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HERMITE-HADAMARD TYPE INEQUALITIES WITH APPLICATIONS

Abstract. In this article first, we give an integral identity and prove some Hermite-Hadamard type inequalities for the function f such that $|f''|^q$ is convex or concave for $q \geq 1$. Second, by using these results, we present applications to f-divergence measures. At the end, we obtain some bounds for special means of real numbers and new error estimates for the trapezoidal formula.

Key words: convex functions, f-divergence, Hermite-Hadamard's inequality, means, trapezoidal formula.

AMS Mathematics Subject Classification: 26D15.

1. Introduction

The following class of functions is well known in the literature and is usually defined in the following way: a function $f : \mathtt{I} \to \mathbb{R}$, defined on the interval $\mathtt{I}$ in $\mathbb{R}$, is said to be convex on $\mathtt{I}$ if the inequality

$$f(\lambda x + (1-\lambda)y) \leq \lambda f(x) + (1-\lambda)f(y) \tag{1}$$

holds for all $x, y \in \mathtt{I}$ and $\lambda \in [0,1]$. Also, we say that f is concave, if the inequality in (1) is reversed. Geometrically, the convexity of the function f implies that if there are any three distinct points $\mathtt{R}$, $\mathtt{S}$ and $\mathtt{T}$ located on the graph of the convex function f with $\mathtt{S}$ lies between $\mathtt{R}$ and $\mathtt{T}$, then we have the point $\mathtt{S}$ lies on or below the chord joining the points $\mathtt{R}$ and $\mathtt{T}$.

Many important inequalities have been obtained for the class of convex functions, when the idea of convexity was introduced more than a hundred years ago. But among those one of the most prominent is the so called Hermite-Hadamard's inequality (or Hadamard's inequality). This double inequality is stated as follows (see for example [16]):

Let $\mathtt{I}$ be an interval in $\mathbb{R}$ and $f : \mathtt{I} \subseteq \mathbb{R} \to \mathbb{R}$ be a convex function defined on $\mathtt{I}$ such that $a, b \in \mathtt{I}$ with $a < b$. Then the inequalities

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)dx \leq \frac{f(a)+f(b)}{2} \tag{2}$$

hold. If the function f is concave on $\mathtt{I}$, then both the inequalities in (2) hold in the reverse direction. It gives an estimate from both sides of the mean, i.e. from above and below of the mean value of a convex function. It is also a matter of great interest and one has to note that some of the classical inequalities for means can be obtained from Hadamard's inequality under the utility of peculiar convex functions f. These inequalities for convex functions play a crucial role in analysis as well as in other areas of pure and applied mathematics.

For recent results, generalizations and refinements related to Hermite-Hadamard inequality see [1, 2, 3, 5, 8, 9, 10, 11, 12, 13, 14, 15, 16, 25, 26, 19, 20, 21, 22, 27, 28, 30] and the references given therein. In 1998, Dragomir and Agarwal proved the following lemma and established some significant results for the class of differentiable convex mappings which are closely annexed with Hadamard's inequality. This important result is stated as follows:

Lemma 1 ([14])**.** *Let $f : \mathcal{I}^\circ \subset \mathbb{R} \to \mathbb{R}$ be a differentiable mapping on $\mathcal{I}^\circ$, $a, b \in \mathcal{I}^\circ$ with $a < b$. If $f' \in \mathtt{L}[a, b]$, then the following identity holds:*

$$\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)dx = \frac{b-a}{2}\int_0^1 (1-2t)f'(ta+(1-t)b)dt. \tag{3}$$

Here $\mathcal{I}^\circ$ denotes the interior of $\mathtt{I}$.

The following two results are the ultimate consequences of Lemma 1, which have been presented in [14].

Theorem 1. *Under the assumptions of Lemma 1 and the convexity of function $|f'|$ on $[a, b]$, we have the following inequality:*

$$\left|\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)dx\right| \leq \frac{(b-a)(|f'(a)|+|f'(b)|)}{8}. \tag{4}$$

Theorem 2. *Suppose that all the assumptions of Lemma 1 are satisfied. Furthermore, if the mapping $|f'|^{\frac{p}{p-1}}$ $(p > 1)$ is convex on $[a, b]$, then the following inequality holds:*

$$\begin{aligned}\left|\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)dx\right| \\ \leq \frac{b-a}{2(p+1)^{\frac{1}{p}}}\left[\frac{|f'(a)|^{\frac{p}{p-1}}+|f'(b)|^{\frac{p}{p-1}}}{2}\right]^{\frac{p-1}{p}}.\end{aligned} \tag{5}$$

In 2000, Pearce and Pečarić [27] employed Lemma 1, and proved the following theorem.

Theorem 3. *Suppose that all the assumptions of Lemma 1 hold. Furthermore, if the mapping $|f'|^q$ $(q \geq 1)$ is concave on $[a,b]$, then the following inequality is valid:*

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)dx\right| \leq \frac{b-a}{4}\left|f'\left(\frac{a+b}{2}\right)\right|. \tag{6}$$

The main purpose of this paper is to give integral identity and to present some Hermite-Hadamard type inequalities for convex as well as concave functions (Theorems 4-8). Then we discuss the importance of our results (Corollaries 1-5). Also, in the next section we present applications to f-divergence measures. At the last section, applications to some special means of real numbers and estimates for the error term of trapezoidal formula are given.

2. Main results

We begin this section with the proof of our first main result in the following theorem.

Theorem 4. *Let $f : \mathcal{I} \subset \mathbb{R} \to \mathbb{R}$ be a twice differentiable function on $\mathcal{I}^\circ$ such that $f'' \in \mathcal{L}[a,b]$ where $a,b \in \mathcal{I}^\circ$ with $a<b$. If $|f''|$ is convex function on $[a,b]$, then we have the following inequality:*

$$\begin{aligned}\left|\frac{f'(x)\left((x-a)^2-(b-x)^2\right)+2f(b)(b-x)+2f(a)(x-a)}{2(b-a)}\right.\\ \left.-\frac{1}{b-a}\int_a^b f(u)du\right|\\ \leq \frac{(x-a)^3}{b-a}\left[\frac{3|f''(a)|+5|f''(x)|}{24}\right]+\frac{(b-x)^3}{b-a}\left[\frac{3|f''(b)|+5|f''(x)|}{24}\right]\end{aligned} \tag{7}$$

for all $x \in [a,b]$.

Proof. Integrating by parts, we have

$$\begin{aligned}&\frac{(x-a)^3}{2(b-a)}\int_0^1\left(1-t^2\right)f''(ta+(1-t)x)dt\\ &\quad+\frac{(b-x)^3}{2(b-a)}\int_0^1\left(1-t^2\right)f''(tb+(1-t)x)dt\\ &=\frac{(x-a)^3}{2(b-a)}\left[\left|\frac{\left(1-t^2\right)f'(ta+(1-t)x)}{a-x}\right|_0^1-\int_0^1\frac{(-2t)f'(ta+(1-t)x)}{a-x}dt\right]\end{aligned}$$

$$
\begin{aligned}
&+\frac{(b-x)^3}{2(b-a)}\left[\left|\frac{\left(1-t^2\right)f'(tb+(1-t)x)}{b-x}\right|_0^1-\int_0^1\frac{(-2t)\,f'(tb+(1-t)x)}{b-x}dt\right]\\
&=\frac{(x-a)^3}{2(b-a)}\left[\frac{-f'(x)}{a-x}+2\int_0^1\frac{tf'(ta+(1-t)x)}{a-x}dt\right]\\
&+\frac{(b-x)^3}{2(b-a)}\left[\frac{-f'(x)}{b-x}+2\int_0^1\frac{tf'(tb+(1-t)x)}{b-x}dt\right]\\
&=\frac{(x-a)^3}{2(b-a)}\left[\frac{-f'(x)}{a-x}+2\left[\left|\frac{tf(ta+(1-t)x)}{(a-x)^2}\right|_0^1-\int_0^1\frac{f(ta+(1-t)x)}{(a-x)^2}dt\right]\right]\\
&+\frac{(b-x)^3}{2(b-a)}\left[\frac{-f'(x)}{b-x}+2\left[\left|\frac{tf(tb+(1-t)x)}{(b-x)^2}\right|_0^1-\int_0^1\frac{f(tb+(1-t)x)}{(b-x)^2}dt\right]\right].
\end{aligned}
$$

By applying the limits of integration and changing the variables we get

$$
\begin{aligned}
&\frac{f'(x)\left((x-a)^2-(b-x)^2\right)+2f(b)(b-x)+2f(a)(x-a)}{2(b-a)}\\
&\quad-\frac{1}{b-a}\int_a^b f(u)du\\
&=\frac{(x-a)^3}{2(b-a)}\int_0^1\left(1-t^2\right)f''(ta+(1-t)x)dt\\
&\quad+\frac{(b-x)^3}{2(b-a)}\int_0^1\left(1-t^2\right)f''(tb+(1-t)x)dt
\end{aligned}
\tag{8}
$$

Now from (8) and the well-known triangular inequality of real numbers, we have

$$
\begin{aligned}
&\left|\frac{f'(x)\left((x-a)^2-(b-x)^2\right)+2f(b)(b-x)+2f(a)(x-a)}{2(b-a)}-\frac{1}{b-a}\int_a^b f(u)du\right|\\
&\le\frac{(x-a)^3}{2(b-a)}\int_0^1\left(1-t^2\right)\mid f''(ta+(1-t)x)\mid dt\\
&\quad+\frac{(b-x)^3}{2(b-a)}\int_0^1\left(1-t^2\right)\mid f''(tb+(1-t)x)\mid dt\\
&\le\frac{(x-a)^3}{2(b-a)}\int_0^1\left(1-t^2\right)[t\mid f''(a)\mid+(1-t)\mid f''(x)\mid]\,dt\ \text{(by convexity of } |f''|)\\
&\quad+\frac{(b-x)^3}{2(b-a)}\int_0^1\left(1-t^2\right)[t\mid f''(b)\mid+(1-t)\mid f''(x)\mid]\,dt\\
&=\frac{(x-a)^3}{b-a}\left[\frac{3|f''(a)|+5|f''(x)|}{24}\right]+\frac{(b-x)^3}{b-a}\left[\frac{3|f''(b)|+5|f''(x)|}{24}\right].
\end{aligned}
$$

This completes the desired proof. ■

Corollary 1. *Under the assumptions of Theorem 4, we have the following inequality:*

$$\begin{aligned}(9)\qquad &\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(u)du\right| \\ &\leq \frac{(b-a)^2}{8}\left[\frac{3|f''(a)|+10|f''\left(\frac{a+b}{2}\right)|+3|f''(b)|}{24}\right] \\ &\leq \frac{(b-a)^2}{24}\left[|f''(a)|+|f''(b)|\right].\end{aligned}$$

Proof. By choosing $x=\frac{a+b}{2}$ in inequality (7), we get the first inequality in (9) and then using the convexity of $|f''|$, we obtain the second inequality. ■

Theorem 5. *Let $f:\mathcal{I}\subset\mathbb{R}\to\mathbb{R}$ be a twice differentiable function on $\mathcal{I}^\circ$ and $a,b\in\mathcal{I}^\circ$ with $a<b$. If $f''\in\mathcal{L}[a,b]$ and $|f''|^q$ is convex function on $[a,b]$, for some fixed $q>1$ such that $p^{-1}+q^{-1}=1$, then we have*

$$\begin{aligned}(10)\qquad &\left|\frac{f'(x)\left((x-a)^2-(b-x)^2\right)+2f(b)(b-x)+2f(a)(x-a)}{2(b-a)}\right. \\ &\left.-\frac{1}{b-a}\int_a^b f(u)du\right| \leq \left(\frac{\Gamma(\frac{1}{2})\Gamma(p+1)}{\Gamma(p+\frac{3}{2})}\right)^{\frac{1}{p}} \\ &\times\frac{(x-a)^3\left[|f''(a)|^q+|f''(x)|^q\right]^{\frac{1}{q}}+(b-x)^3\left[|f''(b)|^q+|f''(x)|^q\right]^{\frac{1}{q}}}{4(b-a)}\end{aligned}$$

for each $x\in[a,b]$.

Proof. Considering (8) and using the famous Hölder inequality, it follows that

$$\begin{aligned}&\left|\frac{f'(x)\left((x-a)^2-(b-x)^2\right)+2f(b)(b-x)+2f(a)(x-a)}{2(b-a)}-\frac{1}{b-a}\int_a^b f(u)du\right| \\ &\leq\frac{(x-a)^3}{2(b-a)}\left(\int_0^1\left(1-t^2\right)^p dt\right)^{\frac{1}{p}}\left(\int_0^1 |\, f''(ta+(1-t)x)\,|^q\, dt\right)^{\frac{1}{q}} \\ &\quad+\frac{(b-x)^3}{2(b-a)}\left(\int_0^1\left(1-t^2\right)^p dt\right)^{\frac{1}{p}}\left(\int_0^1 |\, f''(tb+(1-t)x)\,|^q\, dt\right)^{\frac{1}{q}}.\end{aligned}$$

Using the convexity of $|f''|^q$, we get

$$\begin{aligned}\int_0^1 |\, f''(ta+(1-t)x)\,|^q\, dt &\leq \int_0^1\left(t\,|\,f''(a)\,|^q+(1-t)\,|\,f''(x)\,|^q\right)dt \\ &= \frac{|f''(a)|^q+|f''(x)|^q}{2},\end{aligned}$$

similarly

$$\int_0^1 \mid f''(tb+(1-t)x) \mid^q dt \leq \int_0^1 \left(t \mid f''(b) \mid^q + (1-t) \mid f''(x) \mid^q\right) dt = \frac{|f''(b)|^q + |f''(x)|^q}{2}$$

and using the beta function, we have

$$\left(\int_0^1 \left(1-t^2\right)^p dt\right)^{\frac{1}{p}} = \left(\frac{1}{2}\int_0^1 (1-x)^p x^{-\frac{1}{2}} dx\right)^{\frac{1}{p}} = \left(\frac{1}{2}\frac{\Gamma(\frac{1}{2})\Gamma(p+1)}{\Gamma(p+\frac{3}{2})}\right)^{\frac{1}{p}}.$$

The combination of all the above inequalities and facts lead us to the required conclusion. ■

Corollary 2. *Under the assumptions of Theorem 5, the following inequality holds:*

$$\begin{aligned}(11)\quad &\left|\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(u)du\right| \\ &\leq \frac{(b-a)^2}{32}\left(\frac{\Gamma(\frac{1}{2})\Gamma(p+1)}{\Gamma(p+\frac{3}{2})}\right)^{\frac{1}{p}}\left[\left(|f''(a)|^q + |f''\left(\frac{a+b}{2}\right)|^q\right)^{\frac{1}{q}}\right. \\ &\quad \left. + \left(|f''(b)|^q + |f''\left(\frac{a+b}{2}\right)|^q\right)^{\frac{1}{q}}\right] \\ &\leq \frac{(b-a)^2}{32}\left(1+\frac{2}{2^{\frac{1}{q}}}\right)\left(\frac{\Gamma(\frac{1}{2})\Gamma(p+1)}{\Gamma(p+\frac{3}{2})}\right)^{\frac{1}{p}}\left(|f''(a)| + |f''(b)|\right).\end{aligned}$$

Proof. By putting $x = \frac{a+b}{2}$ in the above inequality (10) in Theorem 5, we get the first inequality in (11). The second inequality is obtained by using the convexity of $|f''|^q$ and the fact that: $\sum_{k=1}^n (\alpha_k + \beta_k)^s \leq \sum_{k=1}^n (\alpha_k)^s + \sum_{k=1}^n (\beta_k)^s$ for $(0 \leq s \leq 1)$, $\alpha_1, \alpha_2, \alpha_3, ..., \alpha_n \geq 0$; $\beta_1, \beta_2, \beta_3, ..., \beta_n \geq 0$. ■

Theorem 6. *Let $f : I \subset \mathbb{R} \to \mathbb{R}$ be a twice differentiable function on I° and $a, b \in I^\circ$ with $a < b$. If $f'' \in L[a,b]$ and $|f''|^q$ is concave function on $[a,b]$, for some fixed $q > 1$ and $p = \frac{q}{q-1}$, then the following inequality holds:*

$$\begin{aligned}(12)\quad &\left|\frac{f'(x)\left((x-a)^2 - (b-x)^2\right) + 2f(b)(b-x) + 2f(a)(x-a)}{2(b-a)}\right. \\ &\left. - \frac{1}{b-a}\int_a^b f(u)du\right|\end{aligned}$$

$$\leq \left[\frac{\Gamma(\frac{1}{2})\Gamma(p+1)}{2\Gamma(p+\frac{3}{2})}\right]^{\frac{1}{p}} \left[\frac{(x-a)^3\left|f''\left(\frac{x+a}{2}\right)\right| + (b-x)^3\left|f''\left(\frac{x+b}{2}\right)\right|}{2(b-a)}\right]$$

for each $x \in [a,b]$.

Proof. As in Theorem 5, taking (8) and then apply the famous Hölder inequality for $q>1$ and $p=\frac{q}{q-1}$, we have

$$\begin{aligned}
&\left|\frac{f'(x)\left((x-a)^2-(b-x)^2\right)+2f(b)(b-x)+2f(a)(x-a)}{2(b-a)}\right. \\
&\qquad \left. -\frac{1}{b-a}\int_a^b f(u)du\right| \\
&\leq \frac{(x-a)^3}{2(b-a)}\left(\int_0^1 \left(1-t^2\right)^p dt\right)^{\frac{1}{p}}\left(\int_0^1 \mid f''(ta+(1-t)x)\mid^q dt\right)^{\frac{1}{q}} \\
&\quad + \frac{(b-x)^3}{2(b-a)}\left(\int_0^1 \left(1-t^2\right)^p dt\right)^{\frac{1}{p}}\left(\int_0^1 \mid f''(tb+(1-t)x)\mid^q dt\right)^{\frac{1}{q}}.
\end{aligned}$$

Since, $|f''|^q$ is concave on $[a,b]$, therefore by applying Jensen's integral inequality for the concave function $|f''|^q$ we get:

$$\begin{aligned}
\int_0^1 \mid f''(ta+(1-t)x)\mid^q dt &\leq \left|f''\left(\int_0^1 (ta+(1-t)x)dt\right)\right|^q \\
&= \left|f''\left(\frac{x+a}{2}\right)\right|^q.
\end{aligned}$$

Similarly,

$$\int_0^1 \mid f''(tb+(1-t)x)\mid^q dt \leq \left|f''\left(\frac{x+b}{2}\right)\right|^q$$

and also from above, we have

$$\left(\int_0^1 \left(1-t^2\right)^p dt\right)^{\frac{1}{p}} = \left(\frac{1}{2}\int_0^1 (1-x)^p x^{-\frac{1}{2}}dx\right)^{\frac{1}{p}} = \left(\frac{1}{2}.\frac{\Gamma(\frac{1}{2})\Gamma(p+1)}{\Gamma(p+\frac{3}{2})}\right)^{\frac{1}{p}}.$$

Combining all the above inequalities and facts, we get the desired inequality in (12). ■

Corollary 3. *Under the assumptions of Theorem 6, we have the following:*

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(u)du\right| \tag{13}$$

$$\leq \left[\frac{\Gamma(\frac{1}{2})\Gamma(p+1)}{2\Gamma(p+\frac{3}{2})}\right]^{\frac{1}{p}} \frac{(b-a)^2}{16}\left[\left|f''\left(\frac{3a+b}{4}\right)\right| + \left|f''\left(\frac{a+3b}{4}\right)\right|\right]$$

$$\leq \left[\frac{\Gamma(\frac{1}{2})\Gamma(p+1)}{2\Gamma(p+\frac{3}{2})}\right]^{\frac{1}{p}} \frac{(b-a)^2}{8}\left|f''\left(\frac{a+b}{2}\right)\right|.$$

Proof. By setting $x = \frac{a+b}{2}$ in the inequality (12), we get the first inequality in (13). The second inequality in (13) is obtained by using the concavity of $|f''|$. ■

Theorem 7. *Let $f : I \subset \mathbb{R} \to \mathbb{R}$ be a twice differentiable function on I° and $a, b \in I^\circ$ with $a < b$. If $f'' \in L[a,b]$ and $|f''|^q$ is convex function on $[a,b]$, for some fixed $q > 1$, then the following inequality holds:*

$$\begin{aligned}(14)\qquad &\left|\frac{f'(x)\left((x-a)^2-(b-x)^2\right)+2f(b)(b-x)+2f(a)(x-a)}{2(b-a)}\right.\\ &\left.-\frac{1}{b-a}\int_a^b f(u)du\right|\\ &\leq \left(\frac{1}{8}\right)^{\frac{1}{q}}\left(\frac{(x-a)^3[3|f''(a)|^q+5|f''(x)|^q]^{\frac{1}{q}}}{3(b-a)}\right.\\ &\qquad\left.+\frac{(b-x)^3[3|f''(b)|^q+5|f''(x)|^q]^{\frac{1}{q}}}{3(b-a)}\right)\end{aligned}$$

for each $x \in [a,b]$.

Proof. Likewise Theorem 5, again consider (8) and then apply the famous power-mean inequality for $q > 1$, we have

$$\begin{aligned}&\left|\frac{f'(x)\left((x-a)^2-(b-x)^2\right)+2f(b)(b-x)+2f(a)(x-a)}{2(b-a)}-\frac{1}{b-a}\int_a^b f(u)du\right|\\ &\leq \frac{(x-a)^3}{2(b-a)}\int_0^1 \left(1-t^2\right)\mid f''(ta+(1-t)x)\mid dt\\ &\quad+\frac{(b-x)^3}{2(b-a)}\int_0^1 \left(1-t^2\right)\mid f''(tb+(1-t)x)\mid dt\\ &\leq \frac{(x-a)^3}{2(b-a)}\left(\int_0^1(1-t^2)dt\right)^{1-\frac{1}{q}}\left(\int_0^1(1-t^2)\mid f''(ta+(1-t)x)\mid^q dt\right)^{\frac{1}{q}}\\ &\quad+\frac{(b-x)^3}{2(b-a)}\left(\int_0^1(1-t^2)dt\right)^{1-\frac{1}{q}}\left(\int_0^1(1-t^2)\mid f''(tb+(1-t)x)\mid^q dt\right)^{\frac{1}{q}}.\end{aligned}$$

Since, $|f''|^q$ is convex on $[a,b]$, therefore we have

$$\int_0^1 (1-t^2) \mid f''(ta+(1-t)x) \mid^q dt \leq \int_0^1 (1-t^2)[t|f''(a)|^q + (1-t)|f(x)|^q]dt$$
$$= \frac{3|f''(a)|^q + 5|f''(x)|^q}{12}.$$

Similarly,

$$\int_0^1 (1-t^2) \mid f''(tb+(1-t)x) \mid^q dt \leq \frac{3|f''(b)|^q + 5|f''(x)|^q}{12}$$

and also we have

$$\int_0^1 (1-t^2)dt = \frac{2}{3}.$$

Combining all the above inequalities and facts, we get the desired inequality in (14). ■

Corollary 4. *Under the assumptions of Theorem 7, we have the following inequality:*

$$\left|\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(u)du\right|$$
$$\leq \left(\frac{1}{8}\right)^{\frac{1}{q}} \frac{(b-a)^2}{24}\left[\left\{3|f''(a)|^q + 5\left|f''\left(\frac{a+b}{2}\right)\right|^q\right\}^{\frac{1}{q}}\right.$$
$$\left. + \left\{3\left|f''(b)\right|^q + 5\left|f''\left(\frac{a+b}{2}\right)\right|^q\right\}^{\frac{1}{q}}\right]$$
$$\leq \frac{(b-a)^2}{24}\left(\left(\frac{3}{8}\right)^{\frac{1}{q}} + 2\left(\frac{5}{16}\right)^{\frac{1}{q}}\right)\left[|f''(a)| + |f''(b)|\right]. \tag{15}$$

Proof. By putting $x = \frac{a+b}{2}$, in inequality (14), we get the first inequality in (15). The second inequality is obtained by using the convexity of $|f''|^q$ and the following fact: $\sum_{k=1}^n (\alpha_k + \beta_k)^s \leq \sum_{k=1}^n (\alpha_k)^s + \sum_{k=1}^n (\beta_k)^s$ for $(0 \leq s \leq 1)$, $\alpha_1, \alpha_2, \alpha_3, ..., \alpha_n \geq 0$; $\beta_1, \beta_2, \beta_3, ..., \beta_n \geq 0$. ■

Theorem 8. *Let $f : \mathcal{I} \subset \mathbb{R} \to \mathbb{R}$ be a twice differentiable function on $\mathcal{I}^\circ$ and $a, b \in \mathcal{I}^\circ$ with $a < b$. If $f'' \in \mathcal{L}[a,b]$ and $|f''|^q$ is concave function on $[a,b]$, for some fixed $q > 1$, then the following inequality holds:*

$$\left|\frac{f'(x)\left((x-a)^2 - (b-x)^2\right) + 2f(b)(b-x) + 2f(a)(x-a)}{2(b-a)}\right.$$
$$\left. - \frac{1}{b-a}\int_a^b f(u)du\right| \tag{16}$$

$$\leq \frac{(x-a)^3|f''\left(\frac{3a+5x}{8}\right)| + (b-x)^3|f''\left(\frac{3b+5x}{8}\right)|}{3(b-a)}$$

for each $x \in [a,b]$.

Proof. By power mean inequality, we have

$$\begin{aligned}\big(t|f''(a)| + (1-t)|f''(b)|\big)^q &\leq t|f''(a)|^q + (1-t)|f''(b)|^q \\ &\leq |f''(ta+(1-t)b)|^q, \\ &\quad \text{(by concavity of } |f''|^q)\end{aligned}$$

and therefore

$$|f''(ta+(1-t)b)| \geq t|f''(a)| + (1-t)|f''(b)|,$$

this shows that $|f''|$ is also concave.

Now applying triangular inequality on (8) and then using Jensen's integral inequality, we have

$$\begin{aligned}&\left|\frac{f'(x)\left((x-a)^2-(b-x)^2\right)+2f(b)(b-x)+2f(a)(x-a)}{2(b-a)} - \frac{1}{b-a}\int_a^b f(u)du\right| \\ &\quad\leq \frac{(x-a)^3}{2(b-a)}\int_0^1 \left(1-t^2\right) \mid f''(ta+(1-t)x) \mid dt \\ &\quad\quad + \frac{(b-x)^3}{2(b-a)}\int_0^1 \left(1-t^2\right) \mid f''(tb+(1-t)x) \mid dt \\ &\quad\leq \frac{(x-a)^3}{2(b-a)}\left(\int_0^1 (1-t^2)dt\right)\left|f''\left(\frac{\int_0^1(1-t^2)(ta+(1-t)x)dt}{\int_0^1(1-t^2)dt}\right)\right| \\ &\quad\quad + \frac{(b-x)^3}{2(b-a)}\left(\int_0^1 (1-t^2)dt\right)\left|f''\left(\frac{\int_0^1(1-t^2)(tb+(1-t)x)dt}{\int_0^1(1-t^2)dt}\right)\right| \\ &\quad= \frac{(x-a)^3|f''\left(\frac{3a+5x}{8}\right)| + (b-x)^3|f''\left(\frac{3b+5x}{8}\right)|}{3(b-a)}.\end{aligned}$$

■

Corollary 5. *Under the assumptions of Theorem 8, we have the following inequality:*

$$\begin{aligned}&\left|\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(u)du\right| \\ &\quad\leq \frac{(b-a)^2}{24}\left[\left|f''\left(\frac{5a+11b}{16}\right)\right| + \left|f''\left(\frac{11a+5b}{16}\right)\right|\right] \\ &\quad\leq \frac{(b-a)^2}{12}\left|f''\left(\frac{a+b}{2}\right)\right|.\end{aligned} \tag{17}$$

Proof. By taking $x = \frac{a+b}{2}$ in the inequality (16), we get the first inequality in (17). The second inequality is obtained by the concavity of $|f''|$. ■

3. Applications to f-divergence measures

One of the basic problems in various applications of Probability Theory is finding an appropriate measure of distance between any two probability distributions. For this purpose, a lot of divergence measures have been proposed and extensively studied by Kullback and Leibler [23], Renyi [29], Havrda and Charvat [17], Burbea and Rao [6], Lin [24], Csisźar[7], Ali and Silvey [4], Shioya and Da-te [31] and others (see for example [18] and the references therein). But here, we will consider only two of them, and in this connection we define the following terms.

Let the set χ and the σ-finite measure μ be given and consider the set of all probability densities on μ to be defined on $\Omega := \{p | \, p : \chi \to \mathbb{R}, \, p(x) > 0, \, \int_\chi p(x)d\mu(x) = 1\}$.

Let $f : (0, \infty) \to \mathbb{R}$ be given function and consider $\mathtt{D}_f(p, q)$ be defined by

$$\mathtt{D}_f(p, q) := \int_\chi p(x) f\left[\frac{q(x)}{p(x)}\right] d\mu(x), \quad p, q \in \Omega. \tag{18}$$

If f is convex function, then (18) is known as the Csisźar f-divergence [7].

In [31], Shioya and Da-te introduced the Hermite-Hadamard (HH) divergence

$$\mathtt{D}^f_{HH}(p, q) := \int_\chi p(x) \frac{\int_1^{\frac{q(x)}{p(x)}} f(t)dt}{\frac{q(x)}{p(x)} - 1} d\mu(x), \quad p, q \in \Omega, \tag{19}$$

where f is convex function on $(0, \infty)$ with $f(1) = 0$. In [31], the authors gave the property of HH divergence that $\mathtt{D}^f_{HH}(p, q) \geq 0$ with the equality holds if and only if $p = q$.

Proposition 1. *Let all the assumptions of Theorem 4 hold with $I = (0, \infty)$ and $f(1) = 0$. If $p, q \in \Omega$, then the following inequality holds:*

$$\left|\frac{1}{2}\mathtt{D}_f(p, q) - \mathtt{D}^f_{HH}(p, q)\right| \leq \frac{1}{24}\left[\left|f''(1)\right| \int_\chi \frac{(q(x) - p(x))^2}{p(x)} d\mu(x) + \int_\chi \frac{(q(x) - p(x))^2}{p(x)} \left|f''\left(\frac{q(x)}{p(x)}\right)\right| d\mu(x)\right]. \tag{20}$$

Proof. Let $X_1 = \{x \in \chi : q(x) > p(x)\}$, $X_2 = \{x \in \chi : q(x) < p(x)\}$ and $X_3 = \{x \in \chi : q(x) = p(x)\}$.

If $x \in X_3$, then obviously equality holds in (20). Now if $x \in X_1$, then by using Corollary 1 for $a = 1$, $b = \frac{q(x)}{p(x)}$, multiplying both hand sides of the obtained results by $p(x)$ and then integrating over X_1, we get

$$\begin{aligned}(21)\qquad &\left|\frac{1}{2}\int_{X_1} p(x) f\left[\frac{q(x)}{p(x)}\right] d\mu(x) - \int_{X_1} p(x)\frac{\int_1^{\frac{q(x)}{p(x)}} f(t)dt}{\frac{q(x)}{p(x)}-1} d\mu(x)\right| \\ &\leq \frac{1}{24}\left[\left|f''(1)\right| \int_{X_1} \frac{(q(x)-p(x))^2}{p(x)} d\mu(x) \right. \\ &\quad \left. + \int_{X_1} \frac{(q(x)-p(x))^2}{p(x)} \left|f''\left(\frac{q(x)}{p(x)}\right)\right| d\mu(x)\right].\end{aligned}$$

Similarly if $x \in X_2$, then by using Corollary 1 for $a = \frac{q(x)}{p(x)}$, $b = 1$, multiplying both sides by $p(x)$ and then integrating over X_2, we get

$$\begin{aligned}(22)\qquad &\left|\frac{1}{2}\int_{X_2} p(x) f\left[\frac{q(x)}{p(x)}\right] d\mu(x) - \int_{X_2} p(x)\frac{\int_1^{\frac{q(x)}{p(x)}} f(t)dt}{\frac{q(x)}{p(x)}-1} d\mu(x)\right| \\ &\leq \frac{1}{24}\left[\left|f''(1)\right| \int_{X_2} \frac{(q(x)-p(x))^2}{p(x)} d\mu(x) \right. \\ &\quad \left. + \int_{X_2} \frac{(q(x)-p(x))^2}{p(x)} \left|f''\left(\frac{q(x)}{p(x)}\right)\right| d\mu(x)\right].\end{aligned}$$

By adding inequalities (21) and (22) and then using triangular inequality we get (20). ■

Proposition 2. *Let all the assumptions of Theorem 5 hold with $I = (0,\infty)$ and $f(1) = 0$. If $p, q \in \Omega$, then the following inequality holds:*

$$\begin{aligned}(23)\quad &\left|\frac{1}{2}\mathsf{D}_f(p,q) - \mathsf{D}^f_{HH}(p,q)\right| \\ &\leq \frac{1}{32}\left(1+\frac{2}{2^{\frac{1}{q}}}\right)\left(\frac{\Gamma(\frac{1}{2})\Gamma(p+1)}{\Gamma(p+\frac{3}{2})}\right)^{\frac{1}{p}} \left[\left|f''(1)\right| \int_{\chi} \frac{(q(x)-p(x))^2}{p(x)} d\mu(x) \right. \\ &\quad \left. + \int_{\chi} \frac{(q(x)-p(x))^2}{p(x)} \left|f''\left(\frac{q(x)}{p(x)}\right)\right| d\mu(x)\right].\end{aligned}$$

Proof. The proof is similar as to that of Proposition 1 but use Corollary 2 instead of Corollary 1. ■

Proposition 3. *Let all the assumptions of Theorem 6 hold with* $I = (0,\infty)$ *and* $f(1)=0$. *If* $p,q \in \Omega$, *then the following inequality holds:*

$$\left|\frac{1}{2}D_f(p,q) - D^f_{HH}(p,q)\right| \leq \left[\frac{\Gamma(\frac{1}{2})\Gamma(p+1)}{2\Gamma(p+\frac{3}{2})}\right]^{\frac{1}{p}} \times \left(\frac{1}{8}\right)\left[\int_\chi \frac{(q(x)-p(x))^2}{p(x)}\left|f''\left(\frac{p(x)+q(x)}{2p(x)}\right)\right| d\mu(x)\right].$$

Proof. The proof is similar as to that of Proposition 1 but use Corollary 3 instead of Corollary 1. ■

Proposition 4. *Let all the assumptions of Theorem 7 hold with* $I = (0,\infty)$ *and* $f(1)=0$. *If* $p,q \in \Omega$, *then we have the inequality:*

$$\begin{aligned}(24)\qquad &\left|\frac{1}{2}D_f(p,q) - D^f_{HH}(p,q)\right| \\ &\leq \frac{1}{24}\left(\left(\frac{3}{8}\right)^{\frac{1}{q}} + 2\left(\frac{5}{16}\right)^{\frac{1}{q}}\right)\left[|f''(1)|\int_\chi \frac{(q(x)-p(x))^2}{p(x)}d\mu(x)\right. \\ &\quad \left. + \int_\chi \frac{(q(x)-p(x))^2}{p(x)}\left|f''\left(\frac{q(x)}{p(x)}\right)\right| d\mu(x)\right].\end{aligned}$$

Proof. The proof is similar as to that of Proposition 1 but use Corollary 4 instead of Corollary 1. ■

Proposition 5. *Let all the assumptions of Theorem 8 hold with* $I = (0,\infty)$ *and* $f(1)=0$. *If* $p,q \in \Omega$, *then we have the inequality:*

$$\begin{aligned}(25)\qquad &\left|\frac{1}{2}D_f(p,q) - D^f_{HH}(p,q)\right| \\ &\leq \frac{1}{12}\left[\int_\chi \frac{(q(x)-p(x))^2}{p(x)}\left|f''\left(\frac{p(x)+q(x)}{2p(x)}\right)\right| d\mu(x)\right].\end{aligned}$$

Proof. The proof is similar as to that of Proposition 1 but use Corollary 5 instead of Corollary 1. ■

4. Applications to some special means and trapezoidal formula

In this section, we are concerned with the applications of our main results to some special means of real numbers as well as new error estimates of

trapezoidal formula. For this purpose, we consider the following means of positive real numbers:

(*i*) The arithmetic mean:

$$\mathtt{A} = \mathtt{A}(a,b) = \frac{a+b}{2}, \quad a,b>0.$$

(*ii*) The logarithmic mean:

$$\mathtt{L}(a,b) = \frac{b-a}{\ln b - \ln a}, \quad a \neq b, \quad a,b>0.$$

(*iii*) The generalized logarithmic mean:

$$\mathtt{L}_n(a,b) = \left[\frac{b^{n+1}-a^{n+1}}{(b-a)(n+1)}\right]^{\frac{1}{n}}, \quad n \in \mathbb{Z}\setminus\{-1,0\}, \quad a,b>0, \quad a\neq b.$$

In the following propositions applications of the above results to certain means of real numbers have been incorporated:

Proposition 6. *Let $0<a<b$, $n\in\mathbb{Z}$, $|n(n-1)|\geq 3$ and $q>1$ such that $p^{-1}+q^{-1}=1$, then we have*

$$\begin{aligned}
&|\mathtt{A}(a^n,b^n)-\mathtt{L}_n^n(a,b)| \\
&\quad \leq |n(n-1)|\frac{(b-a)^2}{16}\left(1+\frac{2}{2^{\frac{1}{q}}}\right)\left[\frac{\Gamma(\frac{1}{2})\Gamma(p+1)}{\Gamma(p+\frac{3}{2})}\right]^{\frac{1}{p}}\mathtt{A}\left(a^{n-2},b^{n-2}\right), \\
&|\mathtt{A}(a^n,b^n)-\mathtt{L}_n^n(a,b)| \\
&\quad \leq |n(n-1)|\frac{(b-a)^2}{12}\left(\left(\frac{3}{8}\right)^{\frac{1}{q}}+2\left(\frac{5}{16}\right)^{\frac{1}{q}}\right)\mathtt{A}\left(a^{n-2},b^{n-2}\right).
\end{aligned}$$

Proof. Consider the function $f(x)=x^n$, $x>0$, $|n(n-1)|\geq 3$, $n\in\mathbb{Z}$. Then clearly f satisfies the conditions of Theorem 5. Therefore using this function in Corollaries 2 and 4, we obtain the required inequalities. ■

Proposition 7. *Let $0<a<b$ and $q>1$ such that $p^{-1}+q^{-1}=1$, then we have*

$$\begin{aligned}
&|\mathtt{A}(a^{-1},b^{-1})-\mathtt{L}^{-1}(a,b)| \\
&\quad \leq \frac{(b-a)^2}{8}\left(1+\frac{2}{2^{\frac{1}{q}}}\right)\left(\frac{\Gamma(\frac{1}{2})\Gamma(p+1)}{\Gamma(p+\frac{3}{2})}\right)^{\frac{1}{p}}\mathtt{A}\left(a^{-3},b^{-3}\right), \\
&|\mathtt{A}(a^{-1},b^{-1})-\mathtt{L}^{-1}(a,b)| \\
&\quad \leq \frac{(b-a)^2}{6}\left(\left(\frac{3}{8}\right)^{\frac{1}{q}}+2\left(\frac{5}{16}\right)^{\frac{1}{q}}\right)\mathtt{A}\left(a^{-3},b^{-3}\right).
\end{aligned}$$

Proof. Consider the function $f(x) = \frac{1}{x}, x > 0$. Then clearly f satisfies the conditions of Theorem 5. Therefore using this function in Corollaries 2 and 4, we obtain the required inequalities. ■

Next, we provide some new error estimates for the trapezoidal formula. For this, we proceed as follows:

Let d be a division $a = x_0 < x_1 < \ldots < x_{n-1} < x_n = b$ of the interval $[a, b]$ and consider the quadrature formula

$$\int_a^b f(x)dx = \mathtt{T}(f, d) + \mathtt{E}(f, d),$$

where

$$\mathtt{T}(f, d) = \sum_{i=0}^{n-1} \frac{f(x_i) + f(x_{i+1})}{2} (x_{i+1} - x_i)$$

for the trapezoidal version and $\mathtt{E}(f, d)$ denotes the associated approximation error.

Proposition 8. *Let $f : I \subset \mathbb{R} \to \mathbb{R}$ be twice differentiable function on I° and $a, b \in I^\circ$ with $a < b$. If $f'' \in L[a, b]$ and $|f''|^q$ is convex function on $[a, b]$ with $q > 1$ and $p^{-1} + q^{-1} = 1$, then we have:*

$$|E(f, d)| \leq \left(1 + \frac{2}{2^{\frac{1}{q}}}\right) \left(\frac{\Gamma(\frac{1}{2})\Gamma(p+1)}{\Gamma(p+\frac{3}{2})}\right)^{\frac{1}{p}} \times \sum_{i=0}^{n-1} \frac{(x_{i+1} - x_i)^3}{32} \left[|f''(x_i)| + |f''(x_{i+1})|\right].$$

Proof. Applying Corollary 2 on the subintervals $[x_i, x_{i+1}]$ $(i = 0, 1, 2, \ldots, n-1)$ of the division, we have

$$\left|\frac{f(x_i) + f(x_{i+1})}{2} - \frac{1}{x_{i+1} - x_i} \int_{x_i}^{x_{i+1}} f(x)dx\right| \leq \left(1 + \frac{2}{2^{\frac{1}{q}}}\right) \left(\frac{\Gamma(\frac{1}{2})\Gamma(p+1)}{\Gamma(p+\frac{3}{2})}\right)^{\frac{1}{p}} \frac{(x_{i+1} - x_i)^2}{32} \left[|f''(x_i)| + |f''(x_{i+1})|\right]$$

hence from above

$$\begin{aligned}
&\left|\int_a^b f(x)dx - \mathtt{T}(f, d)\right| \\
&\quad = \left|\sum_{i=0}^{n-1} \left\{\int_{x_i}^{x_{i+1}} f(x)dx - \frac{f(x_i) + f(x_{i+1})}{2} (x_{i+1} - x_i)\right\}\right| \\
&\quad \leq \sum_{i=0}^{n-1} \left|\left\{\int_{x_i}^{x_{i+1}} f(x)dx - \frac{f(x_i) + f(x_{i+1})}{2} (x_{i+1} - x_i)\right\}\right|
\end{aligned}$$

$$\leq \left(1+\frac{2}{2^{\frac{1}{q}}}\right)\left(\frac{\Gamma(\frac{1}{2})\Gamma(p+1)}{\Gamma(p+\frac{3}{2})}\right)^{\frac{1}{p}}\sum_{i=0}^{n-1}\frac{(x_{i+1}-x_i)^3}{32}\left[|f''(x_i)|+|f''(x_{i+1})|\right].$$

■

Proposition 9. *Let $f : I \subset \mathbb{R} \to \mathbb{R}$ be twice differentiable function on I° and $a, b \in I^\circ$ with $a < b$. If $f'' \in L[a, b]$ and $|f''|^q$ is concave function on $[a, b]$ with $q > 1$ and $p^{-1} + q^{-1} = 1$, then we have:*

$$|E(f,d)| \leq \left(\frac{\Gamma(\frac{1}{2})\Gamma(p+1)}{2\Gamma(p+\frac{3}{2})}\right)^{\frac{1}{p}}\sum_{i=0}^{n-1}\frac{(x_{i+1}-x_i)^3}{8}\left|f''\left(\frac{x_i+x_{i+1}}{2}\right)\right|$$

Proof. The proof is analogous as to that of Proposition 8 but use Corollary 3 instead of Corollary 2. ■

Proposition 10. *Let $f : I \subset \mathbb{R} \to \mathbb{R}$ be twice differentiable function on I° and $a, b \in I^\circ$ with $a < b$. If $f'' \in L[a, b]$ and $|f''|^q$ is concave function on $[a, b]$ with $q > 1$ and $p^{-1} + q^{-1} = 1$, then we have:*

$$|E(f,d)| \leq \sum_{i=0}^{n-1}\frac{(x_{i+1}-x_i)^3}{12}\left|f''\left(\frac{x_i+x_{i+1}}{2}\right)\right|.$$

Proof. The proof is similar as to that of Proposition 8 but use Corollary 5 instead of Corollary 2. ■

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