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**HERMITE-HADAMARD TYPE FRACTIONAL
 INTEGRAL INEQUALITIES FOR GENERALIZED
 $(r; g, s, m, \varphi)$ -PREINVEX FUNCTIONS**

ABSTRACT. In the present paper, a new class of generalized $(r; g, s, m, \varphi)$ -preinvex functions is introduced and some new integral inequalities for the left hand side of Gauss-Jacobi type quadrature formula involving generalized $(r; g, s, m, \varphi)$ -preinvex functions are given. Moreover, some generalizations of Hermite-Hadamard type inequalities for generalized $(r; g, s, m, \varphi)$ -preinvex functions via Riemann-Liouville fractional integrals are established. These results not only extend the results appeared in the literature (see [1],[2]), but also provide new estimates on these types.

KEY WORDS: Hermite-Hadamard type inequality, Hölder's inequality, Minkowski's inequality, Cauchy's inequality, power mean inequality, Riemann-Liouville fractional integral, s -convex function in the second sense, m -invex, P -function.

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1. Introduction and preliminaries

The following notation is used throughout this paper. We use I to denote an interval on the real line $\mathbb{R} = (-\infty, +\infty)$ and I° to denote the interior of I . For any subset $K \subseteq \mathbb{R}^n$, K° is used to denote the interior of K . \mathbb{R}^n is used to denote a n -dimensional vector space. The set of integrable functions on the interval $[a, b]$ is denoted by $L_1[a, b]$.

The following inequality, named Hermite-Hadamard inequality, is one of the most famous inequalities in the literature for convex functions.

Theorem 1. *Let $f : I \subseteq \mathbb{R} \rightarrow \mathbb{R}$ be a convex function on I and $a, b \in I$ with $a < b$. Then the following inequality holds:*

$$(1) \quad f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a} \int_a^b f(x) dx \leq \frac{f(a) + f(b)}{2}.$$

Fractional calculus (see [14]), was introduced at the end of the nineteenth century by Liouville and Riemann, the subject of which has become a rapidly growing area and has found applications in diverse fields ranging from physical sciences and engineering to biological sciences and economics.

Definition 1. Let $f \in L_1[a, b]$. The Riemann-Liouville integrals $J_{a+}^\alpha f$ and $J_{b-}^\alpha f$ of order $\alpha > 0$ with $a \geq 0$ are defined by

$$J_{a+}^\alpha f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (x-t)^{\alpha-1} f(t) dt, \quad x > a$$

and

$$J_{b-}^\alpha f(x) = \frac{1}{\Gamma(\alpha)} \int_x^b (t-x)^{\alpha-1} f(t) dt, \quad b > x,$$

where $\Gamma(\alpha) = \int_0^{+\infty} e^{-u} u^{\alpha-1} du$. Here $J_{a+}^0 f(x) = J_{b-}^0 f(x) = f(x)$.

In the case of $\alpha = 1$, the fractional integral reduces to the classical integral.

Due to the wide application of fractional integrals, some authors extended to study fractional Hermite-Hadamard type inequalities for functions of different classes (see [13],[14]).

Now, let us recall some definitions of various convex functions.

Definition 2 (see [4]). A nonnegative function $f : I \subseteq \mathbb{R} \rightarrow [0, +\infty)$ is said to be P -function or P -convex, if

$$f(tx + (1-t)y) \leq f(x) + f(y), \quad \forall x, y \in I, t \in [0, 1].$$

Definition 3 (see [5]). A function $f : [0, +\infty) \rightarrow \mathbb{R}$ is said to be s -convex in the second sense, if

$$(2) \quad f(\lambda x + (1-\lambda)y) \leq \lambda^s f(x) + (1-\lambda)^s f(y)$$

for all $x, y \geq 0$, $\lambda \in [0, 1]$ and $s \in (0, 1]$.

It is clear that a 1-convex function must be convex on $[0, +\infty)$ as usual. The s -convex functions in the second sense have been investigated in (see [5]).

Definition 4 (see [6]). A set $K \subseteq \mathbb{R}^n$ is said to be invex with respect to the mapping $\eta : K \times K \rightarrow \mathbb{R}^n$, if $x + t\eta(y, x) \in K$ for every $x, y \in K$ and $t \in [0, 1]$.

Notice that every convex set is invex with respect to the mapping $\eta(y, x) = y - x$, but the converse is not necessarily true. For more details (see [6],[7]).

Definition 5 (see [8]). *The function f defined on the invex set $K \subseteq \mathbb{R}^n$ is said to be preinvex with respect η , if for every $x, y \in K$ and $t \in [0, 1]$, we have that*

$$f(x + t\eta(y, x)) \leq (1 - t)f(x) + tf(y).$$

The concept of preinvexity is more general than convexity since every convex function is preinvex with respect to the mapping $\eta(y, x) = y - x$, but the converse is not true.

The Gauss-Jacobi type quadrature formula has the following

$$(3) \quad \int_a^b (x - a)^p (b - x)^q f(x) dx = \sum_{k=0}^{+\infty} B_{m,k} f(\gamma_k) + R_m^* |f|,$$

for certain $B_{m,k}$, γ_k and rest $R_m^* |f|$ (see [9]). Recently, Liu (see [10]) obtained several integral inequalities for the left hand side of (3) under the Definition 2 of P -function. Also in (see [11]), Özdemir et al. established several integral inequalities concerning the left-hand side of (3) via some kinds of convexity.

Motivated by these results, in Section , the notion of generalized $(r; g, s, m, \varphi)$ -preinvex function is introduced and some new integral inequalities for the left hand side of (3) involving generalized $(r; g, s, m, \varphi)$ -preinvex functions are given. In Section , some generalizations of Hermite-Hadamard type inequalities for generalized $(r; g, s, m, \varphi)$ -preinvex functions via fractional integrals are given. These general inequalities give us some new estimates for the left hand side of Gauss-Jacobi type quadrature formula and Hermite-Hadamard type fractional integral inequalities.

2. New integral inequalities for generalized $(r; g, s, m, \varphi)$ -preinvex functions

Definition 6 (see [3]). *A set $K \subseteq \mathbb{R}^n$ is said to be m -invex with respect to the mapping $\eta : K \times K \times (0, 1] \rightarrow \mathbb{R}^n$ for some fixed $m \in (0, 1]$, if $mx + t\eta(y, x, m) \in K$ holds for each $x, y \in K$ and any $t \in [0, 1]$.*

Remark 1. In Definition 6, under certain conditions, the mapping $\eta(y, x, m)$ could reduce to $\eta(y, x)$.

Definition 7 (see [12]). *A positive function f on the invex set K is said to be logarithmically preinvex, if*

$$f(u + t\eta(v, u)) \leq f^{1-t}(u) f^t(v)$$

for all $u, v \in K$ and $t \in [0, 1]$.

Definition 8 (see [12]). *The function f on the invex set K is said to be r -preinvex with respect to η , if*

$$f(u + t\eta(v, u)) \leq M_r(f(u), f(v); t)$$

holds for all $u, v \in K$ and $t \in [0, 1]$, where

$$M_r(x, y; t) = \begin{cases} [(1-t)x^r + ty^r]^{\frac{1}{r}}, & \text{if } r \neq 0 \\ x^{1-t}y^t, & \text{if } r = 0, \end{cases}$$

is the weighted power mean of order r for positive numbers x and y .

We next give new definition, to be referred as generalized $(r; g, s, m, \varphi)$ -preinvex function.

Definition 9. *Let $K \subseteq \mathbb{R}$ be an open m -invex set with respect to η : $K \times K \times (0, 1] \rightarrow \mathbb{R}$, $g : [0, 1] \rightarrow [0, 1]$ be a differentiable function and $\varphi : I \rightarrow K$ is a continuous function. The function $f : K \rightarrow (0, +\infty)$ is said to be generalized $(r; g, s, m, \varphi)$ -preinvex with respect to η , if*

$$(4) \quad f(m\varphi(x) + g(t)\eta(\varphi(y), \varphi(x), m)) \leq M_r(f(\varphi(x)), f(\varphi(y)), m, s; t)$$

holds for any fixed $s, m \in (0, 1]$ and for all $x, y \in I, t \in [0, 1]$, where

$$M_r(f(\varphi(x)), f(\varphi(y)), m, s; t) = \begin{cases} [m(1-g(t))^s f^r(\varphi(x)) + g^s(t) f^r(\varphi(y))]^{\frac{1}{r}}, & \text{if } r \neq 0 \\ [f(\varphi(x))]^{m(1-g(t))^s} [f(\varphi(y))]^{g^s(t)}, & \text{if } r = 0, \end{cases}$$

is the weighted power mean of order r for positive numbers $f(\varphi(x))$ and $f(\varphi(y))$.

Remark 2. In Definition 9, it is worthwhile to note that the class of generalized $(r; g, s, m, \varphi)$ -preinvex function is a generalization of the class of s -convex in the second sense function given in Definition 3. Also, for $r = 1, g(t) = t, \forall t \in [0, 1]$ and $\varphi(x) = x, \forall x \in I$, we get the notion of generalized (s, m) -preinvex function (see [3]).

Example 1. Let $f(x) = -|x|, g(t) = t, \varphi(x) = x, r = s = 1$ and

$$\eta(y, x, m) = \begin{cases} y - mx, & \text{if } x \geq 0, y \geq 0 \\ y - mx, & \text{if } x \leq 0, y \leq 0 \\ mx - y, & \text{if } x \geq 0, y \leq 0 \\ mx - y, & \text{if } x \leq 0, y \geq 0. \end{cases}$$

Then $f(x)$ is a generalized $(1; t, 1, m, x)$ -preinvex function of with respect to $\eta : \mathbb{R} \times \mathbb{R} \times (0, 1] \rightarrow \mathbb{R}$ and any fixed $m \in (0, 1]$. However, it is obvious that $f(x) = -|x|$ is not a convex function on \mathbb{R} .

In this section, in order to prove our main results regarding some new integral inequalities involving generalized $(r; g, s, m, \varphi)$ -preinvex functions, we need the following new interesting lemma:

Lemma 1. *Let $\varphi : I \rightarrow K$ be a continuous function and $g : [0, 1] \rightarrow [0, 1]$ is a differentiable function. Assume that $f : K = [m\varphi(a), m\varphi(a) + \eta(\varphi(b), \varphi(a), m)] \rightarrow \mathbb{R}$ is a continuous function on K° with respect to $\eta : K \times K \times (0, 1] \rightarrow \mathbb{R}$, for $m\varphi(a) < m\varphi(a) + \eta(\varphi(b), \varphi(a), m)$. Then for any fixed $m \in (0, 1]$ and $p, q > 0$, we have*

$$\begin{aligned} & \int_{m\varphi(a)}^{m\varphi(a)+\eta(\varphi(b),\varphi(a),m)} (x - m\varphi(a))^p (m\varphi(a) + \eta(\varphi(b), \varphi(a), m) - x)^q f(x) dx \\ &= \eta(\varphi(b), \varphi(a), m)^{p+q+1} \\ & \quad \times \int_0^1 g^p(t)(1 - g(t))^q f(m\varphi(a) + g(t)\eta(\varphi(b), \varphi(a), m)) d[g(t)]. \end{aligned}$$

Proof. It is easy to observe that

$$\begin{aligned} & \int_{m\varphi(a)}^{m\varphi(a)+\eta(\varphi(b),\varphi(a),m)} (x - m\varphi(a))^p (m\varphi(a) + \eta(\varphi(b), \varphi(a), m) - x)^q f(x) dx \\ &= \eta(\varphi(b), \varphi(a), m) \int_0^1 (m\varphi(a) + g(t)\eta(\varphi(b), \varphi(a), m) - m\varphi(a))^p \\ & \quad \times (m\varphi(a) + \eta(\varphi(b), \varphi(a), m) - m\varphi(a) - g(t)\eta(\varphi(b), \varphi(a), m))^q \\ & \quad \times f(m\varphi(a) + g(t)\eta(\varphi(b), \varphi(a), m)) d[g(t)] \\ &= \eta(\varphi(b), \varphi(a), m)^{p+q+1} \\ & \quad \times \int_0^1 g^p(t)(1 - g(t))^q f(m\varphi(a) + g(t)\eta(\varphi(b), \varphi(a), m)) d[g(t)]. \end{aligned}$$

■

Theorem 2. *Let $\varphi : I \rightarrow K$ be a continuous function and $g : [0, 1] \rightarrow [0, 1]$ is a differentiable function. Assume that $f : K = [m\varphi(a), m\varphi(a) + \eta(\varphi(b), \varphi(a), m)] \rightarrow (0, +\infty)$ is a continuous function on K° with $m\varphi(a) < m\varphi(a) + \eta(\varphi(b), \varphi(a), m)$. Let $k > 1$ and $0 < r \leq 1$. If $f^{\frac{k}{k-1}}$ is generalized $(r; g, s, m, \varphi)$ -preinvex function on an open m -invex set K with respect to $\eta : K \times K \times (0, 1] \rightarrow \mathbb{R}$ for any fixed $s, m \in (0, 1]$, then for any fixed $p, q > 0$,*

$$\begin{aligned} (5) \quad & \int_{m\varphi(a)}^{m\varphi(a)+\eta(\varphi(b),\varphi(a),m)} (x - m\varphi(a))^p (m\varphi(a) \\ & \quad + \eta(\varphi(b), \varphi(a), m) - x)^q f(x) dx \\ & \leq |\eta(\varphi(b), \varphi(a), m)|^{p+q+1} \left(\frac{r}{s+r} \right)^{\frac{k-1}{k}} B^{\frac{1}{k}}(g(t); k, p, q) \end{aligned}$$

$$\begin{aligned} & \times \left[m \left((1 - g(0))^{\frac{s}{r}+1} - (1 - g(1))^{\frac{s}{r}+1} \right)^r f^{\frac{rk}{k-1}}(\varphi(a)) \right. \\ & \left. + \left(g^{\frac{s}{r}+1}(1) - g^{\frac{s}{r}+1}(0) \right)^r f^{\frac{rk}{k-1}}(\varphi(b)) \right]^{\frac{k-1}{rk}}, \end{aligned}$$

where $B(g(t); k, p, q) = \int_0^1 g^{kp}(t)(1 - g(t))^{kq} d[g(t)]$.

Proof. Let $k > 1$ and $0 < r \leq 1$. Since $f^{\frac{k}{k-1}}$ is generalized $(r; g, s, m, \varphi)$ -preinvex function on K , combining with Lemma 1, Hölder inequality and Minkowski inequality for all $t \in [0, 1]$ and for any fixed $s, m \in (0, 1]$, we get

$$\begin{aligned} & \int_{m\varphi(a)}^{m\varphi(a)+\eta(\varphi(b), \varphi(a), m)} (x - m\varphi(a))^p (m\varphi(a) + \eta(\varphi(b), \varphi(a), m) - x)^q f(x) dx \\ & \leq |\eta(\varphi(b), \varphi(a), m)|^{p+q+1} \left[\int_0^1 g^{kp}(t)(1 - g(t))^{kq} d[g(t)] \right]^{\frac{1}{k}} \\ & \quad \times \left[\int_0^1 |f(m\varphi(a) + g(t)\eta(\varphi(b), \varphi(a), m))|^{\frac{k}{k-1}} d[g(t)] \right]^{\frac{k-1}{k}} \\ & \leq |\eta(\varphi(b), \varphi(a), m)|^{p+q+1} B^{\frac{1}{k}}(g(t); k, p, q) \\ & \quad \times \left[\int_0^1 \left(m(1 - g(t))^s f^{\frac{rk}{k-1}}(\varphi(a)) + g^s(t) f^{\frac{rk}{k-1}}(\varphi(b)) \right)^{\frac{1}{r}} d[g(t)] \right]^{\frac{k-1}{k}} \\ & \leq |\eta(\varphi(b), \varphi(a), m)|^{p+q+1} B^{\frac{1}{k}}(g(t); k, p, q) \\ & \quad \times \left[\left(\int_0^1 m^{\frac{1}{r}} (1 - g(t))^{\frac{s}{r}} f^{\frac{k}{k-1}}(\varphi(a)) d[g(t)] \right)^r \right. \\ & \quad \left. + \left(\int_0^1 g^{\frac{s}{r}}(t) f^{\frac{k}{k-1}}(\varphi(b)) d[g(t)] \right)^r \right]^{\frac{k-1}{rk}} \\ & = |\eta(\varphi(b), \varphi(a), m)|^{p+q+1} \left(\frac{r}{s+r} \right)^{\frac{k-1}{k}} B^{\frac{1}{k}}(g(t); k, p, q) \\ & \quad \times \left[m \left((1 - g(0))^{\frac{s}{r}+1} - (1 - g(1))^{\frac{s}{r}+1} \right)^r f^{\frac{rk}{k-1}}(\varphi(a)) \right. \\ & \quad \left. + \left(g^{\frac{s}{r}+1}(1) - g^{\frac{s}{r}+1}(0) \right)^r f^{\frac{rk}{k-1}}(\varphi(b)) \right]^{\frac{k-1}{rk}}. \end{aligned}$$

■

Corollary 1. *Under the same conditions as in Theorem 2 for $r = 1$ and $g(t) = t$, we get (see [1], Theorem 2.2).*

Theorem 3. *Let $\varphi : I \rightarrow K$ be a continuous function and $g : [0, 1] \rightarrow [0, 1]$ is a differentiable function. Assume that $f : K = [m\varphi(a), m\varphi(a) + \eta(\varphi(b), \varphi(a), m)] \rightarrow (0, +\infty)$ is a continuous function on K° with $m\varphi(a) < m\varphi(a) + \eta(\varphi(b), \varphi(a), m)$. Let $l \geq 1$ and $0 < r \leq 1$. If f^l is generalized $(r; g, s, m, \varphi)$ -preinvex function on an open m -invex set K with respect to $\eta : K \times K \times (0, 1] \rightarrow \mathbb{R}$ for any fixed $s, m \in (0, 1]$, then for any fixed $p, q > 0$,*

$$(6) \int_{m\varphi(a)}^{m\varphi(a)+\eta(\varphi(b),\varphi(a),m)} (x - m\varphi(a))^p (m\varphi(a) + \eta(\varphi(b), \varphi(a), m) - x)^q f(x) dx \\ \leq |\eta(\varphi(b), \varphi(a), m)|^{p+q+1} B^{\frac{l-1}{l}}(g(t); p, q) \\ \times \left[m f^{rl}(\varphi(a)) B^r \left(g(t); p, q + \frac{s}{r} \right) + f^{rl}(\varphi(b)) B^r \left(g(t); p + \frac{s}{r}, q \right) \right]^{\frac{1}{rl}},$$

where $B(g(t); p, q) = \int_0^1 g^p(t)(1 - g(t))^q d[g(t)]$.

Proof. Let $l \geq 1$ and $0 < r \leq 1$. Since f^l is generalized $(r; g, s, m, \varphi)$ -preinvex function on K , combining with Lemma 1, the well-known power mean inequality and Minkowski inequality for all $t \in [0, 1]$ and for any fixed $s, m \in (0, 1]$, we get

$$\int_{m\varphi(a)}^{m\varphi(a)+\eta(\varphi(b),\varphi(a),m)} (x - m\varphi(a))^p (m\varphi(a) + \eta(\varphi(b), \varphi(a), m) - x)^q f(x) dx \\ = \eta(\varphi(b), \varphi(a), m)^{p+q+1} \\ \times \int_0^1 \left[g^p(t)(1 - g(t))^q \right]^{\frac{l-1}{l}} \left[g^p(t)(1 - g(t))^q \right]^{\frac{1}{l}} \\ \times f(m\varphi(a) + g(t)\eta(\varphi(b), \varphi(a), m)) d[g(t)] \\ \leq |\eta(\varphi(b), \varphi(a), m)|^{p+q+1} \left[\int_0^1 g^p(t)(1 - g(t))^q d[g(t)] \right]^{\frac{l-1}{l}} \\ \times \left[\int_0^1 g^p(t)(1 - g(t))^q |f(m\varphi(a) + g(t)\eta(\varphi(b), \varphi(a), m))|^l d[g(t)] \right]^{\frac{1}{l}} \\ \leq |\eta(\varphi(b), \varphi(a), m)|^{p+q+1} B^{\frac{l-1}{l}}(g(t); p, q) \\ \times \left[\int_0^1 g^p(t)(1 - g(t))^q \left(m(1 - g(t))^s f^{rl}(\varphi(a)) \right) \right]$$

$$\begin{aligned}
& \left. + g^s(t) f^{rl}(\varphi(b)) \right)^{\frac{1}{r}} d[g(t)] \Bigg]^{\frac{1}{l}} \\
& \leq |\eta(\varphi(b), \varphi(a), m)|^{p+q+1} B^{\frac{l-1}{l}}(g(t); p, q) \\
& \quad \times \left[\left(\int_0^1 m^{\frac{1}{r}} g^p(t) (1-g(t))^{q+\frac{s}{r}} f^l(\varphi(a)) d[g(t)] \right)^r \right. \\
& \quad \left. + \left(\int_0^1 g^{p+\frac{s}{r}}(t) (1-g(t))^q f^l(\varphi(b)) d[g(t)] \right)^r \right]^{\frac{1}{rl}} \\
& = |\eta(\varphi(b), \varphi(a), m)|^{p+q+1} B^{\frac{l-1}{l}}(g(t); p, q) \\
& \quad \times \left[m f^{rl}(\varphi(a)) B^r \left(g(t); p, q + \frac{s}{r} \right) + f^{rl}(\varphi(b)) B^r \left(g(t); p + \frac{s}{r}, q \right) \right]^{\frac{1}{rl}}.
\end{aligned}$$

■

Corollary 2. *Under the same conditions as in Theorem 3 for $r = 1$ and $g(t) = t$, we get (see [1], Theorem 2.3).*

3. Hermite-Hadamard type fractional integral inequalities for generalized $(r; g, s, m, \varphi)$ -preinvex functions

In this section, we prove our main results regarding some generalizations of Hermite-Hadamard type inequalities for generalized $(r; g, s, m, \varphi)$ -preinvex functions via fractional integrals.

Theorem 4. *Let $\varphi : I \rightarrow K$ be a continuous function and $g : [0, 1] \rightarrow [0, 1]$ is a differentiable function. Suppose $K \subseteq \mathbb{R}$ be an open m -invex subset with respect to $\eta : K \times K \times (0, 1] \rightarrow \mathbb{R}$ for any fixed $s, m \in (0, 1]$ with $m\varphi(a) < m\varphi(a) + \eta(\varphi(b), \varphi(a), m)$. Assume that $f : K = [m\varphi(a), m\varphi(a) + \eta(\varphi(b), \varphi(a), m)] \rightarrow (0, +\infty)$ be generalized $(r; g, s, m, \varphi)$ -preinvex function on an open m -invex set K° . Then for $\alpha > 0$ and $0 < r \leq 1$, we have*

$$\begin{aligned}
(7) \quad & \frac{1}{\eta^\alpha(\varphi(b), \varphi(a), m)} \int_{m\varphi(a)+g(0)\eta(\varphi(b), \varphi(a), m)}^{m\varphi(a)+g(1)\eta(\varphi(b), \varphi(a), m)} (t - m\varphi(a))^{\alpha-1} f(t) dt \\
& \leq \left[m f^r(\varphi(a)) B^r \left(g(t); \alpha - 1, \frac{s}{r} \right) \right. \\
& \quad \left. + f^r(\varphi(b)) \left(\frac{r}{\alpha r + s} \right)^r \left(g^{\frac{s}{r}+\alpha}(1) - g^{\frac{s}{r}+\alpha}(0) \right)^r \right]^{\frac{1}{r}}.
\end{aligned}$$

Proof. Let $0 < r \leq 1$. Since f is generalized $(r; g, s, m, \varphi)$ -preinvex function on an open m -invex set K° , combining with Minkowski inequality for all $t \in [0, 1]$ and for any fixed $s, m \in (0, 1]$, we get

$$\begin{aligned} & \frac{1}{\eta^\alpha(\varphi(b), \varphi(a), m)} \int_{m\varphi(a)+g(0)\eta(\varphi(b), \varphi(a), m)}^{m\varphi(a)+g(1)\eta(\varphi(b), \varphi(a), m)} (t - m\varphi(a))^{\alpha-1} f(t) dt \\ &= \int_0^1 g^{\alpha-1}(t) f(m\varphi(a) + g(t)\eta(\varphi(b), \varphi(a), m)) d[g(t)] \\ &\leq \int_0^1 g^{\alpha-1}(t) \left[m(1 - g(t))^s f^r(\varphi(a)) + g^s(t) f^r(\varphi(b)) \right]^{\frac{1}{r}} d[g(t)] \\ &\leq \left\{ \left[\int_0^1 g^{\alpha-1+\frac{s}{r}}(t) f(\varphi(b)) d[g(t)] \right]^r \right. \\ &\quad \left. + \left[\int_0^1 m^{\frac{1}{r}} g^{\alpha-1}(t) (1 - g(t))^{\frac{s}{r}} f(\varphi(a)) d[g(t)] \right]^r \right\}^{\frac{1}{r}} \\ &= \left[m f^r(\varphi(a)) B^r \left(g(t); \alpha - 1, \frac{s}{r} \right) \right. \\ &\quad \left. + f^r(\varphi(b)) \left(\frac{r}{\alpha r + s} \right)^r \left(g^{\frac{s}{r} + \alpha}(1) - g^{\frac{s}{r} + \alpha}(0) \right)^r \right]^{\frac{1}{r}}. \end{aligned}$$

■

Corollary 3. Under the same conditions as in Theorem 4 for $m = s = 1$, $\varphi(x) = x$, $\eta(\varphi(b), \varphi(a), m) = \eta(b, a)$ and $g(t) = t$, we get (see [2], Theorem 3.1).

Theorem 5. Let $\varphi : I \rightarrow K$ be a continuous function and $g : [0, 1] \rightarrow [0, 1]$ is a differentiable function. Suppose $K \subseteq \mathbb{R}$ be an open m -invex subset with respect to $\eta : K \times K \times (0, 1] \rightarrow \mathbb{R}$ for any fixed $s, m \in (0, 1)$ with $m\varphi(a) < m\varphi(a) + \eta(\varphi(b), \varphi(a), m)$. Assume that $f, h : K = [m\varphi(a), m\varphi(a) + \eta(\varphi(b), \varphi(a), m)] \rightarrow (0, +\infty)$ are respectively generalized $(r; g, s, m, \varphi)$ -preinvex function and generalized $(l; g, s, m, \varphi)$ -preinvex function on an open m -invex set K° . Then for $\alpha > 0$, $r > 1$ and $r^{-1} + l^{-1} = 1$, we have

$$(8) \quad \frac{1}{\eta^\alpha(\varphi(b), \varphi(a), m)} \int_{m\varphi(a)+g(0)\eta(\varphi(b), \varphi(a), m)}^{m\varphi(a)+g(1)\eta(\varphi(b), \varphi(a), m)} (t - m\varphi(a))^{\alpha-1} f(t) h(t) dt \\ \leq \frac{1}{2} \left\{ \left[m f^r(\varphi(a)) B^{\frac{r}{2}} \left(g(t); \frac{2(\alpha-1)}{r}, \frac{2s}{r} \right) \right. \right.$$

$$\begin{aligned}
& + f^r(\varphi(b)) \left(\frac{r}{2(\alpha-1+s)+r} \right)^{\frac{r}{2}} \left(g^{\frac{2(\alpha-1+s)}{r}+1}(1) - g^{\frac{2(\alpha-1+s)}{r}+1}(0) \right)^{\frac{r}{2}} \Bigg]^{\frac{2}{r}} \\
& + \left[mh^l(\varphi(a)) B^{\frac{l}{2}} \left(g(t); \frac{2(\alpha-1)}{l}, \frac{2s}{l} \right) \right. \\
& \left. + h^l(\varphi(b)) \left(\frac{l}{2(\alpha-1+s)+l} \right)^{\frac{l}{2}} \left(g^{\frac{2(\alpha-1+s)}{l}+1}(1) - g^{\frac{2(\alpha-1+s)}{l}+1}(0) \right)^{\frac{l}{2}} \right]^{\frac{l}{2}} \Bigg]^{\frac{2}{l}} \Bigg\}.
\end{aligned}$$

Proof. Let $r > 1$ and $r^{-1} + l^{-1} = 1$. Since f and h are respectively generalized $(r; g, s, m, \varphi)$ -preinvex function and generalized $(l; g, s, m, \varphi)$ -preinvex function on an open m -invex set K° , combining with Cauchy and Minkowski inequalities for all $t \in [0, 1]$ and for any fixed $s, m \in (0, 1]$, we get

$$\begin{aligned}
& \frac{1}{\eta^\alpha(\varphi(b), \varphi(a), m)} \int_{m\varphi(a)+g(0)\eta(\varphi(b), \varphi(a), m)}^{m\varphi(a)+g(1)\eta(\varphi(b), \varphi(a), m)} (t - m\varphi(a))^{\alpha-1} f(t)h(t) dt \\
& = \int_0^1 g^{(\alpha-1)(\frac{1}{r}+\frac{1}{l})}(t) f(m\varphi(a) + g(t)\eta(\varphi(b), \varphi(a), m)) \\
& \quad \times h(m\varphi(a) + g(t)\eta(\varphi(b), \varphi(a), m)) d[g(t)] \\
& \leq \int_0^1 g^{(\alpha-1)(\frac{1}{r}+\frac{1}{l})}(t) \left[m(1-g(t))^s f^r(\varphi(a)) + g^s(t) f^r(\varphi(b)) \right]^{\frac{1}{r}} \\
& \quad \times \left[m(1-g(t))^s h^l(\varphi(a)) + g^s(t) h^l(\varphi(b)) \right]^{\frac{1}{l}} d[g(t)] \\
& \leq \frac{1}{2} \left\{ \int_0^1 \left[g^{\alpha-1+s}(t) f^r(\varphi(b)) + mg^{\alpha-1}(t)(1-g(t))^s f^r(\varphi(a)) \right]^{\frac{2}{r}} d[g(t)] \right. \\
& \quad \left. + \int_0^1 \left[g^{\alpha-1+s}(t) h^l(\varphi(b)) + mg^{\alpha-1}(t)(1-g(t))^s h^l(\varphi(a)) \right]^{\frac{2}{l}} d[g(t)] \right\} \\
& \leq \frac{1}{2} \left[\left\{ \left(\int_0^1 g^{\frac{2(\alpha-1+s)}{r}}(t) f^2(\varphi(b)) d[g(t)] \right)^{\frac{r}{2}} \right. \right. \\
& \quad \left. \left. + \left(\int_0^1 m^{\frac{2}{r}} g^{\frac{2(\alpha-1)}{r}}(t)(1-g(t))^{\frac{2s}{r}} f^2(\varphi(a)) d[g(t)] \right)^{\frac{r}{2}} \right\}^{\frac{2}{r}} \right. \\
& \quad \left. + \left\{ \left(\int_0^1 g^{\frac{2(\alpha-1+s)}{l}}(t) h^2(\varphi(b)) d[g(t)] \right)^{\frac{l}{2}} \right. \right. \\
& \quad \left. \left. + \left(\int_0^1 m^{\frac{2}{l}} g^{\frac{2(\alpha-1)}{l}}(t)(1-g(t))^{\frac{2s}{l}} h^2(\varphi(a)) d[g(t)] \right)^{\frac{l}{2}} \right\}^{\frac{2}{l}} \right]^{\frac{2}{r}}
\end{aligned}$$

$$\begin{aligned}
&= \frac{1}{2} \left\{ \left[m f^r(\varphi(a)) B^{\frac{r}{2}} \left(g(t); \frac{2(\alpha-1)}{r}, \frac{2s}{r} \right) \right. \right. \\
&\quad \left. \left. + f^r(\varphi(b)) \left(\frac{r}{2(\alpha-1+s)+r} \right)^{\frac{r}{2}} \left(g^{\frac{2(\alpha-1+s)}{r}+1}(1) - g^{\frac{2(\alpha-1+s)}{r}+1}(0) \right)^{\frac{r}{2}} \right]^{\frac{2}{r}} \right. \\
&\quad \left. + \left[m h^l(\varphi(a)) B^{\frac{l}{2}} \left(g(t); \frac{2(\alpha-1)}{l}, \frac{2s}{l} \right) \right. \right. \\
&\quad \left. \left. + h^l(\varphi(b)) \left(\frac{l}{2(\alpha-1+s)+l} \right)^{\frac{l}{2}} \left(g^{\frac{2(\alpha-1+s)}{l}+1}(1) - g^{\frac{2(\alpha-1+s)}{l}+1}(0) \right)^{\frac{l}{2}} \right]^{\frac{2}{l}} \right\}.
\end{aligned}$$

■

Corollary 4. *Under the same conditions as in Theorem 5 for $m = s = 1$, $\varphi(x) = x$, $\eta(\varphi(b), \varphi(a), m) = \eta(b, a)$ and $g(t) = t$, we get (see [2], Theorem 3.3).*

Theorem 6. *Let $\varphi : I \rightarrow K$ be a continuous function and $g : [0, 1] \rightarrow [0, 1]$ is a differentiable function. Suppose $K \subseteq \mathbb{R}$ be an open m -invex subset with respect to $\eta : K \times K \times (0, 1] \rightarrow \mathbb{R}$ for any fixed $s, m \in (0, 1]$ with $m\varphi(a) < m\varphi(a) + \eta(\varphi(b), \varphi(a), m)$. Assume that $f, h : K = [m\varphi(a), m\varphi(a) + \eta(\varphi(b), \varphi(a), m)] \rightarrow (0, +\infty)$ are respectively generalized $(r; g, s, m, \varphi)$ -preinvex function and generalized $(l; g, s, m, \varphi)$ -preinvex function on an open m -invex set K° . Then for $\alpha > 0$, $r > 1$ and $r^{-1} + l^{-1} = 1$, we have*

$$\begin{aligned}
(9) \quad &\frac{1}{\eta^\alpha(\varphi(b), \varphi(a), m)} \int_{m\varphi(a)+g(0)\eta(\varphi(b), \varphi(a), m)}^{m\varphi(a)+g(1)\eta(\varphi(b), \varphi(a), m)} (t - m\varphi(a))^{\alpha-1} f(t)h(t)dt \\
&\leq \left\{ \frac{f^r(\varphi(b))}{s+\alpha} (g^{s+\alpha}(1) - g^{s+\alpha}(0)) + m f^r(\varphi(a)) B(g(t); \alpha-1, s) \right\}^{\frac{1}{r}} \\
&\quad + \left\{ \frac{h^l(\varphi(b))}{s+\alpha} (g^{s+\alpha}(1) - g^{s+\alpha}(0)) + m h^l(\varphi(a)) B(g(t); \alpha-1, s) \right\}^{\frac{1}{l}}.
\end{aligned}$$

Proof. Let $r > 1$ and $r^{-1} + l^{-1} = 1$. Since f and h are respectively generalized $(r; g, s, m, \varphi)$ -preinvex function and generalized $(l; g, s, m, \varphi)$ -preinvex function on an open m -invex set K° , combining with Hölder inequality for all $t \in [0, 1]$ and for any fixed $s, m \in (0, 1]$, we get

$$\frac{1}{\eta^\alpha(\varphi(b), \varphi(a), m)} \int_{m\varphi(a)+g(0)\eta(\varphi(b), \varphi(a), m)}^{m\varphi(a)+g(1)\eta(\varphi(b), \varphi(a), m)} (t - m\varphi(a))^{\alpha-1} f(t)h(t)dt$$

$$\begin{aligned}
&= \int_0^1 g^{(\alpha-1)(\frac{1}{r}+\frac{1}{l})}(t) f(m\varphi(a) + g(t)\eta(\varphi(b), \varphi(a), m)) \\
&\quad \times h(m\varphi(a) + g(t)\eta(\varphi(b), \varphi(a), m)) d[g(t)] \\
&\leq \left\{ \int_0^1 \left[g^{\alpha-1+s}(t) f^r(\varphi(b)) + m g^{\alpha-1}(t) (1-g(t))^s f^r(\varphi(a)) \right]^{\frac{1}{r}} \right. \\
&\quad \left. \times \left[g^{\alpha-1+s}(t) h^l(\varphi(b)) + m g^{\alpha-1}(t) (1-g(t))^s h^l(\varphi(a)) \right]^{\frac{1}{l}} d[g(t)] \right\} \\
&\leq \left\{ \int_0^1 \left[g^{\alpha-1+s}(t) f^r(\varphi(b)) + m g^{\alpha-1}(t) (1-g(t))^s f^r(\varphi(a)) \right] d[g(t)] \right\}^{\frac{1}{r}} \\
&\quad + \left\{ \int_0^1 \left[g^{\alpha-1+s}(t) h^l(\varphi(b)) + m g^{\alpha-1}(t) (1-g(t))^s h^l(\varphi(a)) \right] d[g(t)] \right\}^{\frac{1}{l}} \\
&= \left\{ \frac{f^r(\varphi(b))}{s+\alpha} (g^{s+\alpha}(1) - g^{s+\alpha}(0)) + m f^r(\varphi(a)) B(g(t); \alpha-1, s) \right\}^{\frac{1}{r}} \\
&\quad + \left\{ \frac{h^l(\varphi(b))}{s+\alpha} (g^{s+\alpha}(1) - g^{s+\alpha}(0)) + m h^l(\varphi(a)) B(g(t); \alpha-1, s) \right\}^{\frac{1}{l}}.
\end{aligned}$$

■

Corollary 5. *Under the same conditions as in Theorem 6 for $m = s = 1$, $\varphi(x) = x$, $\eta(\varphi(b), \varphi(a), m) = \eta(b, a)$ and $g(t) = t$, we get (see [2], Theorem 3.9).*

Remark 3. For different choices of positive values $r, l = \frac{1}{2}, \frac{1}{3}, 2$, etc., for any fixed $s, m \in (0, 1]$, for a particular choices of a differentiable function $g(t) = e^{-t}, \ln(t+1), \sin(\frac{\pi t}{2}), \cos(\frac{\pi t}{2})$, etc, and a particular choices of a continuous function $\varphi(x) = e^x$ for all $x \in \mathbb{R}$, x^n for all $x > 0$ and for all $n \in \mathbb{N}$, etc, by Theorem 4, Theorem 5 and Theorem 6 we can get some special kinds of Hermite-Hadamard type fractional inequalities.

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