we have

$$\begin{aligned}\psi(d(x_{m_k+1},\ x_{n_k+1})) &= \psi(d(Tx_{m_k},\ Tx_{n_k}))\\ &\leqslant \psi(M_{\psi_1}(x_{m_k},\ x_{n_k}))-\phi(M_{\psi_1}(x_{m_k},\ x_{n_k})),\end{aligned} \tag{12}$$

where

$$\begin{aligned}M_{\psi_1}(x_{m_k},x_{n_k}) &= \psi_1(d(x_{m_k},\ x_{n_k}),d(x_{m_k},\ Tx_{m_k}),\ d(x_{n_k},\ Tx_{n_k}),\\ &\quad \frac{d(Tx_{m_k},\ x_{n_k})+d(x_{m_k},\ Tx_{n_k})}{2},\\ &\quad \frac{d(x_{n_k},Tx_{n_k})[1+d(x_{m_k},Tx_{m_k})]}{1+d(x_{n_k},x_{m_k})},\\ &\quad \frac{d(Tx_{m_k},x_{n_k})[1+d(x_{m_k},Tx_{n_k})]}{1+d(x_{m_k},x_{n_k})}),\\ &= \psi_1(d(x_{m_k},\ x_{n_k}),d(x_{m_k},\ x_{m_k+1}),\ d(x_{n_k},\ x_{n_k+1}),\\ &\quad \frac{d(x_{m_k+1},\ x_{n_k})+d(x_{m_k},\ x_{n_k+1})}{2},\end{aligned} \tag{13}$$

$$\frac{d(x_{n_k}, x_{n_k+1})[1 + d(x_{m_k}, x_{m_k+1})]}{1 + d(x_{n_k}, x_{m_k})},$$
$$\frac{d(x_{m_k+1}, x_{n_k})[1 + d(x_{m_k}, x_{n_k+1})]}{1 + d(x_{m_k}, x_{n_k})}).$$

On taking limits as $k \to \infty$ in (13) and using (i)-(iv) of Lemma 2, we have

$$\lim_{k\to\infty} M_{\psi_1}(x_{m_k}, x_{n_k}) = \psi_1(\epsilon, 0, 0, \epsilon, 0, \epsilon) \leqslant \epsilon. \tag{14}$$

Now, using the continuity and non-decreasing property of ψ, from (12) and (14) we have

$$\begin{aligned}\psi(\epsilon) = \lim_{k\to\infty} \psi(d(x_{m_k+1},\ x_{n_k+1})) &\leqslant \lim_{k\to\infty} \psi(d(M_{\psi_1}(m_k,\ x_{n_k})) \\ &\quad - \lim_{k\to\infty} \phi(M_{\psi_1}(x_{m_k},\ x_{n_k})), \\ &\leqslant \psi(\epsilon) - \lim_{k\to\infty} \phi(M_{\psi_1}(x_{m_k},\ x_{n_k})),\end{aligned}$$

and hence $\lim_{k\to\infty} \phi(M_{\psi_1}(x_{m_k},\ x_{n_k})) = 0$, which implies that

$$\lim_{k\to\infty} M_{\psi_1}(x_{m_k},\ x_{n_k}) = \psi_1(\epsilon, 0, 0, \epsilon, 0, \epsilon) = 0.$$

Hence from property (ii) of ψ_1, we have $\epsilon = 0$, a contradiction. So we conclude that $\{x_n\}$ is a Cauchy sequence in X.

Since $\alpha(x_n, x_{n+1}) \geqslant 1$ for all n and (X, d) is α-complete, it follows that there exists $z \in X$ such that $\lim_{n\to\infty} x_n = z$.

Since T is continuous, we have $\lim_{n\to\infty} Tx_n = Tz$, so that

$$Tz = T \lim_{n\to\infty} x_n = \lim_{n\to\infty} Tx_n = \lim_{n\to\infty} x_{n+1} = z.$$

■

Theorem 6. *Let (X, d) be a metric space and $T : X \to X$ be a generalized (α, ψ, ϕ)-rational contractive mapping. Suppose that the following conditions hold:*

(i) (X, d) is α-complete metric space,
(ii) T is a triangular α-admissible mapping,
(iii) there exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geqslant 1$ and
(iv) if $\{x_n\}$ is a sequence in X such that $x_n \to z$ and $\alpha(x_n, x_{n+1}) \geqslant 1$ for all $n \in \mathbb{N} \cup \{0\}$, then $\alpha(x_n, z) \geqslant 1$ for all $n \in \mathbb{N} \cup \{0\}$.

Then T has a fixed point in X.

Proof. From the similar arguments as in the proof of Theorem 5, we obtain that the sequence $\{x_n\}$ is Cauchy and $\alpha(x_n, x_{n+1}) \geqslant 1$ for all $n \in$

$\mathbb{N}\cup\{0\}$. Since (X,d) is an α-complete metric space, there exists $z\in X$ such that $\lim_{n\to\infty} x_n = z$. From *(iv)* we have $\alpha(x_n,z)\geqslant 1$ for all $n\in\mathbb{N}\cup\{0\}$.

Now, we show that z is a fixed point of T. We claim that

$$\begin{aligned}(15)\qquad &\frac{1}{2}d(x_n,Tx_n)\leqslant d(x_n,z)\quad \text{or}\\ &\frac{1}{2}d(x_{n+1},Tx_{n+1})\leqslant d(Tx_n,z),\quad \forall n\in\mathbb{N}\cup\{0\}.\end{aligned}$$

Suppose not, i.e., there exists $m\in\mathbb{N}\cup\{0\}$ such that $\frac{1}{2}d(x_m,Tx_m) > d(x_m,z)$ and $\frac{1}{2}d(x_{m+1},Tx_{m+1}) > d(Tx_m,z)$. We now consider

$$\begin{aligned} d(x_m,Tx_m) &\leqslant d(x_m,z)+d(z,Tx_m)\\ &< \frac{1}{2}d(x_m,Tx_m)+\frac{1}{2}d(x_{m+1},Tx_{m+1})\\ &\leqslant \frac{1}{2}d(x_m,Tx_m)+\frac{1}{2}d(x_m,Tx_m) = d(x_m,Tx_m),\end{aligned}$$

a contradiction. Hence (15) holds. Suppose $\frac{1}{2}d(x_n,Tx_n)\leqslant d(x_n,z)$, then by (4), we have

$$(16)\qquad \psi(d(Tx_n,Tz))\leqslant \psi(M_{\psi_1}(x_n,z))-\phi(M_{\psi_1}(x_n,z)),$$

where

$$\begin{aligned}(17)\ M_{\psi_1}(x_n,z) &= \psi_1(d(x_n,z),d(x_n,Tx_n),d(z,Tz),\\ &\quad \frac{d(x_n,Tz)+d(z,Tx_n)}{2},\\ &\quad \frac{d(z,Tz)[1+d(x_n,Tx_n)]}{1+d(x_n,z)},\frac{d(z,Tx_n)[1+d(x_n,Tz)]}{1+d(x_n,z)}).\end{aligned}$$

On taking limits as $n\to\infty$ and using the continuity of ψ_1 in (17), we have

$$\lim_{n\to\infty} M_{\psi_1}(x_n,z)=\psi_1(0,0,d(z,Tz),\frac{d(z,Tz)}{2},d(z,Tz),0)\leqslant d(z,Tz),$$

and also on letting $n\to\infty$ and using the continuity of ψ in (16), we obtain

$$\begin{aligned}\psi(d(z,Tz)) &\leqslant \lim_{n\to\infty}\psi(M_{\psi_1}(x_n,z))-\lim_{n\to\infty}\phi(M_{\psi_1}(x_n,z))\\ &\leqslant \psi(d(z,Tz)-\lim_{n\to\infty}\phi(M_{\psi_1}(x_n,z)),\end{aligned}$$

which implies that $\lim_{n\to\infty}\phi(M_{\psi_1}(x_n,z))=0$. Now by using the property of ϕ, we have $\lim_{n\to\infty} M_{\psi_1}(x_n,z)=\psi_1\big(0,0,d(z,Tz),\frac{d(z,Tz)}{2},d(z,Tz),0\big)=0$, which implies that $d(z,Tz)=0$. Hence $Tz=z$ ■

Theorem 7. *In addition to the hypotheses of Theorem 5 (Theorem 6) if $\alpha(u,z) \geqslant 1$ for all $u, z \in F(T)$, where $F(T)$ is the set of all fixed points of T. Then T has a unique fixed point.*

Proof. Let u and z be a fixed points of T. By hypothesis we have $\alpha(u,z) \geqslant 1$ and $0 = \frac{1}{2}d(u,Tu) \leqslant d(u,z)$. Hence, from (4) we have

$$\psi(d(u,z)) = \psi(Tu,Tz) \leqslant \psi(M_{\psi_1}(u,z)) - \phi(M_{\psi_1}(u,z)) \tag{18}$$

where

$$\begin{aligned} M_{\psi_1}(u,z) &= \psi_1(d(u,z), d(u,Tu), d(z,Tz), \frac{d(u,Tz)+d(z,Tu)}{2}, \\ &\quad \frac{d(z,Tz)[1+d(u,Tu)]}{1+d(u,z)}, \frac{d(z,Tu)[1+d(u,Tz)]}{1+d(u,z)}), \\ &= \psi_1(d(u,z), 0, 0, d(u,z), 0, d(u,z)) \leqslant d(u,z). \end{aligned}$$

Now, by using the inequality (18), we have

$$\psi(d(u,z)) \leqslant \psi(M_{\psi_1}(u,z)) - \phi(M_{\psi_1}(u,z)) \leqslant \psi(d(u,z)) - \phi(M_{\psi_1}(u,z)),$$

which implies that $\phi(M_{\psi_1}(u,z)) = 0$ and hence $M_{\psi_1}(u,z) = 0$. i.e., $\psi_1(d(u,z), 0, 0, d(u,z), 0, d(u,z)) = 0$. Now, from property (ii) of ψ_1 we have $d(u,z) = 0$. Hence $u = z$.

Therefore T has a unique fixed point. ■

The following theorems can be proved easily by the similar arguments that are given in the proofs of Theorem 5 (Theorem 6) and Theorem 7.

Theorem 8. *Let (X,d) be a metric space. Let $T : X \to X$ be a selfmapping on X and $\alpha : X \times X \to [0,\infty)$ be a given mapping. Assume that there exist $\psi \in \Psi$ and $\phi \in \Phi$ such that*

(19) *for all* $x, y \in X, \frac{1}{2}d(x,Tx) \leqslant d(x,y)$ *and* $\alpha(x,y) \geqslant 1$ *implies*

$$\begin{aligned} \psi(d(Tx,Ty)) &\leqslant \psi(\psi_2(d(x,y), d(x,fx), d(y,fy), \frac{d(x,Ty)+d(y,Tx)}{2})) \\ &\quad - \phi(\psi_2(d(x,y), d(x,fx), d(y,fy), \frac{d(x,Ty)+d(y,Tx)}{2})), \end{aligned}$$

where $\psi_2 : [0,\infty)^4 \to [0,\infty)$ is continuous and non-decreasing in each coordinate, $\psi_2(t_1,t_2,t_3,t_4) = 0 \implies t_1 = t_2 = t_3 = t_4 = 0$ and $\psi_2(t,t,t,t) \leqslant t$, for all $t > 0$.

Further, suppose that the following conditions hold:

(i) (X,d) is α-complete metric space,

(ii) T is a triangular α-admissible mapping,

(iii) there exists $x_0 \in X$ *such that* $\alpha(x_0, Tx_0) \geqslant 1$.
Assume that one of the following conditions holds:
(a) T *is continuous,*
(b) *if* $\{x_n\}$ *is a sequence in* X *such that* $x_n \to z$ *and* $\alpha(x_n, x_{n+1}) \geqslant 1$ *for all* $n \in \mathbb{N} \cup \{0\}$, *then* $\alpha(x_n, z) \geqslant 1$ *for all* $n \in \mathbb{N} \cup \{0\}$.
Then T *has a fixed point in* X.

Theorem 9. *In addition to the hypotheses of Theorem 8, if* $\alpha(u, z) \geqslant 1$ *for all* $u, z \in F(T)$, *where* $F(T)$ *is the set of all fixed points of* T. *Then* T *has a unique fixed point.*

3. Corollaries and examples

Corollary 1. *Let* (X, d) *be a complete metric space and* $T : X \to X$ *be a selfmapping of* X. *If there exist* $\psi \in \Psi$, $\phi \in \Phi$ *and* $\psi_1 \in \Psi_1$ such that

$$\text{for all } x, y \in X \text{ with } \frac{1}{2}d(x, Tx) \leqslant d(x, y) \text{ implies} \\ \psi(d(Tx, Ty)) \leqslant \psi(M_{\psi_1}(x, y)) - \phi(M_{\psi_1}(x, y)), \tag{20}$$

where M_{ψ_1} is defined as in (5), *then* T *has a unique fixed point in* X.

Proof. By choosing $\alpha(x, y) = 1$ for all $x, y \in X$, clearly the inequality (20) implies the inequality (4) and hence by Theorem 7, the conclusion of the corollary follows. ■

Corollary 2. *Let* (X, d) *be a complete metric space and* $T : X \to X$ *be a selfmapping of* X. *If there exist* $\psi \in \Psi$, $\tau > 0$ *and* $\psi_1 \in \Psi_1$ *such that*

$$\textit{for all } x, y \in X \textit{ with } \frac{1}{2}d(x, Tx) \leqslant d(x, y) \textit{ implies} \\ \tau + \psi(d(Tx, Ty)) \leqslant \psi(M_{\psi_1}(x, y)), \tag{21}$$

where M_{ψ_1} *is defined as in* (5), *then* T *has a unique fixed point in* X.

Proof. The result follows by choosing $\alpha(x, y) = 1$ for all $x, y \in X$ and $\phi(t) = \tau > 0$, clearly the inequality (21) implies the inequality (4) and hence by Theorem 7, the conclusion of the corollary follows. ■

Remark 2. If we choose $\psi_1 : [0, \infty)^6 \to [0, \infty)$, $\psi_1(t_1, t_2, t_3, t_4, t_5, t_6) = \max\{t_1, t_2, t_3, t_4, t_5, t_6\}$, by nondecreasing property of ψ, we have $\tau + \psi(d(Tx, Ty)) \leqslant \psi(d(x, y)) \leqslant \psi(M_{\psi_1}(x, y))$. Hence Theorem 4 follows as a corollary to Corollary 2.

Corollary 3. *Let (X,d) be a metric space. Let $T : X \to X$ be a self mapping of X and $\alpha : X \times X \to [0,\infty)$ be a given mapping. Assume that there exist $\psi \in \Psi$, $\phi \in \Phi$ and $\psi_1 \in \Psi_1$ such that*

$$\text{for all } x,y \in X \text{ with } \frac{1}{2}d(x,Tx) \leqslant d(x,y) \text{ implies} \quad \alpha(x,y)\psi(d(Tx,Ty)) \leqslant \psi(M_{\psi_1}(x,y)) - \phi(M_{\psi_1}(x,y)), \tag{22}$$

where M_{ψ_1} is defined as in (5). Further, suppose that the following conditions hold:

(i) (X,d) is α-complete metric space,

(ii) T is a triangular α-admissible mapping,

(iii) there exists $x_0 \in X$ such that $\alpha(x_0,Tx_0) \geqslant 1$.

Assume that one of the following conditions hold:

(a) T is continuous,

(b) if $\{x_n\}$ is a sequence in X such that $x_n \to z$ and $\alpha(x_n,x_{n+1}) \geqslant 1$ for all $n \in \mathbb{N}\cup\{0\}$, then $\alpha(x_n,z) \geqslant 1$ for all $n \in \mathbb{N}\cup\{0\}$.

Then T has a fixed point in X.

Proof. Let $x,y \in X$ with $\alpha(x,y) \geqslant 1$. Then, from (22) we have

$$\psi(d(Tx,Ty)) \leqslant \alpha(x,y)\psi(d(Tx,Ty)) \leqslant \psi(M_{\psi_1}(x,y)) - \phi(M_{\psi_1}(x,y)), \tag{23}$$

which is the inequality (4). Hence the inequality (22) implies (4). Hence by applying Theorem 5 (Theorem 6) T has a fixed point. ■

Corollary 4. *Let (X,d) be a complete metric space and $T : X \to X$ be a selfmapping of X. If there exist $\psi \in \Psi$ and $\phi \in \Phi$ such that for all $x,y \in X$*

$$\frac{1}{2}d(x,Tx) \leqslant d(x,y) \implies \psi(d(Tx,Ty)) \leqslant \psi(M(x,y)) - \phi(M(x,y)), \tag{24}$$

where $M(x,y = \max\{d(x,y), d(x,Tx), d(y,Ty), \frac{d(x,Ty)+d(y,Tx)}{2}\}$, then T has a unique fixed point in X.

Proof. By choosing $\alpha(x,y) = 1$ for all $x,y \in X$ and defining $\psi_2 : [0,\infty)^4 \to [0,\infty)$ by $\psi_2(t_1,t_2,t_3,t_4) = \max\{t_1,t_2,t_3,t_4\}$, we have $\psi_2(d(x,y), d(x,fx), d(y,fy), \frac{d(x,Ty)+d(y,Tx)}{2}) = \max\{d(x,y), d(x,fx), d(y,fy), \frac{d(x,Ty)}{2} + \frac{d(y,Tx)}{2}\}$ so that the inequality (24) implies the inequality (19) and T satisfies all the hypotheses of Theorem 9 and hence by Theorem 9, the conclusion of the corollary follows. ■

Remark 3. Since $\{\varphi : [0,\infty) \to [0,\infty) | \ \varphi$ is lower semi-continuous function with $\varphi(t) = 0$ if and only if $t = 0\} \subset \Phi$, we have Theorem 3 follows as a corollary to Corollary 4.

The following is an example in support of Theorem 5.

Example 2. Let $X = [0, \infty)$ with the usual metric d. We define $T : X \to X$ by

$$Tx = \begin{cases} \frac{x}{8} & \text{if } x \in [0, 2] \\ 2x - \frac{31}{8} & \text{if } x \in [2, \infty) \end{cases}$$

and $\alpha : X \times X \to [0, \infty)$ by

$$\alpha(x, y) = \begin{cases} 2 & \text{if } x \in [0, 1] \\ 0 & \text{otherwise.} \end{cases}$$

Since for any $x, y \in X$ with $\alpha(x, y) \geqslant 1 \iff x, y \in [0, 1]$ where $\frac{x}{8} = Tx$, $Ty = \frac{y}{8} \in [0, 1]$, and hence $\alpha(Tx, Ty) \geqslant 1$. Let $\alpha(x, y) \geqslant 1$ and $\alpha(y, z) \geqslant 1$ for $x, y, z \in X$ this implies that $x, y, z \in [0, 1]$, so that $\alpha(x, z) \geqslant 1$. Therefore T is a triangular α-admissible mapping. Clearly for any $x_0 \in [0, 1]$, $\alpha(x_0, Tx_0) \geqslant 1$.

We now show that T is a generalized (α, ψ, ϕ)-rational contractive mapping. For this purpose, we choose

$$\psi, \phi : [0, \infty) \to [0, \infty) \text{ by } \psi(t) = 2t \text{ and } \phi(t) = \begin{cases} 1 & \text{if } t = 0 \\ \frac{t}{2} & \text{if } t \neq 0, \end{cases}$$

and $\psi_1 : [0, \infty)^6 \to [0, \infty)$ by $\psi_1(t_1, t_2, t_3, t_4, t_5, t_6) = \max\{t_1, t_2, t_3, t_4, t_5.t_6\}$.

Since $\alpha(x, y) \geqslant 1$ if and only if $x, y \in [0, 1]$, we verify the inequality (4) for $x, y \in [0, 1]$ with $\frac{1}{2}d(x, Tx) \leqslant d(x, y)$. Let $x, y \in [0, 1]$. We assume without loss of generality that $x \leqslant y$. Let $\frac{1}{2}d(x, Tx) \leqslant d(x, y)$ i.e., $\frac{23}{16}x \leqslant y$. Now,

$$\begin{aligned} \psi(d(Tx, Ty)) &= \psi(d(\frac{y}{8}, \frac{x}{8})) = \psi(|\frac{y}{8} - \frac{x}{8}|) = 2|\frac{y}{8} - \frac{x}{8}| \leqslant \frac{5}{2}|y - x| \\ &= \frac{5}{2}d(x, y) \leqslant \frac{5}{2}M_{\psi_1}(x, y) = 2M_{\psi_1} - \frac{1}{2}M_{\psi_1}(x, y), \\ &= \psi(M_{\psi_1}(x, y)) - \phi(M_{\psi_1}(x, y)), \end{aligned}$$

so that the inequality (4) holds. Hence T satisfies all the hypotheses of Theorem 5, and $x = 0$ and $x = \frac{31}{8}$ are two fixed points of T.

The following example is in support of Theorem 7.

Example 3. Let $X = [0, \infty)$ with the usual metric. We define $T : X \to X$ by

$$Tx = \begin{cases} \frac{x}{4} & \text{if } x \in [0, 1] \\ x + 4 & \text{if } x \in (1, \infty) \end{cases}$$

and $\alpha : X \times X \to [0, \infty)$ by

$$\alpha(x,y) = \begin{cases} 1 & \text{if} \quad x \in [0,1] \\ 0 & \text{otherwise.} \end{cases}$$

It is easy to see that T is triangular α-admissible and for any $x_0 \in [0,1]$, $\alpha(x_0, Tx_0) \geqslant 1$.

Let $\{x_n\}$ be a sequence in X with $\alpha(x_n, x_{n+1}) \geqslant 1$ for all $n \in \mathbb{N} \cup \{0\}$ and $x_n \to x$ as $n \to \infty$, by the definition of α, we have $\{x_n\} \subset [0,1]$. Therefore$x \in [0,1]$. Hence $\alpha(x_n, x) \geqslant 1$ for all $n \in \mathbb{N} \cup \{0\}$.

We now show that T is a generalized (α, ψ, ϕ)-rational contractive mapping. For this purpose, we choose

$$\psi, \phi : [0,\infty) \to [0,\infty) \text{ by } \psi(t) = \frac{4}{3}t \text{ and } \phi(t) = \begin{cases} 2 & \text{if } t = 0 \\ \frac{t}{3} & \text{if } t \neq 0 \end{cases}$$

and $\psi_1 : [0,\infty)^6 \to [0,\infty)$ by $\psi_1(t_1, t_2, t_3, t_4, t_5, t_6) = \max\{t_1, t_2, t_3, t_4, t_5.t_6\}$.

Since $\alpha(x,y) \geqslant 1$ if and only if $x, y \in [0,1]$, we verify the inequality (4) for $x, y \in [0,1]$ with $\frac{1}{2}d(x,Tx) \leqslant d(x,y)$. Let $x, y \in [0,1]$. We assume without loss of generality that $x \leqslant y$. Let $\frac{1}{2}d(x,Tx) \leqslant d(x,y)$ i.e., $\frac{11}{8}x \leqslant y$. Now,

$$\begin{aligned} \psi(d(Tx,Ty)) &= \psi(d(\frac{y}{4}, \frac{x}{4})) = \psi(|\frac{y}{4} - \frac{x}{4}|) = \frac{4}{3}|\frac{y}{4} - \frac{x}{4}| \\ &= \frac{1}{3}|y - x| \leqslant |y - x| = d(x,y) \\ &\leqslant M_{\psi_1}(x,y) = \frac{4}{3}M_{\psi_1}(x,y) - \frac{1}{3}M_{\psi_1}(x,y), \\ &= \psi(M_{\psi_1}(x,y)) - \phi(M_{\psi_1}(x,y)). \end{aligned}$$

Hence T satisfies all the hypotheses of Theorem 7 and $x = 0$ is the unique fixed point of T.

Here we observe that for $x = 0$ and $y = 2$, we have $\frac{1}{2}d(0,T0) = 0 \leqslant 2 = |2-0| = d(0,2)$ and $M(0,2) = \max\{d(0,2), d(0,T0), d(2,T2), \frac{d(0,T2)+d(2,T0)}{2}\}$ $= 4$, hence $\psi(d(T0,T2)) = \psi(d(0,6)) = \psi(6) \nleq \psi(4) \nleq \psi(4) - \phi(4)$, for any continuous and nondecreasing ψ, and lower semi-continuous ϕ with $\phi(t) = 0$ if and only if $t = 0$. Hence Theorem 3 is not applicable.

Hence Example 3 and Remark 3 suggest that Corollary 4 is a generalization of Theorem 3 which in turn Theorem 9 is a generalization of Theorem 3.

Similarly $F(d(T0,T2)) = F(6) \nleq F(2) \nleq F(2) - \tau$ for any continuous and strictly increasing map F and $\tau > 0$. Hence Theorem 4 is also not applicable.

Hence Example 3 and Remark 2 suggest that Corollary 2 is a generalization of Theorem 4 which in turn Theorem 7 is a generalization of Theorem 4.

Acknowledgements. The authors sincerely thank the referees for their valuable suggestions and comments which improved the presentation of the paper.

References

[1] Ahmad J., Al-Rawashdeh A., Azam A., New fixed point theorems for generalized F-contractions in complete metric spaces, *Fixed Point Theory Appl.,* 80(2015).

[2] Alber Ya.I., Guerre-Delabriere S., Principle of weakly contractive maps in Hilbert spaces, *New Results in Operator Theory and Its Applications, Oper. Theory Adv. Appl.* 98, *Birkhauser, Basel,* (1997), 7-22.

[3] Ansari A.H., Note on φ-ψ-contractive type mappings and related fixed point, *Proceedings of the 2nd Regional Conference on Math. and Appl.,* (2014), 377-380.

[4] Babu G.V.R., Sailaja P.D., A fixed point theorem of generalized weakly contractive maps in orbitally complete metric spaces, *Thai J. of Math.,* 9(1) 2011, 1-10.

[5] Chandok S., Tas K., Ansari A.H., Some Fixed Point Results for TAC-type contractive mappings, *J. Funct. Spaces,* Volume 2016, Article ID 1907676, 6 pages.

[6] Czerwik S., Contraction mappings in b-metric spaces, *Acta Math. Inform. Univ. Ostraviensis,* 1(1993), 5-11.

[7] Doric D., Common fixed point for generalized (ψ, φ)-weak contractions, *Appl. Math. Lett.,* 22(2009), 1896-1900.

[8] Dutta P.N., Choudhury B.S., A generalization of contraction principle in metric spaces, *Fixed Point Theory Appl.,* (2008), Article ID 406368, 8 pages.

[9] Hieu N.T., Dung N.V., Some fixed point results for generalized rational type contraction mappings in partially ordered $b-$metric spaces, *Facta Univ. Ser. Math. Inform.,* 30(1)(2015), 49-66.

[10] Huang H., Ansari A.H., Dolicanin-Dekic D., Radenovic S., Some fixed point results for rational type and subrational type contractive mappings, *Acta Univ. Sapientiae Math.,* 9(1)(2017), 185-201.

[11] Huang H., Deng G., Chen Z., Radenovic S., On some recent fixed point results for $\alpha-$admissible mappings in $b-$metric spaces, *J. Comput. Anal. Appl.,* 25(2)(2018), 255-268.

[12] Hussain N., Kutbi M.A., Salimi P., Fixed point theory in α-complete metric spaces with applications, *Abstr. Appl. Anal.,* (2014), Article ID 280817.

[13] Jaggi D.S., Some unique fixed point theorems, *Indian J. of Pure and Appl. Math.,* 8(1977), 223-230.

[14] Karapinar E., Kumam P., Salimi P., On $\alpha - \psi$ Meir-Keeler contractive mappings, *Fixed Point Theory Appl.,* 94(2013), 12 pages.

[15] Khan M.S., Swaleh M.S., Sessa S., Fixed point theorems by altering distances between the points, *Bull. Aust. Math. Soc.,* 30(1984), 1-9.

[16] Kirk W.A., Srinivasan P.S., Veeramani P., Fixed points for mappings satisfying cyclical contractive conditions, *Fixed Point Theory,* 4(2003), 79-89.

[17] Pansuwon A., Sintunavarat W., Parvaneh V., Cho Y.J., Some fixed point theorems for (α, θ, k)-contractive multi-valued mappings with some applications, *Fixed Point Theory Appl.,* 132(2015).
[18] Piri H., Kumam P., Some fixed point theorems concerning F-contraction in complete metric spaces, *Fixed Point Theory Appl.,* 2014, Article ID 210 (2014).
[19] Piri H., Rahrovi S., Generalized multivalued F-weak contraction on complete metric spaces, *Sahand Commun. Math. Anal.* (*SCMA*)*,* 2(2)(2015), 1-11.
[20] Rhoades B.E., Some theorems on weakly contractive maps, *Nonlinear Anal.,* 47(2001), 2683-2693.
[21] Samet B. Vetro C., Vetro P., Fixed point theorems for $\alpha - \psi$-contractive type mappings, *Nonlinear Anal.,* 75(2012), 2154-2165.
[22] Singh S.L., Kamal R., De la Sen M., Chugh R., A fixed point theorem for generalized weak contractions, *Filomat,* 29(7)(2015), 1481-1490.
[23] Wardowski D., Fixed point theory of a new type of contractive mappings in complete metric spaces, *Fixed Point Theory Appl.,* 2012, Article ID 94 (2012).
[24] Wardowski D., Van Dung N., Fixed points of F-weakly contractions on complete metric spaces, *Demonstr. Math.,* Vol. XLVII, (1)(2014).

G. V. Ravindranadh Babu
Department of Mathematics
Andhra University
Visakhapatnam-530 003, India
e-mail: gvr_babu@hotmail.com

M. Dula Tolera
Department of Mathematics
Andhra University
Visakhapatnam-530 003, India,
or
Department of Mathematics
Wollega University
Nekemte-395, Ethiopia
e-mail: dulamosissa@gmail.com

Received on 24.10.2016 and, in revised form, on 16.11.2017.