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MEROMORPHIC SOLUTIONS OF LINEAR DIFFERENCE EQUATIONS WITH POLYNOMIAL COEFFICIENTS*

Abstract. We study the growth of the transcendental meromorphic solution $f(z)$ of the linear difference equation:

$$\sum_{j=0}^{n} p_j(z)f(z+j) = q(z),$$

where $q(z)$, $p_0(z)$, …, $p_n(z)$ $(n \geq 1)$ are polynomials such that $p_0(z)p_n(z) \not\equiv 0$, and obtain some necessary conditions guaranteeing that the order of $f(z)$ satisfies $\sigma(f) \geq 1$ using a difference analogue of the Wiman-Valiron theory. Moreover, we give the form of $f(z)$ with two Borel exceptional values when two of $p_0(z)$, …, $p_n(z)$ have the maximal degrees.

Key words: growth, difference equation, Borel exceptional value.

AMS Mathematics Subject Classification: 30D35, 39A10.

1. Introduction and main results

Let $f(z)$ be a meromorphic function in the whole complex plane $\mathbb{C}$, we shall use the standard notations of Nevanlinna's theory (see, e.g., [9, 15]), such as the characteristic function $T(r,f)$. Moreover, we will use the notation $S(r,f)$ to denote any quantity that satisfies $S(r,f) = o(1)T(r,f)$ as $r \to \infty$ outside of a possible exceptional set of finite logarithmic measure. And we will use the notation $\sigma(f)$ to denote the order of growth of $f(z)$ and the notations $\lambda(f)$ and $\lambda(1/f)$ to denote the exponent of convergence of the zeros and poles of $f(z)$, respectively. We define the difference operators of $f(z)$ by $\Delta f(z) = f(z+1) - f(z)$ and $\Delta^n f(z) = \Delta(\Delta^{n-1} f(z)) = \sum_{i=0}^{n}(-1)^{n-i}\binom{n}{i} f(z+i)$, where n (≥ 2) is an integer.

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In 1935, Whittaker [13] proved that the difference equation $f(z+1) = \psi(z)f(z)$ admits a meromorphic solution of order $\sigma(f) \leq \sigma(\psi)+1$, where $\psi(z)$ is a finite order entire function. In the 1980s, some mathematicians (see, e.g. [1, 12, 14]) obtained more existence theorems about the meromorphic solutions of difference equations. At the beginning of the 21st century, Halburd and Korhonen [6] and Chiang and Feng [3] proved a difference analogue of the logarithmic derivative lemma independently, which provides an efficient tool to study the properties of complex difference equations. By using this new result, Chiang and Feng [3] investigated the growth of meromorphic solutions for higher order linear difference equation

$$\sum_{j=0}^{n} p_j(z)f(z+j) = 0, \tag{1}$$

where $p_j(z)$, $j = 0, \ldots, n$ $(n \geq 1)$ are entire functions or polynomials. They proved the following two theorems.

Theorem 1 (see [3]). *Let $p_0(z)$, …, $p_n(z)$ be polynomials such that there exists an integer l, $0 \leq l \leq n$, such that*

$$\deg(p_l) > \max_{0\leq j\leq n, j\neq l}\{\deg(p_j)\}.$$

If $f(z)$ is a meromorphic solution of (1), then $\sigma(f) \geq 1$.

Theorem 2 (see [3]). *Let $p_0(z)$, …, $p_n(z)$ be entire functions such that there exists an integer l, $0 \leq l \leq n$, such that*

$$\sigma(p_l) > \max_{0\leq j\leq n, j\neq l}\{\sigma(p_j)\}.$$

If $f(z)$ is a meromorphic solution of (1), then $\sigma(f) \geq \sigma(p_l)+1$.

Remark 1. Laine and Yang [10] completed the proof of Theorem 2 by showing that the conclusion of Theorem 2 still holds if there exists an integer l, $0 \leq l \leq n$ so that among those having the maximal order $\sigma = \max_{0\leq l\leq n}\sigma(p_l)$, exactly p_l has its type strictly greater than the others.

By proving Theorem 1 and Theorem 2, Chiang and Feng [3] have shown that Whittaker's conclusion $\sigma(f) \leq \sigma(\psi)+1$ can be replaced by $\sigma(f) = \sigma(\psi)+1$ (see [3, Corollary 9.3]). Some mathematicians(see, e.g., [2, 4, 8, 11, 16]) then made their efforts to improve Theorem 1 by weakening the conditions. We recall from [4] and [11] the following two results, where λ_f denotes $\max\{\lambda(f), \lambda(1/f)\}$ for simplicity.

Theorem 3 (see [4]). *Let $q_0(z)$, …, $q_n(z)$ be polynomials such that $q_0(z)q_n(z) \not\equiv 0$ and*

$$\deg(q_0) \geq \max_{1\leq j\leq n}\{\deg(q_j)\}.$$

If $f(z)$ is a transcendental meromorphic solution of the following difference equation

$$\sum_{j=0}^{n} q_j(z)\Delta^j f(z) = 0, \tag{2}$$

then $\sigma(f) \geq 1$.

Theorem 4 (see [11]). *Let $q(z)$, $p_0(z)$, …, $p_n(z)$ be polynomials such that $p_0(z)p_n(z) \not\equiv 0$ and*

$$\deg\left(\sum_{j=0}^{n} p_j(z)\right) = \max_{0\leq j\leq n}\{\deg(p_j)\} \geq 1.$$

If $f(z)$ is a transcendental meromorphic solution of the following difference equation

$$\sum_{j=0}^{n} p_j(z) f(z+j) = q(z), \tag{3}$$

then $\sigma(f) \geq 1$. Moreover, if $f(z)$ has finite order, then $1 \leq \sigma(f) \leq 1 + \lambda_f$.

Theorem 3 improves Theorem 1 because we can use the relation $g(z+l) = \sum_{j=0}^{l}\binom{l}{j}\Delta^j g(z)$, $l = 0, \ldots, n$ to rewrite (1) as the form of (2) and it follows that the only coefficient with the maximal degree in Theorem 1 implies $q_0(z)$ satisfies the condition of Theorem 3. Now we use the same relation to rewrite (3) as

$$\sum_{j=0}^{n} q_j(z)\Delta^j f(z) = q(z), \tag{4}$$

where $q(z)$, $q_0(z)$, …, $q_n(z)$ are polynomials such that $q_0(z)q_n(z) \not\equiv 0$. We study the growth of transcendental meromorphic solution $f(z)$ of (4) and give two conditions ensuring that $f(z)$ has order of growth no less than 1. We prove the following Theorem 5.

Theorem 5. *Let $q(z)$, $q_0(z)$, …, $q_n(z)$ be polynomials such that $q_0(z)q_n(z) \not\equiv 0$ and*

$$\deg(q_0) \geq \max_{1\leq j\leq n}\{\deg(q_j)\}, \tag{5}$$

or

$$\deg(q_1) \geq \max_{0\leq j\leq n, j\neq 1}\{\deg(q_j)\}. \tag{6}$$

If $f(z)$ is a transcendental meromorphic solution of (4)*, then $\sigma(f) \geq 1$.*

From the processing of rewriting (3) to (4), we easily see that $q_1(z)$ in (4) corresponds to $\sum_{j=0}^{n} jp_j(z)$, where p_j, $j = 0, \dots, n$ are the coefficients of (3). Therefore, we have the following corollary from Theorem 5.

Corollary 1. *Let $p_0(z), \dots, p_n(z)$ be polynomials such that $p_0(z)p_n(z) \not\equiv 0$ and*

$$\deg\left(\sum_{j=0}^{n} jp_j(z)\right) = d = \max_{0\leq j\leq n}\{\deg(p_j)\} \geq 1.$$

If $f(z)$ is a transcendental meromorphic solution of (3)*, then $\sigma(f) \geq 1$.*

Example 1. Ishizaki and Yanagihara [7] proved that the following linear difference equation

$$(6z^2 + 19z + 15)\Delta^3 f(z) + (z + 3)\Delta^2 f(z) - \Delta f(z) - f(z) = 0$$

admits an entire function with order 1/3. This example shows that none of the two conditions in Theorem 5 can be ignored.

In the rest of this paper, we give another result on the growth of transcendental meromorphic solution of (3) and present the form of $f(z)$ which has two Borel exceptional values in the case that two of the coefficients of (3) have the maximal degrees. We prove the following Theorem 6.

Theorem 6. *Let $q(z), p_0(z), \dots, p_n(z)$ be polynomials such that $p_0(z)p_n(z) \not\equiv 0$ and l and s $(0 \leq l, s \leq n)$ be two distinct integers such that p_l and p_s satisfy*

$$\deg(p_l) = \deg(p_s) > \max_{0\leq j\leq n, j\neq l,s}\{\deg(p_j)\}.$$

If $f(z)$ is a transcendental meromorphic solution of (3)*, then $\sigma(f) \geq 1$. Moreover, if $f(z)$ is of finite order and has two Borel exceptional values α $(\neq \infty)$ and β $(\neq \alpha)$, then we have*

(i) if $\beta = \infty$, then $q(z) - \alpha\sum_{j=0}^{n} p_j(z) \equiv 0$ and $f(z) = h(z)e^{az+b} + \alpha$;

(ii) if $\beta \neq \infty$, then $q(z) = \alpha\sum_{j=0}^{n} p_j(z) \equiv 0$ and $f(z) = \frac{\beta-\alpha}{1-h(z)e^{az+b}} + \alpha$,

where a $(\neq 0)$ and b are two constants and $h(z)$ is a nonzero rational function.

Example 2. If $f(z)$ is a period 1 function, that is, $h(z)$ is a nonzero constant and $a = 2ki\pi$, $k \in \mathbb{Z} \setminus \{0\}$, then it is easy to see that $f(z)$ always satisfies equation (3) when $q(z) \equiv 0$ and $\sum_{j=0}^{n} p_j(z) \equiv 0$. Therefore, both the two cases of Theorem 6 can occur. In the non-periodic case, for example, the function $f(z) = ze^{2i\pi z}$ has two Borel exceptional values 0 and ∞ and satisfies the following difference equation

$$(z+2)f(z+2) - (z+4)f(z+1) + f(z) = 0.$$

2. Some lemmas

Let $f(z) = \sum_{n=0}^{\infty} a_n z^n$ be an entire function, we denote the *maximum modulus* of $f(z)$ on $r > 0$ by $M(r,f) = \max_{|z|=r} |f(z)|$ and the *central index* of $f(z)$ by $\nu(r,f)$, which is defined as the greatest exponent of the maximal term of $f(z)$. The following Lemma 1 obtained recently can be regarded as a difference analogue of the classical Wiman-Valiron theory (see, e.g. [9]).

Lemma 1 (see [5]). *Let f be a transcendental entire function of order $\sigma(f) = \sigma < 1$, let $0 < \varepsilon < \min\{1/8, 1-\sigma\}$ and z be such that $|z| = r$, where*

$$|f(z)| > M(r,f)\nu(r,f)^{-1/8+\varepsilon}$$

holds. Then for each positive integer k, there exists a set $E \subset (1,\infty)$ that has finite logarithmic measure, such that for all $r \notin [0,1] \cup E$,

$$\frac{\Delta^k f(z)}{f(z)} = \left(\frac{\nu(r,f)}{z}\right)^k (1 + R_k(z)),$$

where $R_k(z) = O(\nu(r,f)^{-\kappa+\varepsilon})$ and $\kappa = \min\{1/8, 1-\sigma\}$.

Lemma 2 (see [9]). *If $f(z)$ is an entire function of order $\sigma(f) = \sigma$, then*

$$\sigma = \limsup_{r\to\infty} \frac{\log \nu(r,f)}{\log r}.$$

Lemma 3 (see [3]). *Let $f(z)$ be a meromorphic function with order $\sigma(f) = \sigma < \infty$, and let η be a fixed non-zero complex number. Then for each $\varepsilon > 0$, we have*

$$T(r, f(z+\eta)) = T(r,f) + O(r^{\sigma-1+\varepsilon}) + O(\log r).$$

Lemma 4 (see [9]). *Let $f(z)$ be a meromorphic function. Then for all irreducible rational functions in f,*

$$R(z,f) = \frac{P(z,f)}{Q(z,f)} = \frac{\sum_i^p a_i(z) f^i}{\sum_j^q b_j(z) f^j},$$

such that the meromorphic coefficients $a_i(z)$, $b_j(z)$ satisfy $T(r, a_i(z)) = S(r,f)$, $i = 0,1,\ldots,p$ and $T(r, b_i(z)) = S(r,f)$, $i = 0,1,\ldots,q$, we have

$$T(r, R(z,f)) = \max\{p,q\} T(r,f) + S(r,f).$$

3. Proof of Theorem 5

Proof. On the contrary, we suppose that $\sigma(f) = \sigma < 1$. From the proof of Theorem 4 in [11], we know that $f(z)$ has only finitely many poles. Therefore, there exists a rational function $S(z)$ such that $F(z) = f(z) - S(z)$ is transcendental entire. Substituting $f(z) = F(z) + S(z)$ into (4), we get

$$\sum_{j=0}^{n} q_j(z)\Delta^j F(z) = Q(z), \tag{7}$$

where $Q(z) = q(z) - \sum_{j=0}^{n} q_j(z)\Delta^j S(z)$ is a rational function. Since $F(z)$ is transcendental, we may choose an infinite sequence z_k such that $|z_k| = r_k$ and $|F(z_k)| = M(r_k, F)$. Let $0 < \varepsilon < \kappa = \min\{1/8, 1 - \sigma\}$. By Lemma 1, we have

$$\frac{\Delta^j F(z_k)}{F(z_k)} = \left(\frac{\nu(r_k, F)}{z_k}\right)^j (1 + o(1)) \tag{8}$$

holds for all $j = 1, \ldots, n$. Dividing $q_0(z)F(z)$ on both sides of (7) and substituting (8) into the resulting equation gives

$$\sum_{j=1}^{n} \frac{q_j(z_k)}{q_0(z_k)} \left(\frac{\nu(r_k, F)}{z_k}\right)^j (1 + o(1)) + 1 = \frac{Q(z_k)}{q_0(z_k)F(z_k)}. \tag{9}$$

By the condition (5) and the fact that $F(z)$ is transcendental, we have

$$\frac{Q(z_k)}{q_0(z_k)F(z_k)} = o(1), \quad \frac{q_j(z_k)}{q_0(z_k)} = O(1)$$

for $j = 1, \ldots, n$ as $|z_k| = r_k \to \infty$. Moreover, from Lemma 2, we know that $\frac{\nu(r_k,F)}{r_k} = o(1)$ as $|z_k| = r_k \to \infty$. Hence (9) is a contradiction when we let $|z_k| = r_k \to \infty$. This implies that $\sigma(f) \geq 1$ when equation (5) holds.

Consider the case that (6) holds. From the above reasoning we see that $\deg(q_0) < \deg(q_1)$ since we have assumed $\sigma(f) = \sigma < 1$. By dividing $q_1(z)F(z)$ on both sides of (7) and substituting (8) into the resulting equation, we get

$$\sum_{j=2}^{n} \frac{q_j(z_k)}{q_1(z_k)} \left(\frac{\nu(r_k, F)}{z_k}\right)^j (1 + o(1)) + \frac{\nu(r_k, F)}{z_k}(1 + o(1)) + \frac{q_0(z)}{q_1(z_k)} = \frac{Q(z_k)}{q_1(z_k)F(z_k)}, \tag{10}$$

From (6) and the fact that $F(z)$ is transcendental, we have

$$\frac{Q(z_k)}{q_1(z_k)F(z_k)} = o(1), \quad \frac{q_j(z_k)}{q_1(z_k)} = O(1), \quad \frac{q_0(z_k)}{q_1(z_k)} = o(1)$$

for $j = 2, \ldots, n$ as $|z_k| = r_k \to \infty$. Note that $\frac{\nu(r_k,F)}{r_k} = o(1)$ as $|z_k| = r_k \to \infty$ by Lemma 2. These results lead (10) to the following

$$\frac{\nu(r_k, F)}{r_k} \leq K \sum_{j=2}^{n} \left(\frac{\nu(r_k, F)}{r_k}\right)^j \leq nK \left(\frac{\nu(r_k, F)}{r_k}\right)^2,$$

where K is some positive value, which implies that $\sigma(f) = \sigma(F) \geq 1$ by Lemma 2, a contradiction to our assumption. So we must have $\sigma(f) \geq 1$ when equation (6) holds. This completes the proof. ■

4. Proof of Theorem 6

Proof. (i) We first prove that $\sigma(f) \geq 1$. Let a_l and a_s be, respectively, the leading coefficients of $p_l(z)$ and $p_s(z)$ with degree $d \geq 1$. If $\sigma(f) < 1$, then from Theorem 4 and Corollary 1, we know that $\deg(p_l(z) + p_s(z)) \leq d - 1$ and $\deg(lp_l(z) + sp_s(z)) \leq d - 1$, which implies that $a_l + a_s = 0$ and $la_l + sa_s = 0$. It follows that $a_l = a_s = 0$, which contradicts the fact that $p_l(z)$ and $p_s(z)$ both have the maximal degrees. Hence $\sigma(f) \geq 1$.

(ii) When $f(z)$ has two Borel exceptional values, we discuss the following two cases:

Case 1. $\beta = \infty$. By Hadamard's theory, $f(z)$ assumes the form: $f(z) = h(z)e^{g(z)} + \alpha$, where $g(z)$ is a polynomial with $\deg(g(z)) = \sigma(f) = k \geq 1$ and $h(z)$ satisfies $\lambda_h = \sigma(h) < \sigma(f) = k$. Substituting this equation into (3) and extracting $e^{g(z)}$ on the left-hand side of the resulting equation gives

$$e^{g(z)} \left(\sum_{j=0}^{n} p_j(z)H(z+j)\right) = q(z) - \alpha \sum_{j=0}^{n} p_j(z), \tag{11}$$

where $H(z+j) = h(z+j)e^{g(z+j)-g(z)}$, $j = 0, \ldots, n$. Denote $g(z) = b_k z^k + b_{k-1}z^{k-1} + \ldots + b_0$, where b_k ($\neq 0$), $\ldots$, b_0 are constants. Then we have

$$g(z+j) - g(z) = b_k kjz^{k-1} + g_j(z), \quad j = 1, \ldots, n,$$

where $g_j(z) \equiv 0$ when $k = 1$ or $g_j(z)$ are polynomials with degree $\deg(g_j(z)) \leq k - 2$ when $k \geq 2$. From Lemma 3, we know that $\sigma(H(z+j)) < k$ for $j = 0, \ldots, n$. If $q(z) - \alpha \sum_{j=0}^{n} p_j(z) \not\equiv 0$, then by Lemma 4, we get from (11) that $T(r, e^{g(z)}) = S(r, e^{g(z)})$, which is impossible. Therefore, $q(z) - \alpha \sum_{j=0}^{n} p_j(z) \equiv 0$ and it follows that

$$\sum_{j=0}^{n} p_j(z)e^{g(z+j)-g(z)}h(z+j) = 0. \tag{12}$$

If $k \geq 2$, then from the definition of the type $\tau(f)$ (see, e.g., [15]) for an entire function $f(z)$ with order $0 < \sigma(f) < \infty$, we easily get $\tau(p_j(z)e^{g(z+j)-g(z)}) = kj|b_k|$ for $j = 1, \ldots, n$. Obviously, $kn|b_k| > \ldots > k|b_k|$. However, from Remark 1 we know that $\sigma(h) \geq k$, a contradiction to that $\sigma(h) < k$. Hence $k = 1$ and so $\lambda_h < \sigma(f) = 1$. Note that now $p_l(z)$ and $p_s(z)$ still have the maximal degrees since $e^{g(z+j)-g(z)}$, $j = 1, \ldots, n$ are all nonzero constants. If $h(z)$ is transcendental, then from the first part, we get $\lambda_h = \sigma(h) \geq 1$, a contradiction again. So $h(z)$ must be rational and hence $f(z)$ assumes the form: $f(z) = h(z)e^{az+b} + \alpha$, where a $(\neq 0)$ and b are two constants and $h(z)$ is a rational function.

Case 2. $\beta \neq \infty$. In this case, $f(z)$ satisfies the equation

$$\frac{f(z)-\beta}{f(z)-\alpha} = h(z)e^{g(z)},$$

where $g(z)$ and $h(z)$ are defined as above. It follows that $f(z) = \frac{\beta-\alpha}{1-h(z)e^{g(z)}}$ $+ \alpha$ and substituting this equation into (3) yields

$$\sum_{j=0}^{n} \frac{p_j(z)}{1-h(z+j)e^{g(z+j)}} = \frac{q(z)-\alpha\sum_{j=0}^{n} p_j(z)}{\beta-\alpha}. \tag{13}$$

For simplicity, denote the polynomial on the right-hand side of (13) by $A(z)$. By multiplying $\prod_{j=0}^{n}(1-H(z+j)e^{g(z)})$ on both sides of (13), we get

$$B_1(z)e^{(n-1)g(z)}+\ldots+B_{n-1}(z)e^{g(z)}+B_n = A(z)\prod_{j=0}^{n}(1-H(z+j)e^{g(z)}), \tag{14}$$

where B_1, …, B_n are meromorphic functions with order less than k. Moreover,

$$B_{n-1}(z) = \sum_{j=0}^{n}\left[p_j(z)\left(H(z+j)-\sum_{i=0}^{n}H(z+i)\right)\right], \quad B_n(z) = \sum_{j=0}^{n} p_j(z).$$

If $A(z) \not\equiv 0$, then the right-hand side of (14) is a polynomial in $e^{g(z)}$ with coefficients of order less than k. By Lemma 4, we get from (14) that $nT(r, e^{g(z)}) \leq (n-1)T(r, e^{g(z)}) + S(r, e^{g(z)})$, which is impossible. Therefore, $A(z) \equiv 0$. Now we have

$$B_1(z)e^{(n-1)g(z)} + \ldots + B_{n-1}(z)e^{g(z)} + B_n = 0. \tag{15}$$

Since (15) is a polynomial in $e^{g(z)}$ of degree n with coefficients of order less than k, we conclude from Lemma 4 that $B_1(z) \equiv \ldots \equiv B_n(z) \equiv 0$. In particular,

$$\begin{aligned} B_{n-1}(z) &= \sum_{j=0}^{n} p_j(z)H(z+j) - \left(\sum_{j=0}^{n} p_j(z)\right)\left(\sum_{i=0}^{n} H(z+i)\right) \\ &= \sum_{i=0}^{n} p_j(z)H(z+j) \equiv 0, \end{aligned}$$

which is the equation (12) since $H(z+j) = h(z+j)e^{g(z+j)-g(z)}$, $j = 0, \ldots, n$. This implies that $\sigma(f) = 1$ and $h(z)$ is a rational function and so $f(z)$ assumes the form: $f(z) = \frac{\beta-\alpha}{1-h(z)e^{az+b}} + \alpha$, where a ($\neq 0$) and b are two constants and $h(z)$ is a rational function. This completes the proof. ■

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