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EXTENSION OF SOME RESULTS ON THE (SSIE) AND THE (SSE) OF THE FORM $F \subset \mathcal{E} + F'_x$ and $\mathcal{E} + F_x = F$

Abstract. Given any sequence $a = (a_n)_{n \geq 1}$ of positive real numbers and any set E of complex sequences, we write E_a for the set of all sequences $y = (y_n)_{n \geq 1}$ such that $y/a = (y_n/a_n)_{n \geq 1} \in E$. In this paper we deal with the solvability of the (SSIE) of the form $\ell_\infty \subset \mathcal{E} + F'_x$ where $\mathcal{E}$ is a linear space of sequences and F' is either c_0, or ℓ_∞ and we solve the (SSIE) $c_0 \subset \mathcal{E} + s_x$ for $\mathcal{E} \subset (s_\alpha)_\Delta$ and $\alpha \in c_0$. Then we study the (SSIE) $c \subset \mathcal{E} + s_x^{(c)}$ and the (SSE) $\mathcal{E} + s_x^{(c)} = c$. Then we apply the previous results to the solvability of the (SSE) of the form $(\ell_r^p)_\Delta + F_x = F$ for $p \geq 1$ and F is any of the sets c_0, c, or ℓ_∞. These results extend some of those given in [8] and [9].

Key words: BK space, matrix transformations, multiplier of sequence spaces, sequence spaces inclusion equations, sequence spaces inclusion equations with operator.

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1. Introduction

We write ω for the set of all complex sequences $y = (y_n)_{n \geq 1}$, ℓ_∞, c and c_0 for the sets of all bounded, convergent and null sequences, respectively. We write cs for the set of all convergent series and $\ell^p = \{y \in \omega : \sum_{n=1}^{\infty} |y_n|^p < \infty\}$ for $1 \leq p < \infty$. If y, $z \in \omega$, then we write $yz = (y_n z_n)_{n \geq 1}$. Let $U = \{y \in \omega : y_n \neq 0\}$ and $U^+ = \{y \in \omega : y_n > 0\}$. We write $z/u = (z_n/u_n)_{n \geq 1}$ for all $z \in \omega$ and all $u \in U$, in particular $1/u = e/u$, where e is the sequence with $e_n = 1$ for all n. Finally, if $a \in U^+$ and E is any subset of ω, then we put $E_a = (1/a)^{-1} * E = \{y \in \omega : y/a \in E\}$. Let E and F be subsets of ω. In [1], the sets s_a, s_a^0 and $s_a^{(c)}$ were defined for positive sequences a by $(1/a)^{-1} * E$ and $E = \ell_\infty, c_0, c$, respectively. In [2] the sum $E_a + F_b$ and the product $E_a * F_b$ were defined where E, F are any of the symbols s, s^0, or $s^{(c)}$. Then in [5] the solvability was determined of sequences spaces inclusion equations

$G_b \subset E_a + F_x$ where $E,\ F,\ G \in \left\{s^0, s^{(c)}, s\right\}$ and some applications were given to sequence spaces inclusions with operators. Recall that the spaces w_∞ and w_0 of strongly bounded and summable sequences are the sets of all y such that $\left(n^{-1}\sum_{k=1}^{n}|y_k|\right)_n$ is bounded and tends to zero, respectively. These spaces were studied by Maddox [19] and Malkowsky, Rakočević [18]. In [11] we gave some properties of well-known operators defined on the sets $W_a = (1/a)^{-1} * w_\infty$ and $W_a^0 = (1/a)^{-1} * w_0$. In this paper we deal with special *sequence spaces inclusion equations* (*SSIE*), (*resp. sequence spaces equations* (*SSE*)), which are determined by an inclusion, (resp. identity), for which each term is a *sum* or *a sum of products of sets of the form* $(E_a)_T$ *and* $(E_{f(x)})_T$ where f maps U^+ to itself, E is any linear space of sequences and T is a triangle. Some results on (SSE) and (SSIE) were stated in [3], [7], [16], [5], [15], [6], [12], [13].

In this paper for given linear spaces of sequences $\mathcal{E}$, F and F' we consider the (SSIE) $F \subset \mathcal{E} + F'_x$ as a perturbed inclusion equation of the elementary inclusion equation $F \subset F'_x$. In this way it is interesting to determine what are the linear spaces of sequences $\mathcal{E}$ such that $F \subsetneqq \mathcal{E}$ for which the elementary and the perturbed inclusions equations have the same solutions. In a similar way the (SSE) $\mathcal{E} + F_x = F$ can be considered as the perturbed equation of the equation $F_x = F$. Our aim is to extend some of the known results on the solvability of the (SSIE) of the form $F \subset \mathcal{E} + F'_x$ stated in [15], [6], [5], [13], [7], [8], [9]. In [8] writing D_r for the diagonal matrix with $(D_r)_{nn} = r^n$, we dealt with the solvability of the (SSIE) using the operator of the first difference Δ, defined by $c \subset D_r * E_\Delta + c_x$ with $E = c_0$, or s_1. Then we dealt with the (SSIE) $c \subset D_r * E_{C_1} + s_x^{(c)}$ with $E = c_0$, c or s_1, and $s_1 \subset D_r * E_{C_1} + s_x$ with $E = c$ or s_1, where C_1 is the Cesàro operator defined by $(C_1)_n y = \left(\sum_{k=1}^{n} y_k\right)/n$. In [10] we solved the (SSE) with operator and $(E_r)_\Delta + F_x = F_u$ for $r,\ u > 0$ where E, F are any of the sets c_0, c, ℓ_∞ and the (SSE) $\left(W_r^0\right)_\Delta + s_x^{(c)} = s_u^{(c)}$. In [9] we dealt with the class of (SSIE) of the form $F \subset E_a + F'_x$ where $F \in \{c_0, \ell^p, w_0, w_\infty\}$ and $E,\ F' \in \{c_0, c, \ell_\infty, \ell^p, w_0, w_\infty\}$, $(p \geq 1)$. In this paper we extend the results stated in [8], [9]. In this way we deal with the (SSIE) $c_0 \subset \mathcal{E} + s_x$ where $\mathcal{E} \subset (s_\alpha)_\Delta$ for $\alpha \in c_0$ and we solve the (SSIE) of the form $\ell_\infty \subset \mathcal{E} + F'_x$ where F' is either c_0, or ℓ_∞. Then we study the (SSIE) $c \subset \mathcal{E} + s_x^{(c)}$ and the (SSE) $\mathcal{E} + s_x^{(c)} = c$ with $\mathcal{E} \subset (s_\alpha)_\Delta$ with $\alpha \in cs^+$.

This paper is organized as follows. In Section 2 we recall some well-known results on sequence spaces and matrix transformations. In Section 3 we recall some results on the multipliers and on the characterizations of matrix transformations. In Section 4 we recall some general results on the solvability of the (SSIE) of the form $F \subset E_a + F'_x$. In Section 5 we deal with the solvability of the (SSIE) of the form $\ell_\infty \subset \mathcal{E} + F'_x$ where F' is either c_0, or

ℓ_∞. In Section 6 we solve the (SSIE) $c_0 \subset \mathcal{E} + s_x$. In Section 7 we study the (SSIE) $c \subset \mathcal{E} + s_x^{(c)}$ and the (SSE) $\mathcal{E} + s_x^{(c)} = c$. In Section 8 we apply results of the previous sections to the solvability of the (SSE) of the form $(\ell_r^p)_\Delta + F_x = F$.

2. Premilinaries and notations

An FK space is a *complete linear metric space*, for which convergence implies *coordinatewise convergence*. A *BK space* is a Banach space of sequences that is an *FK space*. A BK space E *is said to have AK if for every sequence* $y = (y_k)_{k\geq 1} \in E$, *then* $y = \lim_{p\to\infty} \sum_{k=1}^{p} y_k e^{(k)}$, *where* $e^{(k)} = (0, ..., 0, 1, 0, ...)$, 1 *being in the* $k-th$ *position*.

Let $\mathbb{R}$ be the set of all real numbers. For any given infinite matrix $A = (\mathbf{a}_{nk})_{n,k\geq 1}$ we define the operators $A_n = (\mathbf{a}_{nk})_{k\geq 1}$ for any integer $n \geq 1$, by $A_n y = \sum_{k=1}^{\infty} \mathbf{a}_{nk} y_k$, where $y = (y_k)_{k\geq 1}$, and the series are assumed convergent for all n. So we are led to the study of the operator A defined by $Ay = (A_n y)_{n\geq 1}$ mapping between sequence spaces. When A *maps* E *into* F, *where* E *and* F *are subsets of* ω, *we write* $A \in (E, F)$, (cf. [19], [20]). It is well known that if E has AK, then the set $\mathcal{B}(E)$ of all *bounded linear operators* L mapping in E, with norm $\|L\| = \sup_{y\neq 0} \left(\|L(y)\|_E / \|y\|_E\right)$ satisfies the identity $\mathcal{B}(E) = (E, E)$. For any subset F of ω, we write $F_A = \{y \in \omega \,:\, Ay \in F\}$ for the matrix domain of A in F. Then for any given sequence $u = (u_n)_{n\geq 1} \in \omega$ we define the diagonal matrix D_u by $[D_u]_{nn} = u_n$ for all n. It is interesting to rewrite the set E_u using a diagonal matrix. Let E be any subset of ω and $u \in U^+$ we have $E_u = D_u * E = \{y = (y_n)_n \in \omega \,: y/u \in E\}$. We use the sets s_a^0, $s_a^{(c)}$, s_a and $(\ell^p)_a$ defined as follows (cf. [1]). For given $a \in U^+$ and $p \geq 1$ we put $D_a * c_0 = s_a^0$, $D_a * c = s_a^{(c)}$, $D_a * \ell_\infty = s_a$, and $D_a * \ell^p = (\ell^p)_a$. Each of the spaces $D_a * E$, where $E \in \{c_0, c, \ell_\infty\}$ is a *BK space normed* by $\|y\|_{s_a} = \sup_n (|y_n| / a_n)$ and s_a^0 *has AK*. The set ℓ^p, $(p \geq 1)$ normed by $\|y\|_{\ell^p} = \left(\sum_{k=1}^{\infty} |y_k|^p\right)^{1/p}$ is a BK space with AK. If $a = (R^n)_{n\geq 1}$ with $R > 0$, we write s_R, s_R^0, $s_R^{(c)}$, (or c_R) and $(\ell^p)_R$ for the sets s_a, s_a^0, $s_a^{(c)}$ and $(\ell^p)_a$, respectively. We also write D_R for $D_{(R^n)_{n\geq 1}}$. When $R = 1$, we obtain $s_1 = \ell_\infty$, $s_1^0 = c_0$ and $s_1^{(c)} = c$. Notice that the set $S_1 = (s_1, s_1)$ is a Banach algebra with $\|A\|_{S_1} = \sup_n \left(\sum_{k=1}^{\infty} |\mathbf{a}_{nk}|\right)$ and we have $(c_0, s_1) = (c, s_1) = (s_1, s_1) = S_1$. In the following we use the Schur's theorem (cf. [20], Theorem 1.17 (iii)) stated as follows. We have $A \in (s_1, c)$ if and only if $\lim_{n\to\infty} \mathbf{a}_{nk} = l_k$ for all k and for some scalar l_k and $\lim_{n\to\infty} \sum_{k=1}^{\infty} |\mathbf{a}_{nk}| = \sum_{k=1}^{\infty} |l_k|$. We also use the well known properties, stated as follows.

Lemma 1. *Let* a, $b \in U^+$ *and let* E, $F \subset \omega$ *be any linear spaces. We have* $A \in (E_a, F_b)$ *if and only if* $D_{1/b} A D_a \in (E, F)$.

Recall that the infinite matrix $T=(t_{nk})_{n,k\geq 1}$ is a triangle if $t_{nk}=0$ for $k>n$ and $t_{nn}\neq 0$ for all n. Then we obtain the next lemma.

Lemma 2 ([4], Lemma 9, p. 45)**.** *Let T' and T'' be any given triangles and let E, $F\subset\omega$. Then for any given operator T represented by a triangle we have $T\in(E_{T'},F_{T''})$ if and only if $T''TT'^{-1}\in(E,F)$.*

3. Some results on matrix transformations and on the multipliers of special sets

3.1. On the triangles $C(\lambda)$ and $\Delta(\lambda)$ and the sets W_a and W_a^0

For $\lambda\in U$ the infinite matrices $C(\lambda)$ and $\Delta(\lambda)$ are triangles. We have $[C(\lambda)]_{nk}=1/\lambda_n$ for $k\leq n$, and the nonzero entries of $\Delta(\lambda)$ are determined by $[\Delta(\lambda)]_{nn}=\lambda_n$ for all n, and $[\Delta(\lambda)]_{n,n-1}=-\lambda_{n-1}$ for all $n\geq 2$. It can be shown that the matrix $\Delta(\lambda)$ is the inverse of $C(\lambda)$, that is, $C(\lambda)(\Delta(\lambda)y)=\Delta(\lambda)(C(\lambda)y)=y$ for all $y\in\omega$. If $\lambda=e$ we obtain the well known operator of the first difference represented by $\Delta(e)=\Delta$. We then have $\Delta_n y=y_n-y_{n-1}$ for all $n\geq 1$, with the convention $y_0=0$. It is usually written $\Sigma=C(e)$ and then we may write $C(\lambda)=D_{1/\lambda}\Sigma$. Notice that $\Delta=\Sigma^{-1}$. We also have $cs=c_\Sigma$ for the set of all convergent series. The Cesàro operator is defined By $C_1=C((n)_{n\geq 1})$. We use the sets of sequences that are *$a-$strongly bounded and $a-$strongly convergent to zero* defined for $a\in U^+$ by $W_a=\left\{y\in\omega:\|y\|_{W_a}=\sup_n\left(n^{-1}\sum_{k=1}^n|y_k|/a_k\right)<\infty\right\}$ and

$$W_a^0=\left\{y\in\omega:\lim_{n\to\infty}\left(\frac{1}{n}\sum_{k=1}^n|y_k|/a_k\right)=0\right\},$$

(cf. [14], [11]). It can easily be seen that $W_a=\left\{y\in\omega:C_1D_{1/a}|y|\in s_1\right\}$. If $a=(r^n)_{n\geq 1}$ the sets W_a and W_a^0 are denoted by W_r and W_r^0. For $r=1$ we obtain the well-known sets $w_\infty=\left\{y\in\omega:\|y\|_{w_\infty}=\sup_n\left(n^{-1}\sum_{k=1}^n|y_k|\right)<\infty\right\}$ and $w_0=\left\{y\in\omega:\lim_{n\to\infty}\left(n^{-1}\sum_{k=1}^n|y_k|\right)=0\right\}$ called the *spaces of sequences that are strongly bounded and strongly summable to zero by the Cesàro method* (cf. [17]).

3.2. On the multipliers of some sets

First we need to recall some well known results. Let y and z be sequences and let E and F be two subsets of ω, we then write $M(E,F)=\{y\in\omega:\ yz\in F\text{ for all }z\in E\}$, the set $M(E,F)$ is called the *multiplier space of E and F*. In the following we use the next well known results.

Lemma 3. *Let E, $\widetilde{E}$, F and $\widetilde{F}$ be arbitrary subsets of ω. Then (i) $M(E,F)\subset M(\widetilde{E},F)$ for all $\widetilde{E}\subset E$. (ii) $M(E,F)\subset M(E,\widetilde{F})$ for all $F\subset\widetilde{F}$.*

Recall that for a, $b \in U^{+}$ and E and $F \subset \omega$ we have $D_a * E \subset D_b * F$ if and only if $a/b \in M(E,F)$. In the following we use the results stated below.

Lemma 4 ([9], Lemma 6, pp. 214-215). *Let $p \geq 1$. We have:*

i) a) $M(c,c_0) = M(\ell_\infty,c) = M(\ell_\infty,c_0) = c_0$ and $M(c,c) = c$. b) $M(E,\ell_\infty) = M(c_0,F) = \ell_\infty$ for E, $F = c_0$, c, or ℓ_∞. c) $M(c_0,\ell^p) = M(c,\ell^p) = M(\ell_\infty,\ell^p) = \ell^p$. d) $M(\ell^p,F) = \ell_\infty$ for $F \in \{c_0,c,s_1,\ell^p\}$.

ii) a) $M(w_0,F) = M(w_\infty,\ell_\infty) = s_{(1/n)_{n\geq 1}}$ for $F = c_0$, c, or ℓ_∞. b) $M(w_\infty,c_0) = M(w_\infty,c) = s^0_{(1/n)_{n\geq 1}}$. c) $M(\ell_1,w_\infty) = s_{(n)_{n\geq 1}}$ and $M(\ell_1,w_0) = s^0_{(n)_{n\geq 1}}$. d) $M(E,w_0) = w_0$ for $E = s_1$, or c. e) $M(E,w_\infty) = w_\infty$ for $E = c_0$, s_1, or c.

3.3. The equivalence relation $R_{\mathcal{E}}$

We need to recall some results on the equivalence relation $R_{\mathcal{E}}$ which is defined using the multiplier of sequence spaces. For $b \in U^{+}$ and for any subset $\mathcal{E}$ of ω, we denote by $cl^{\mathcal{E}}(b)$ the equivalence class for the equivalence relation $R_{\mathcal{E}}$ defined by $xR_{\mathcal{E}}b$ if $\mathcal{E}_x = \mathcal{E}_b$ for $x \in U^{+}$. It can easily be seen that $cl^{\mathcal{E}}(b)$ is the set of all $x \in U^{+}$ such that $x/b \in M(\mathcal{E},\mathcal{E})$ and $b/x \in M(\mathcal{E},\mathcal{E})$, (cf. [15]). We then have $cl^{\mathcal{E}}(b) = cl^{M(\mathcal{E},\mathcal{E})}(b)$. For instance $cl^{c}(b)$ is the set of all $x \in U^{+}$ such that $s_x^{(c)} = s_b^{(c)}$. This is the set of all sequences $x \in U^{+}$ such that $x_n \sim Cb_n$ $(n \to \infty)$ for some $C > 0$. We denote by $cl^{\infty}(b)$ the class $cl^{\ell_\infty}(b)$. Recall that $cl^{\infty}(b)$ is the set of all $x \in U^{+}$, such that $K_1 \leq x_n/b_n \leq K_2$ for all n and for some K_1, $K_2 > 0$.

4. Some general results on the (SSIE) $F \subset \mathcal{E} + F'_x$

Here we are interested in the study of the set of all positive sequences x that satisfy the inclusion $F \subset \mathcal{E} + F'_x$ where $\mathcal{E}$, F and F' are linear spaces of sequences. We may consider this problem as a *perturbation problem.*

4.1. The perturbed problem

If we know the set $M(F,F')$, then the solutions of the *elementary inclusion* $F'_x \supset F$ are determined by $1/x \in M(F,F')$. Now the question is: let $\mathcal{E}$ be a linear space of sequences. What are the solutions of the *perturbed* inclusion $F'_x + \mathcal{E} \supset F$? An additional question may be the following one: what are the conditions on $\mathcal{E}$ under which the solutions of the elementary and the perturbed inclusions are the same? In the following we write $\mathcal{I}(\mathcal{E},F,F') = \{x \in U^{+} : F \subset \mathcal{E} + F'_x\}$, where E, F and F' are linear spaces of sequences. If $F = F'$ we write $\mathcal{I}(\mathcal{E},F) = \mathcal{I}(\mathcal{E},F,F')$.

4.2. Some known results on the solvability of (SSIE)

For any set χ of sequences we let $\overline{\chi} = \{x \in U^+ : 1/x \in \chi\}$ and we write $\Phi = \{c_0, c, s_1, \ell^p, w_0, w_\infty\}$ with $p \geq 1$. By $c(1)$ we define the set of all sequences $\alpha \in U^+$ that satisfy $\lim_{n\to\infty} \alpha_n = 1$. Then we consider the condition

$$G \subset G_{1/\alpha} \text{ for all } \alpha \in c(1), \tag{1}$$

for any given linear space G of sequences. Notice that condition (1) is satisfied for all $G \in \Phi$. In this part we denote by U_1^+ the set of all sequences α with $0 < \alpha_n \leq 1$ for all n. We consider the condition

$$G \subset G_{1/\alpha} \text{ for all } \alpha \in U_1^+. \tag{2}$$

for any given linear space G of sequences. Then we introduce a linear space of sequences H which contains the spaces E and F'. The proof of the next theorem is based on the fact that if H satisfies the condition in (2) we then have $H_\alpha + H_\beta = H_{\alpha+\beta}$ for all α, $\beta \in U^+$ (cf. [13], Proposition 5.1, pp. 599-600). Notice that c does not satisfy this condition, but each of the sets c_0, ℓ_∞, ℓ^p, $(p \geq 1)$, w_0 and w_∞ satisfies the condition in (2). So we have for instance $s_\alpha^0 + s_\beta^0 = s_{\alpha+\beta}^0$. In the following we write $M(F, F') = \chi$. The next result is used to determine some classes of (SSIE), where we write $\mathcal{I}_a(E, F, F') = \mathcal{I}(E_a, F, F')$ for $a \in U^+$.

Theorem 1 ([9], Theorem 9, p. 216)**.** *Let $a \in U^+$ and let E, F and F' be linear subspaces of ω. Assume*

a) χ satisfies condition (1).

b) There is a linear space of sequences H that satisfies the condition in (2) and conditions α) and β), where α) E, $F' \subset H$, β) $M(F, H) = \chi$.

Then we have: i) $a \in M(\chi, c_0)$ implies $\mathcal{I}_a(E, F, F') = \overline{\chi}$. ii) $a \in \overline{M(F, E)}$ implies $\mathcal{I}_a(E, F, F') = U^+$.

As a direct consequence of the preceding we obtain the following result.

Lemma 5 ([9], Corollary 10, p. 216)**.** *Let $a \in U^+$, let E, F and F' be linear subspaces of ω. Assume χ satisfies condition (1) and assume $E \subset F'$ where F' satisfies the condition in (2). Then we have: i) The condition $a \in M(\chi, c_0)$ implies $\mathcal{I}_a(E, F, F') = \overline{\chi}$, ii) the condition $a \in \overline{M(F, E)}$ implies $\mathcal{I}_a(E, F, F') = U^+$.*

In [8] we have shown the next result on the (SSIE) $c \subset s_a^{(c)} + F'_x$ and $s_1 \subset s_a^{(c)} + F'_x$ with $F' \in \Phi$.

Proposition 1 ([8])**.** *Let $a \in U^+$ and let $F' \in \Phi$. We have: i) $\mathcal{I}_a(c, c, F') = \overline{F'}$ if $a \in c_0$, and $\mathcal{I}_a(c, c, F') = U^+$ if $1/a \in c$. ii) $\mathcal{I}_a(c, s_1, F') = \overline{F'}$ if $a \in c_0$, and $\mathcal{I}_a(c, s_1, F') = U^+$ if $1/a \in c_0$.*

5. Solvability of the (SSIE) of the form $\ell_\infty \subset \mathcal{E} + F'_x$ where F' is either c_0 or ℓ_∞

5.1. Solvability of the (SSIE) of the form $\ell_\infty \subset \mathcal{E} + s_x$

By Proposition 1 *ii*) the (SSIE) $s_1 \subset s_a^{(c)} + s_x$ is equivalent to $x \in \overline{s_1}$ for all $a \in c_0$. In the next theorem we extend this result to the case when $a \in c$. For instance, notice that x is a positive solution of the (SSE) $s_1 \subset c + s_x$ if the next statement holds. The condition $y_n = O(1)$ implies there are u, $v \in \omega$ such that $u_n \to l$ and $v_n/x_n = O(1)$ $(n \to \infty)$ for all $y \in \omega$ and for some scalar l. Now we state a more general result.

Theorem 2. *Let $\mathcal{E} \subset c$ be a linear space of sequences. Then the set $\mathcal{I}(\mathcal{E}, s_1)$ of all positive sequences x such that $s_1 \subset \mathcal{E} + s_x$ is determined by*

$$\mathcal{I}(\mathcal{E}, s_1) = \overline{s_1}.$$

Proof. *i*) Since $\mathcal{E} \subset c$ we obtain $\mathcal{I}(\mathcal{E}, s_1) \subset \mathcal{I}(c, s_1)$. So we begin to show the inclusion $\mathcal{I}(c, s_1) \subset \overline{s_1}$. For this, we assume $x \in \mathcal{I}(c, s_1)$ and $x \notin \overline{s_1}$. Then we have $1/x \notin \ell_\infty$ and there is a strictly increasing sequence $(n_i)_{i \geq 1}$ tending to infinity such that $x_{n_i} \to 0$ $(i \to \infty)$. Now let $h \in \ell_\infty$ be the sequence defined by $h_{n_i} = (-1)^i$ and $h_n = 0$ for all $n \notin \{n_i : i \in \mathbb{N}\}$. Since $\ell_\infty \subset c + s_x$ there are sequences $\varphi \in c$ and $\rho \in \ell_\infty$ such that $h = \varphi + x\rho$ and $(-1)^i = \varphi_{n_i} + \rho_{n_i} x_{n_i}$. This leads to a contradiction since $\rho_{n_i} x_{n_i} \to 0$ and $\varphi_{n_i} + \rho_{n_i} x_{n_i}$ tends to a limit as $i \to \infty$. This implies $\mathcal{I}(c, s_1) \subset \overline{s_1}$. So we have shown the inclusion $\mathcal{I}(\mathcal{E}, s_1) \subset \overline{s_1}$. Conversely, we show $\overline{s_1} \subset \mathcal{I}(c, s_1)$. For this, let $x \in \overline{s_1}$, that is, $1/x \in s_1$. Since $s_1 = M(s_1, s_1)$ we obtain $s_1 \subset s_x$, $s_1 \subset \mathcal{E} + s_x$ and $x \in \mathcal{I}(\mathcal{E}, s_1)$. This shows the inclusion $\overline{s_1} \subset \mathcal{I}(\mathcal{E}, s_1)$ and we conclude $\mathcal{I}(\mathcal{E}, s_1) = \overline{s_1}$. ■

As an immediate consequence of Theorem 2 we obtain the next useful result.

Corollary 1. *i*) *The set $\mathcal{I}(c, s_1)$ of all positive sequences x such that $s_1 \subset c + s_x$ is determined by $\mathcal{I}(c, s_1) = \overline{s_1}$.*

ii) *The set $\mathcal{S}(c, s_1)$ of all positive sequences x such that $c + s_x = s_1$ is determined by $\mathcal{S}(c, s_1) = cl^\infty(e)$.*

Proof. The proof of *i*) is immediate and *ii*) follows from *i*) and the equivalence of $c + s_x \subset s_1$ and $x \in s_1$. ■

In all that follows we write $\lambda^+ = \lambda \bigcap U^+$ for any given subset λ of ω. By Theorem 2 we obtain the following corollary.

Corollary 2. *Let $\alpha \in (cs)^+$ and let $\mathcal{E}$ be a linear space of sequences such that $\mathcal{E} \subset (s_\alpha)_\Delta$. Then the set $\mathcal{I}_{\mathcal{E}}^\infty$ of all positive sequences x such that $s_1 \subset \mathcal{E} + s_x$ is determined by $\mathcal{I}_{\mathcal{E}}^\infty = \overline{s_1}$.*

Proof. First recall that ΣD_α is the triangle defined by $(\Sigma D_\alpha)_{nk} = a_k$ for $k \leq n$. We have $(s_\alpha)_\Delta \subset c$ since by the Schur's theorem $\alpha \in cs$ implies $\Sigma D_\alpha \in (s_1, c)$. So we have $\mathcal{E} + s_x \subset c + s_x$ which implies $\mathcal{I}_\mathcal{E}^\infty \subset \mathcal{I}(c, s_1) \subset \overline{s_1}$. It can easily be seen that $\overline{s_1} \subset \mathcal{I}_\mathcal{E}^\infty$ and $\mathcal{I}_\mathcal{E}^\infty = \overline{s_1}$. This concludes the proof. ■

Corollary 3. *Let $a \in (cs)^+$. The next (SSIE) $\ell_\infty \subset (s_a^0)_\Delta + s_x$, $\ell_\infty \subset (s_a^{(c)})_\Delta + s_x$ and $\ell_\infty \subset (s_a)_\Delta + s_x$, have the same set of solutions that are determined by $\widetilde{\mathcal{I}_\Delta} = \overline{s_1}$.*

Corollary 4. *Let $p > 1$ and let $a^{p/(p-1)} \in (cs)^+$. Then the solutions of the (SSIE) $\ell_\infty \subset (\ell_a^p)_\Delta + s_x$ are determined by $\mathcal{I}_{(\ell_a^p)_\Delta}^\infty = \overline{s_1}$.*

We obtain a direct extension of Proposition 1 in the case $E \in \{c_0, c, \ell_\infty\}$ and $F = F' = \ell_\infty$.

Corollary 5. *Let $a \in U^+$. Then we have: i) If $a \in s_1$ then the solutions of the (SSIE) $\ell_\infty \subset s_a^0 + s_x$ are determined by $\mathcal{I}_a(c_0, s_1, s_1) = \overline{s_1}$. ii) If $a \in c$ then the solutions of the (SSIE) $\ell_\infty \subset s_a^{(c)} + s_x$ are determined by $\mathcal{I}_a(c, s_1, s_1) = \overline{s_1}$. iii) If $a \in c_0$ then the solutions of the (SSIE) $\ell_\infty \subset s_a + s_x$ are determined by $\mathcal{I}_a(s_1, s_1, s_1) = \overline{s_1}$.*

Corollary 6. *Let $a \in (D_{(1/n)_{n\geq 1}} * cs)^+$. The solutions of each of the (SSIE) a) $\ell_\infty \subset \left(W_a^0\right)_\Delta + s_x$, b) $\ell_\infty \subset (W_a)_\Delta + s_x$, are determined by $\mathcal{I}_{(W_a^0)_\Delta}^\infty = \mathcal{I}_{(W_a)_\Delta}^\infty = \overline{s_1}$.*

Proof. We have $\left(W_a^0\right)_\Delta \subset c$ if $\Sigma D_a \in (w_0, c)$. Since $w_0 \subset s_{(n)_{n\geq 1}}^0$ we have $\left(W_a^0\right)_\Delta \subset c$ if $\Sigma D_a \in (s_{(n)_{n\geq 1}}^0, c)$ which is equivalent to $\Sigma D_{(na_n)_{n\geq 1}} \in (c_0, c)$. By the characterization of (c_0, c) we deduce $\left(W_a^0\right)_\Delta \subset c$ if $a \in D_{(1/n)_{n\geq 1}} * cs$ and we apply Theorem 2. This shows $\mathcal{I}_{(W_a^0)_\Delta}^\infty = \overline{s_1}$. The case of b) can be obtained in a similar way. This concludes the proof. ■

Corollary 7. *Let $r > 0$. Then we have: i) The set $\mathcal{I}_{r,w}^\infty$ of all positive sequences x that satisfy $\ell_\infty \subset (W_r)_\Delta + s_x$ is determined by $\mathcal{I}_{r,w}^\infty = \begin{cases} \overline{s_1} & \text{if } r < 1, \\ U^+ & \text{if } r \geq 1. \end{cases}$ ii) The set $\mathcal{I}_{r,w}^0$ of all positive sequences x that satisfy $\ell_\infty \subset \left(W_r^0\right)_\Delta + s_x$ is determined by $\mathcal{I}_w^0 = \mathcal{I}_w^\infty$ for all $r \neq 1$.*

Proof. *i)* The case $r < 1$ follows from Corollary 6 since we have $\sum_{k=1}^\infty kr^k < \infty$. Then the nonzero entries of the triangle $D_{1/r}\Delta$ are defined by $\left(D_{1/r}\Delta\right)_{nn} = -\left(D_{1/r}\Delta\right)_{n,n-1} = r^{-n}$. So the condition $r \geq 1$ implies $D_{1/r}\Delta \in (\ell_\infty, \ell_\infty)$ and the inclusion $(\ell_\infty, \ell_\infty) \subset (\ell_\infty, w_\infty)$ successively implies $D_{1/r}\Delta \in (\ell_\infty, w_\infty)$, $\ell_\infty \subset (W_r)_\Delta$ and $\mathcal{I}_{r,w}^\infty = U^+$. *ii)* can be shown similarly. This completes the proof. ■

5.2. Solvability of the (SSIE) of the form $\ell_\infty \subset \mathcal{E} + s_x^0$

By Proposition 1 *ii*) the (SSIE) $\ell_\infty \subset s_a^{(c)} + F'_x$ where $F' \in \Phi$ is equivalent to $x \in \overline{F'}$ for all $a \in c_0$. Especially we have $\ell_\infty \subset s_a^{(c)} + s_x^0$ with $a \in c_0$ if and only if $\lim_{n\to\infty} x_n = \infty$. In the next theorem we extend this result to the case when $a \in c$.

Theorem 3. *Let $\mathcal{E} \subset c$ be a linear space of sequences. Then the set $\mathbb{I}_{\mathcal{E}}^{\infty} = \mathcal{I}(\mathcal{E}, s_1, c_0)$ of all positive sequences x such that $\ell_\infty \subset \mathcal{E} + s_x^0$ is determined by $\mathbb{I}_{\mathcal{E}}^{\infty} = \overline{c_0}$.*

Proof. As we have seen above we have $\mathbb{I}_{\mathcal{E}}^{\infty} \subset \mathbb{I}_{c}^{\infty}$. So we first show $\mathbb{I}_{c}^{\infty} \subset \overline{c_0}$. Assume there is $x \in \mathbb{I}_{c}^{\infty}$ and $x \notin \overline{c_0}$. Then we have $1/x \notin c_0$ and there is a strictly increasing sequence $(n_i)_{i\geq 1}$ tending to infinity such that $(x_{n_i})_{i\geq 1} \in \ell_\infty$. Now let $h \in \ell_\infty$ be the sequence defined by $h_{n_i} = (-1)^i$ and $h_n = 0$ for all $n \notin \{n_i : i \in \mathbb{N}\}$. Since $\ell_\infty \subset c + s_x^0$ there are sequences $\varphi \in c$ and $\varepsilon \in c_0$ such that $h = \varphi + x\varepsilon$ and $(-1)^i = \varphi_{n_i} + \varepsilon_{n_i} x_{n_i}$ for all i. This leads to a contradiction since $\varepsilon_{n_i} x_{n_i} \to 0$ and $\varphi_{n_i} + \varepsilon_{n_i} x_{n_i}$ tends to a limit as $i \to \infty$. This implies $\mathbb{I}_{c}^{\infty} \subset \overline{c_0}$ and $\mathbb{I}_{\mathcal{E}}^{\infty} \subset \overline{c_0}$. Conversely, we have $x \in \overline{c_0}$ implies $1/x \in c_0$ and since $c_0 = M(s_1, c_0)$ we successively obtain $\ell_\infty \subset s_x^0$, $\ell_\infty \subset \mathcal{E} + s_x^0$ and $x \in \mathbb{I}_{\mathcal{E}}^{\infty}$. This shows the inclusion $\overline{c_0} \subset \mathbb{I}_{\mathcal{E}}^{\infty}$ and we conclude $\mathbb{I}_{\mathcal{E}}^{\infty} = \overline{c_0}$. ∎

As an immediate consequence of Theorem 3 we obtain the next corollary.

Corollary 8. *Let $a \in (cs)^+$ and let $\mathcal{E}$ be a linear space of sequences such that $\mathcal{E} \subset (s_a)_\Delta$. Then the set $\mathbb{I}_{\mathcal{E}}^{\infty}$ of all positive sequences x such that $\ell_\infty \subset \mathcal{E} + s_x^0$ is determined by $\mathbb{I}_{\mathcal{E}}^{\infty} = \overline{c_0}$.*

Proof. We have $(s_a)_\Delta \subset c$ since $a \in cs$ implies $\Sigma D_a \in (s_1, c)$. So we have $\mathcal{E} + s_x^0 \subset c + s_x^0$ which implies $\mathbb{I}_{\mathcal{E}}^{\infty} \subset \mathbb{I}_{c}^{\infty} \subset \overline{c_0}$. Conversely, as we have just seen we have $x \in \overline{c_0}$ successively implies $\ell_\infty \subset s_x^0$, $\ell_\infty \subset \mathcal{E} + s_x^0$ and $\overline{c_0} \subset \mathbb{I}_{\mathcal{E}}^{\infty}$. We conclude $\mathbb{I}_{\mathcal{E}}^{\infty} = \overline{c_0}$. This completes the proof. ∎

Corollary 9. *Let $a \in (cs)^+$. Then the next (SSIE) $\ell_\infty \subset \left(s_a^0\right)_\Delta + s_x^0$, $\ell_\infty \subset (s_a^{(c)})_\Delta + s_x^0$ and $\ell_\infty \subset (s_a)_\Delta + s_x^0$ have the same set of solutions that are determined by $\widetilde{\mathcal{I}_\Delta^0} = \overline{c_0}$.*

Corollary 10. *Let $p > 1$ and $q = p/(p-1)$ and assume $a^q \in (cs)^+$. Then the solutions of the (SSIE) $\ell_\infty \subset (\ell_a^p)_\Delta + s_x^0$ are determined by $\mathcal{I}^0_{(\ell_a^p)_\Delta} = \overline{c_0}$.*

We obtain a direct extension of Proposition 1 in the case $E \in \{c_0, c, \ell_\infty\}$, $F = \ell_\infty$ and $F' = c_0$.

Corollary 11. *Let $a \in U^{+}$. Then we have: i) If $a \in s_1$ then the solutions of the (SSIE) $\ell_\infty \subset s_a^0 + s_x^0$ are determined by $\mathcal{I}_a(c_0, s_1, c_0) = \overline{c_0}$. ii) If $a \in c$ then the solutions of the (SSIE) $\ell_\infty \subset s_a^{(c)} + s_x^0$ are determined by $\mathcal{I}_a(c, s_1, c_0) = \overline{c_0}$. iii) If $a \in c_0$ then the solutions of the (SSIE) $\ell_\infty \subset s_a + s_x^0$ are determined by $\mathcal{I}_a(s_1, s_1, c_0) = \overline{c_0}$.*

By similar arguments as those used in Corollary 6 we obtain the next result.

Corollary 12. *Let $a \in D_{(1/n)_{n\geq 1}} * cs$. The solutions of each of the (SSIE) $\ell_\infty \subset \left(W_a^0\right)_\Delta + s_x^0$ and $\ell_\infty \subset (W_a)_\Delta + s_x^0$, are determined by $\mathcal{I}^0_{(W_a^0)_\Delta} = \mathcal{I}^0_{(W_a)_\Delta} = \overline{c_0}$.*

6. On the (SSIE) $c_0 \subset \mathcal{E} + s_x$

In this part we deal with the (SSIE) $c_0 \subset \mathcal{E} + s_x$ with $\mathcal{E} \subset (s_a)_\Delta$ and $a \in c_0^{+}$. The inclusion $c_0 \subset (s_a)_\Delta + s_x$ is associated with the next statement. For every $y \in \omega$ there are u, $v \in \omega$ with $y = u + v$ such that $(u_n - u_{n-1})/a_n = O(1)$ and $v_n/x_n = O(1)$ $(n \to \infty)$. Notice that if $\sum_k a_k < \infty$ then we have $(s_a)_\Delta \subset c$ since by the Schur's theorem we have $\Sigma D_a \in (\ell_\infty, c)$. Then we have $c \nsubseteq (s_a)_\Delta$ since the inclusion $c \subset (s_a)_\Delta$ is equivalent to $D_{1/a}\Delta \in (c, s_1)$ and to $a \in \overline{s_1}$.

6.1. On the identity $(\chi_a)_\Delta + (\chi_b)_\Delta = (\chi_{a+b})_\Delta$

Lemma 6. *Let a, $b \in U^{+}$. Then we have $(\chi_a)_\Delta + (\chi_b)_\Delta = (\chi_{a+b})_\Delta$ for $\chi = s_1$, or c_0.*

Proof. Since the inclusion $(\chi_{a+b})_\Delta \subset (\chi_a)_\Delta + (\chi_b)_\Delta$ is trivial, it is enough to show $(\chi_a)_\Delta + (\chi_b)_\Delta \subset (\chi_{a+b})_\Delta$. For this, let $y \in (\chi_a)_\Delta + (s_b)_\Delta$. Since $(\chi_\alpha)_\Delta = (\Sigma D_\alpha)\chi$ with $\alpha \in U^{+}$ there are u, $v \in \chi$ such that

$$y_n = \sum_{k=1}^{n} a_k u_k + \sum_{k=1}^{n} b_k v_k = \sum_{k=1}^{n} (a_k + b_k) z_k = (\Sigma D_a + \Sigma D_b)_n z,$$

where $z_k = (a_k u_k + b_k v_k)/(a_k + b_k)$ for all k. Since $0 < a_k/(a_k + b_k) < 1$ and $0 < b_k/(a_k + b_k) < 1$ we have $|z_k| \leq |u_k| + |v_k|$ for all k, and $(|u_k| + |v_k|)_{\geq 1} \in \ell_\infty$ for $\chi = s_1$ and $(|u_k| + |v_k|)_{\geq 1} \in c_0$ for $\chi = c_0$. This shows $y \in (\Sigma D_{a+b})\chi = (\chi_{a+b})_\Delta$ and $(\chi_a)_\Delta + (\chi_b)_\Delta \subset (\chi_{a+b})_\Delta$. This completes the proof. ∎

Remark 1. As a direct consequence of the preceding lemma we have $\Sigma D_a \chi + \Sigma D_b \chi = (\Sigma D_{a+b})\chi$ for $\chi = s_1$, or c_0.

6.2. On the (SSIE) $c_0 \subset \mathcal{E} + s_x$ with $\mathcal{E} \subset (s_\alpha)_\Delta$ and $\alpha \in c_0^+$

For the convenience of the reader we state the next result.

Lemma 7. *Let $r > 0$ and let $\varkappa$ be any of the symbols s, s^0, or $s^{(c)}$. Then we have: i) $(\varkappa_r)_\Delta \nsubseteq c_0$ for all r. ii) $c_0 \subset (\varkappa_r)_\Delta$ if and only if $r \geq 1$. iii) $c_0 \nsubseteq (\varkappa_r)_\Delta$ for all $r < 1$.*

Proof. *i*) We have $\Sigma D_r \notin (c_0, c_0)$ since $\lim_{n\to\infty} (\Sigma D_r)_{nk} = r^k \neq 0$ for all $k \geq 1$. Then the condition $(\varkappa_1, c_0) \subset (c_0, c_0)$ implies $\Sigma D_r \notin (\varkappa_1, c_0)$ and $(\varkappa_r)_\Delta \nsubseteq c_0$. *ii*) The inclusion $c_0 \subset (\varkappa_r)_\Delta$ implies $D_{1/r}\Delta \in (c_0, \varkappa_1)$ and since $(c_0, \varkappa_1) \subset (c_0, s_1)$ we conclude $(1/r^n)_{n\geq 1} \in \ell_\infty$ and $r \geq 1$. Conversely, let $r \geq 1$. Then we have $D_{1/r}\Delta \in (c_0, c_0)$ and since $(c_0, c_0) \subset (c_0, \varkappa_1)$ we obtain $c_0 \subset (\varkappa_r)_\Delta$ where $\varkappa$ is any of the symbols s, s^0, or $s^{(c)}$. *iii*) is a direct consequence of *ii*). This completes the proof. ∎

Now we state a result where we must have in mind the statements in Lemma 7 and the equivalence of $\mathcal{E} \subset (s_\alpha)_\Delta$ and $D_{1/\alpha}\Delta \in (\mathcal{E}, s_1)$. So we obtain an extension of Lemma 7 *iii*) since the condition $\alpha \in c_0^+$ implies $\mathcal{E} \nsubseteq (s_\alpha)_\Delta$ for $\mathcal{E} \in \{c_0, c, \ell_\infty\}$, and we have not the trivial inclusion $c_0 \subset \mathcal{E}$ which implies $c_0 \subset \mathcal{E} + s_x$ for all positive sequences x. In the following we write $(x^-)_n = x_{n-1}$ for $n \geq 2$ and $x_1^- = 1$.

Theorem 4. *Let $\alpha \in c_0^+$ and let $\mathcal{E} \subset (s_\alpha)_\Delta$ be a linear space of sequences. Then the set $\mathbb{I}_{\mathcal{E}}^0 = \mathcal{I}(\mathcal{E}, c_0, s_1)$ of all positive sequences x such that $c_0 \subset \mathcal{E} + s_x$ is determined by*

$$\mathbb{I}_{\mathcal{E}}^0 \cap c = cl^c(e).$$

Proof. First we show $s_x \subset (s_{x+x^-})_\Delta$. Indeed, this inclusion is equivalent to $D_{1/(x+x^-)}\Delta D_x \in (s_1, s_1)$ where we have $\left[D_{1/(x+x^-)}\Delta D_x\right]_{nn} = x_n/(x_{n-1} + x_n)$ and $\left[D_{1/(x+x^-)}\Delta D_x\right]_{n,n-1} = -x_{n-1}/(x_{n-1} + x_n)$ for all n, the other entries being naught. Now we let $x \in \mathbb{I}_{\mathcal{E}}^0 \cap c$. Then we have $x \in c$ and $c_0 \subset (s_\alpha)_\Delta + s_x$. The last inclusion implies

$$c_0 \subset (\Sigma D_\alpha) s_1 + (\Sigma D_{x+x^-}) s_1$$

and by Lemma 6 we obtain

$$(\Sigma D_\alpha) s_1 + (\Sigma D_{x+x^-}) s_1 = (\Sigma D_{\alpha+x+x^-}) s_1 = (s_{\alpha+x+x^-})_\Delta .$$

We deduce $c_0 \subset (s_{\alpha+x+x^-})_\Delta$. So there is $K > 0$ such that $(\alpha_n + x_n + x_{n-1})^{-1} \leq K$ and $x_n + x_{n-1} \geq 1/K - \alpha_n$ for all n. Since $\alpha \in c_0$, there is $M > 0$ such that $x_n + x_{n-1} \geq M$ for all n. Then the condition $x \in c$ implies $\lim_{n\to\infty} (x_n + x_{n-1}) = 2 \lim_{n\to\infty} x_n \geq M$ and $\lim_{n\to\infty} x_n > 0$ which implies $s_x^{(c)} = c$. So we have shown $\mathbb{I}_{\mathcal{E}}^0 \cap c \subset c \cap \overline{c} = cl^c(e)$. Conversely, let $x \in cl^c(e)$.

Then we have $\lim_{n\to\infty} x_n = L$ with $L > 0$. So we have $1/x \in s_1$ which implies $c_0 \subset s_x$, $c_0 \subset \mathcal{E} + s_x$ and since $x \in c$ we conclude $x \in \mathbb{I}^0_{\mathcal{E}} \cap c$. This completes the proof. ∎

6.3. Application to the (SSIE) $F \subset (E_a)_\Delta + F'_x$

In this part we deal with some properties of the (SSIE)

$$F \subset (E_a)_\Delta + F'_x \tag{3}$$

where E, F and F' are linear spaces of sequences

Proposition 2. *Let E, F and F' be linear spaces of sequences that satisfy $F \supset c_0$ and E, $F' \subset \ell_\infty$. Let $\mathcal{I}\left((E_a)_\Delta, F, F'\right) \cap c$ be the set of all convergent and positive sequences x such that (3) holds. If $a \in c_0^+$ then we have:*

$$\mathcal{I}\left((E_a)_\Delta, F, F'\right) \cap c \subset cl^c(e). \tag{4}$$

Moreover if we assume $c \subset M(F, F')$ then

$$\mathcal{I}\left((E_a)_\Delta, F, F'\right) \cap c = cl^c(e). \tag{5}$$

Proof. We have $x \in \mathcal{I}\left((E_a)_\Delta, F, F'\right) \cap c$ implies $c_0 \subset (s_a)_\Delta + s_x$ and by Theorem 4 we obtain $x \in cl^c(e)$. Now we assume $c \subset M(F, F')$. Then the condition $x \in cl^c(e)$ implies $s_x^{(c)} = c$ and there is $L > 0$ such that $\lim_{n\to\infty} 1/x_n = L$ and $1/x \in c$. So we obtain $1/x \in M(F, F')$, $F \subset F'_x$ and $x \in \mathcal{I}\left((E_a)_\Delta, F, F'\right) \cap c$. This shows the identity in (5). This concludes the proof. ∎

Remark 2. As a direct consequence of the preceding proposition we may show that if E is a linear space of sequences such that $c_0 \subset E \subset \ell_\infty$ then the set $S\left((E_r)_\Delta, c\right)$ with $0 < r < 1$ be the set of all positive sequences such that $(E_r)_\Delta + s_x^{(c)} = c$ is determined by $S\left((E_r)_\Delta, c\right) = cl^c(e)$.

7. On the (SSIE) $c \subset \mathcal{E} + s_x^{(c)}$ and the (SSE) $\mathcal{E} + s_x^{(c)} = c$

In this part we consider the (SSIE) $c \subset \mathcal{E} + s_x^{(c)}$ which is associated with the next statement. For every $y \in c$ there are u, $v \in \omega$ with $y = u + v$ such that $u \in \mathcal{E}$ and $v/x \in c$. Then we solve the equation $\mathcal{E} + s_x^{(c)} = c$ where $\mathcal{E} \subset (s_\alpha)_\Delta$ with $\sum_{k=1}^\infty \alpha_k < \infty$.

7.1. On the (SSIE) $c \subset \mathcal{E} + s_x^{(c)}$

We obtain the following lemma.

Lemma 8. *Let $\mathcal{E}$ be a linear space of sequences that satisfies $\mathcal{E} \subset (s_\alpha)_\Delta$ with $\alpha \in (cs)^+$. Then the set $\mathcal{I}^c(\mathcal{E}, c)$ of all positive and convergent sequences x that satisfy $c \subset \mathcal{E} + s_x^{(c)}$ is determined by*

$$\mathcal{I}^c(\mathcal{E}, c) = cl^c(e).$$

Proof. Let $\mathcal{E} \subset (s_\alpha)_\Delta$ with $\alpha \in cs^+$. Then it can easily be seen that the condition $c \subset \mathcal{E} + s_x^{(c)}$ implies $c_0 \subset \mathcal{E} + s_x$. So by Theorem 4 we have

$$\mathcal{I}^c(\mathcal{E}, c) \subset \mathbb{I}_{\mathcal{E}}^0 \cap c \subset cl^c(e). \tag{6}$$

Now since $1/x \in c$ implies $c \subset s_x^{(c)}$ and $c \subset \mathcal{E} + s_x^{(c)}$, by the identity $\overline{c} \cap c = cl^c(e)$ we conclude

$$cl^c(e) \subset \mathcal{I}^c(\mathcal{E}, c). \tag{7}$$

By (6) and (7) we obtain $\mathcal{I}^c(\mathcal{E}, c) = cl^c(e)$. This completes the proof. ■

7.2. On the (SSE) $\mathcal{E} + s_x^{(c)} = c$.

In the following we deal with some (SSE) of the form $\mathcal{E} + F_x = F$ where $\mathcal{E}$ and F are two linear subsets of ω. Recall that x satisfies this (SSE) if and only if $\mathcal{E} \subset F$, $x \in M(F, F)$ and $x \in \mathcal{I}(\mathcal{E}, F)$. The next theorem extends the results on the (SSE) of the form $E_a + s_x^{(c)} = c$ where $E = c_0$, c, or ℓ^p, $(p \geq 1)$ stated in ([6], Proposition 5.1, p. 108) and ([6], Theorem 5.2, p. 108). Indeed, here we consider the equation $\mathcal{E} + s_x^{(c)} = c$ with $\mathcal{E} \subset (s_\alpha)_\Delta$ and $\alpha \in cs^+$. For instance the identity $(s_r)_\Delta = s_a^{(c)}$ for $r < 1$ cannot be obtained for any $a \in U^+$, since it should imply $1/a \in c$ and $a_n/r^n = O(1)$ $(n \to \infty)$ which is contradictory.

Theorem 5. *Let $\mathcal{E}$ be a linear space of sequences that satisfies $\mathcal{E} \subset (s_\alpha)_\Delta$ with $\alpha \in cs$. Then the set $\mathcal{S}(\mathcal{E}, c)$ of all positive sequences x that satisfy the (SSE) $\mathcal{E} + s_x^{(c)} = c$ is determined by $\mathcal{S}(\mathcal{E}, c) = cl^c(e)$.*

Proof. Let $x \in \mathcal{S}(\mathcal{E}, c)$. Then we have $s_x^{(c)} \subset c$, that is, $x \in c$, and $c \subset \mathcal{E} + s_x^{(c)}$. So we have $x \in \mathcal{I}^c(\mathcal{E}, c)$ and by Lemma 8 we obtain $\mathcal{S}(\mathcal{E}, c) \subset \mathcal{I}^c(\mathcal{E}, c) = cl^c(e)$. Conversely, let $x \in cl^c(e)$. Then we have $s_x^{(c)} = c$. Since $\alpha \in cs^+$, by the Schur's theorem we have $\Sigma D_\alpha \in (s_1, c)$. This implies $\mathcal{E} \subset (s_\alpha)_\Delta \subset c$ and $\mathcal{E} + s_x^{(c)} = \mathcal{E} + c = c$. So we obtain $cl^c(e) \subset \mathcal{S}(\mathcal{E}, c)$ and we conclude $\mathcal{S}(\mathcal{E}, c) = cl^c(e)$. This completes the proof. ■

Corollary 13. *The perturbed equations* $\left(s_r^0\right)_\Delta + s_x^{(c)} = c$, $(s_r^{(c)})_\Delta + s_x^{(c)} = c$ *and* $(s_r)_\Delta + s_x^{(c)} = c$ *satisfy* $\mathcal{S}\left(\left(s_r^0\right)_\Delta, c\right) = \mathcal{S}((s_r^{(c)})_\Delta, c) = \mathcal{S}\left((s_r)_\Delta, c\right)$ *and*

$$\mathcal{S}\left((s_r)_\Delta, c\right) = \begin{cases} cl^c\left(e\right) & \text{if } r < 1, \\ \varnothing & \text{if } r \geq 1. \end{cases}$$

Proof. We have $\mathcal{S}\left((s_r)_\Delta, c\right) = cl^c\left(e\right)$ if $r < 1$ by Theorem 5, where $\alpha = \left(r^n\right)_{n\geq 1} \in cs$. Then we have $\left(E_r\right)_\Delta \nsubseteq c$ for all $r \geq 1$ and $E \in \{c_0, c, \ell_\infty\}$. Indeed, the condition $\left(E_r\right)_\Delta \subset c$ should imply $\Sigma D_r \in (c_0, c)$ and $r < 1$. This completes the proof. ■

Corollary 14. *The perturbed (SSE) defined by* $(W_r)_\Delta + s_x^{(c)} = c$ *and* $\left(W_r^0\right)_\Delta + s_x^{(c)} = c$ *satisfy the identities* $\mathcal{S}\left((W_r)_\Delta, c\right) = \mathcal{S}\left(\left(W_r^0\right)_\Delta, c\right) = \mathcal{S}\left((s_r)_\Delta, c\right)$ *where* $\mathcal{S}\left((s_r)_\Delta, c\right)$ *is determined in Corollary 13.*

Proof. We have $(W_r)_\Delta = (w_\infty)_{D_{1/r}\Delta}$ and since $w_\infty \subset s_{(n)_{n\geq 1}}$ we obtain $(W_r)_\Delta \subset (s_{(nr^n)_{n\geq 1}})_\Delta$, then we apply Theorem 5 with $\alpha = \left(nr^n\right)_{n\geq 1} \in cs$. In the same way we have $\left(W_r^0\right)_\Delta \subset (W_r)_\Delta \subset (s_{(nr^n)_{n\geq 1}})_\Delta$. Then we have $\left(E_r\right)_\Delta \nsubseteq c$ for all $r \geq 1$ and $E \in \{w_0, w_\infty\}$. Indeed, the condition $\left(E_r\right)_\Delta \subset c$ should imply $\Sigma D_r \in (w_0, c)$ and $\Sigma D_r \in (c_0, c)$ since $w_0 \supset c_0$ and as above we obtain $r < 1$. This concludes the proof. ■

8. Application to the solvability of the (SSE) of the form $(\ell_r^p)_\Delta + F_x = F$

In this part we apply the results stated in the previous sections and we extend the results stated in [10] where we studied the (SSE) of the form $\left(E_r\right)_\Delta + F_x = F_u$ with r, $u > 0$ and where E, F are any of the sets c_0, c, or ℓ_∞ and the (SSE) $\left(W_r^0\right)_\Delta + s_x^{(c)} = s_b^{(c)}$. Then we study the (SSE) $(\ell_r^p)_\Delta + F_x = F$ where F is any of the sets c_0, c, or ℓ_∞ and $p \geq 1$. In the next result we use the characterization of (ℓ^p, F) where $F = c_0$, c, or ℓ_∞, see for instance ([18], Theorem 1.37, p. 161).

Proposition 3. *Let* $p \geq 1$ *and* $r > 0$, *and let* $\mathcal{S}_p^0$ *be the set of all positive sequences* x *such that* $(\ell_r^p)_\Delta + s_x^0 = c_0$. *Then* $\mathcal{S}_p^0 = \varnothing$.

Proof. The entries of the triangle ΣD_r are defined by $\left(\Sigma D_r\right)_{nk} = r^k$ for $k \leq n$. Then we have $\lim_{n\to\infty} \left(\Sigma D_r\right)_{nk} \neq 0$ for all k, which implies $\Sigma D_r \notin (\ell^p, c_0)$ and $(\ell_r^p)_\Delta \nsubseteq c_0$. We conclude $\mathcal{S}_p^0 = \varnothing$. ■

We also obtain the next result.

Theorem 6. *Let* r, $u > 0$ *and let* $p \geq 1$. *Then we have:*

i) Let $p > 1$. Then the set $\mathcal{S}_p^F = \mathcal{S}\left(\left(\ell_r^p\right)_\Delta, F\right)$ of all positive sequences x such that $\left(\ell_r^p\right)_\Delta + F_x = F$ where F is either of the sets c, or ℓ_∞ is determined by

$$\mathcal{S}_p^F = \begin{cases} cl^F(e) & \text{if } r < 1, \\ \varnothing & \text{if } r \geq 1. \end{cases}$$

ii) a) The set $\mathcal{S}_1^\infty = \mathcal{S}\left(\left(\ell_r^1\right)_\Delta, \ell_\infty\right)$ of all positive sequences x such that $\left(\ell_r^1\right)_\Delta + s_x = s_1$ is determined by

$$\mathcal{S}_1^\infty = \begin{cases} cl^\infty(e) & \text{if } r \leq 1, \\ \varnothing & \text{if } r > 1. \end{cases}$$

b) The set $\mathcal{S}_1^c = \mathcal{S}\left(\left(\ell_r^1\right)_\Delta, c\right)$ satisfies the identity $\mathcal{S}_1^c = cl^c(e)$ for $r < 1$ and $\mathcal{S}_1^c = \varnothing$ for $r > 1$.

Proof. *i)* Case $F = c$. Let $x \in \mathcal{S}_p^c$. Then we have

$$\left(\ell_r^p\right)_\Delta \subset c \tag{8}$$

and

$$x \in c \tag{9}$$

We have (8) if and only if $\Sigma D_r \in (\ell^p, c)$ and by the characterization of (ℓ^p, c) it can easily be shown that the condition in (8) is equivalent to

$$\sup_{n \geq 1} \sum_{k=1}^{n} r^{kq} < \infty \quad \text{with} \quad q = p/(p-1). \tag{10}$$

So we have $\mathcal{S}_p^c \neq \varnothing$ implies $r < 1$ and $\mathcal{S}_p^c = \varnothing$ if $r \geq 1$. Then for $r < 1$ we have $\left(\ell_r^p\right)_\Delta \subset (s_r)_\Delta$ with $(r^n)_{n \geq 1} \in cs$ and we conclude by Theorem 5 that $\mathcal{S}_p^c = cl^c(e)$.

Case $F = \ell_\infty$. Let $x \in \mathcal{S}_p^\infty$. Then we have

$$\left(\ell_r^p\right)_\Delta \subset \ell_\infty, \tag{11}$$

$$x \in \ell_\infty \tag{12}$$

and

$$\ell_\infty \subset \left(\ell_r^p\right)_\Delta + s_x. \tag{13}$$

As we have seen above the condition in (11) is equivalent to $\Sigma D_r \in (\ell^p, \ell_\infty)$ and to (10). So we have $r < 1$. Then by Theorem 2 with $\mathcal{E} = \left(\ell_r^p\right)_\Delta \subset c$

and by (12) the condition in (13) implies $x \in cl^{\infty}(e)$. So we have shown $\mathcal{S}_p^{\infty} \subset cl^{\infty}(e)$ for $r < 1$. Conversely, let $r < 1$ and $x \in cl^{\infty}(e)$. Then we have $s_x = \ell_{\infty}$ and $(\ell_r^p)_{\Delta} \subset \ell_{\infty}$ which imply $(\ell_r^p)_{\Delta} + s_x = (\ell_r^p)_{\Delta} + \ell_{\infty} = \ell_{\infty}$. So we have $cl^{\infty}(e) \subset \mathcal{S}_p^{\infty}$. This concludes the proof of i).

ii) a) Let $x \in \mathcal{S}_1^{\infty}$. Then the conditions in (11), (12) and (13) hold with $p = 1$. The condition in (11) with $p = 1$ is equivalent to $\Sigma D_r \in (\ell^1, \ell_{\infty})$ and to $(r^n)_{n \geq 1} \in \ell_{\infty}$. So we have $\mathcal{S}_1^{\infty} \neq \varnothing$ if $r \leq 1$. For $r < 1$, by Theorem 2 where $\mathcal{E} = (\ell_r^1)_{\Delta} \subset (s_{\alpha})_{\Delta}$ for $\alpha = (r^n)_{n \geq 1} \in c_0$ the inclusion in (13) with $p = 1$ implies $x \in \overline{s_1}$ and since (12) holds we conclude $\mathcal{S}_1^{\infty} \subset cl^{\infty}(e)$. By similar arguments as those used above we obtain $cl^{\infty}(e) \subset \mathcal{S}_1^{\infty}$ for $r < 1$ and we conclude $\mathcal{S}_1^{\infty} = cl^{\infty}(e)$.

Case $r = 1$. We write ℓ^1 for the set ℓ_1^1 and we denote by bv the set ℓ_{Δ}^1 of bounded variation. Now we let $x \in \mathcal{S}(bv, s_1)$. Then we successively have $bv \subset \ell_{\infty}$, since $\Sigma \in (\ell^1, \ell_{\infty})$, $x \in \ell_{\infty}$ and $\ell_{\infty} \subset bv + s_x$. Since we have $\Sigma \in (\ell^1, c)$ we obtain $bv \subset c$ and by Theorem 2 the statement $\ell_{\infty} \subset bv + s_x$ implies $x \in \overline{s_1}$. So we have $\mathcal{S}(bv, s_1) \subset cl^{\infty}(e)$. Conversely, assume $x \in cl^{\infty}(e)$. Then we have $s_x = s_1$ and since $bv \subset \ell_{\infty}$ we obtain $bv + s_x = bv + s_1 = s_1$ and $x \in \mathcal{S}(bv, s_1)$. We conclude $\mathcal{S}(bv, s_1) = cl^{\infty}(e)$.

b) Let $x \in \mathcal{S}_1^c$ and let $r \neq 1$. Then the conditions in (8), (9) hold with $p = 1$ and the condition in (8) is equivalent to $\Sigma D_r \in (\ell^1, c)$ and to $(r^n)_{n \geq 1} \in c$. So we have $r < 1$. As we have seen in i) we conclude by Theorem 5 that $\mathcal{S}_1^c = cl^c(e)$. ■

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