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POINTWISE CONVERGENCE OF FOURIER-LAGUERRE SERIES OF INTEGRABLE FUNCTIONS

Abstract. We extend and improve the some results of Xh. Z. Krasniqi [Int. J. of Anal. and Appl. Vol. 1, 33-39 (2013)], M. L. Mittal and M. V. Singh [Operators, Int. J. of Analysis, Vol. 2015, Article ID 478345, 4 pages] and from many other papers on summability of Fourier-Laguerre series to strong summability proving the estimate of the deviation of the partial sums from considered functions. There also is a remark on summability methods used in cited papers.

Key words: rate of approximation, summability of Fourier -Laguerre series.

AMS Mathematics Subject Classification: 42A24.

1. Introduction

Let L be the class of all real–valued functions, integrable in the Lebesgue sense over $\mathbb{R}^+$ with the norm

$$\|f\| = \|f(\cdot)\| = \int_{\mathbb{R}^+} | f(t) | \, dt$$

and consider the Fourier-Laguerre series

$$S^{(\alpha)} f(x) := \sum_{\nu=0}^{\infty} a_\nu^{(\alpha)}(f) L_\nu^{(\alpha)}(x), \text{ with } \alpha > -1,$$

where

$$L_n^{(\alpha)}(x) = \frac{x^{-\alpha} e^x}{n!} \frac{d^n}{dx^n} \left(x^{n+\alpha} e^{-x} \right) = \sum_{\nu=0}^{n} \frac{(-1)^\nu}{\nu!} \binom{n+\alpha}{n-\nu} x^\nu$$

and

$$a_{\nu}^{(\alpha)}(f)=\frac{1}{\Gamma(\alpha+1)\binom{n+\alpha}{n}}\int_{0}^{\infty}e^{-y}y^{\alpha}L_{\nu}^{(\alpha)}(y)\,f(y)\,dy.$$

Let $A:=(a_{n,k})$ and $B:=(b_{n,k})$ be infinite lower triangular matrices of real numbers such that

$$\begin{aligned}a_{n,k}&\geq 0 \quad\text{and}\quad b_{n,k}\geq 0 \quad\text{when}\quad k=0,1,2,...n\\ a_{n,k}&=0 \quad\text{and}\quad b_{n,k}=0 \quad\text{when}\quad k>n,\end{aligned}$$

$$\sum_{k=0}^{n}a_{n,k}=1 \quad\text{and}\quad \sum_{k=0}^{n}b_{n,k}=1, \quad\text{where}\quad n=0,1,2,....$$

Let define the general linear operator by the $AB-$transformation of partial sums

$$S_{n}^{(\alpha)}f(x)=\sum_{\nu=0}^{n}a_{\nu}^{(\alpha)}(f)L_{\nu}^{(\alpha)}(x)$$

as follows

$$T_{n,A,B}^{(\alpha)}f(x):=\sum_{r=0}^{n}\sum_{k=0}^{r}a_{n,r}b_{r,k}S_{k}^{(\alpha)}f(x)$$

for $n=0,1,2,\ldots$.

The deviation $T_{n,A,B}^{(\alpha)}f(0)-f(0)$ was estimated in the papers [2] and [3] as follows:

Theorem. *Let $f\in L$, $\delta>0$, $\alpha\in\left(-1,-\frac{1}{2}\right)$ and ω be a positive increasing function such that $\omega(n)\to\infty$ as $n\to\infty$, and satisfy the conditions*

$$\frac{u^{-(\alpha+1)}}{\Gamma(\alpha+1)}\int_{0}^{u}e^{-t}t^{\alpha}\left|\Delta_{0}f(t)\right|dt=o\left(\omega\left(\frac{1}{u}\right)\right) \tag{1}$$

as $u\to 0$,

$$\frac{n^{(2\alpha+1)/4}}{\Gamma(\alpha+1)}\int_{\delta}^{n}e^{-\frac{t}{2}}t^{\frac{2\alpha-3}{4}}\left|\Delta_{0}f(t)\right|dt=o(\omega(n)) \tag{2}$$

as $n\to\infty$ and

$$\frac{1}{\Gamma(\alpha+1)}\int_{n}^{\infty}e^{-\frac{t}{2}}t^{\alpha-\frac{1}{3}}\left|\Delta_{0}f(t)\right|dt=o(\omega(n)) \tag{3}$$

as $n\to\infty$, where $\Delta_{0}f(t)=f(t)-f(0)$. If matrices A and B are such that for $q>0$

$$\begin{aligned}a_{n,k}&\geq 0 \quad\textit{and}\quad b_{n,k}=\frac{\binom{n}{k}q^{k}}{(1+q)^{n}} \quad\textit{when}\quad 0\leq k\leq n,\\ a_{n,k}&=0 \quad\textit{and}\quad b_{n,k}=0 \quad\textit{when}\quad k>n,\end{aligned}$$

in [3] or in special case

$$\begin{aligned} a_{n,k} &= \frac{1}{n+1} \quad \textit{and} \quad b_{n,k} = \frac{\binom{n}{k} q^k}{(1+q)^n} \quad \textit{when} \quad 0 \le k \le n, \\ a_{n,k} &= 0 \quad \textit{and} \quad b_{n,k} = 0 \quad \textit{when} \quad k > n, \end{aligned}$$

in [2], then

$$\left| T^{(\alpha)}_{n,A,B}(0) - f(0) \right| = o(\omega(n)).$$

In this paper, we will study the upper bound of the quantity $\left| S^{(\alpha)}_k f(0) - f(0) \right|$ by a positive function ω such that: $\omega(n) \to \infty$ for $n \to \infty$. The following strong means

$$H^{s,\alpha}_{n,A,B} f(x) := \left\{ \sum_{r=0}^{n} \sum_{k=0}^{r} a_{n,r} b_{r,k} \left| S^{(\alpha)}_k f(x) - f(x) \right|^s \right\}^{1/s},$$

for $n = 0, 1, 2, ...$ and $s > 0$ generated by wide family of matrices A and B will also be considered.

From our generalizations we derive some corollaries. Finally we also prove a remark which fulfile the gap in the proofs of mentioned Theorem as well in cited papers [1], [4] and [5].

2. Statement of the results

At the beginning we will present the estimate of the quantity $\left| S^{(\alpha)}_n f(0) - f(0) \right|$. Finally, we will formulate some corollaries and remark.

Theorem 1. *Let $f \in L$, $\delta > 0$, $\alpha \in \left(-1, -\frac{1}{2}\right)$ and ω be a positive nondecreasing function such that $\omega(n) \to \infty$ as $n \to \infty$. If ω satisfies the conditions (1), (2), (3), then*

$$\left| S^{(\alpha)}_n f(0) - f(0) \right| = o(\omega(n)) \quad \textit{as} \quad n \to \infty.$$

Theorem 2. *Let $f \in L$, $\alpha \in \left(-1, -\frac{1}{2}\right)$ and ω be a positive function such that $\omega(n) \to \infty$ as $n \to \infty$. If ω satisfies the conditions (1), (3) and*

$$\text{(4)} \qquad \frac{n^{(2\alpha+1)/4}}{\Gamma(\alpha+1)} \int_{1/n}^{n} e^{-\frac{t}{2}} t^{\frac{2\alpha-3}{4}} \left| \Delta_0 f(t) \right| dt = o(\omega(n))$$

as $n \to \infty$, then

$$\left| S^{(\alpha)}_n f(0) - f(0) \right| = o(\omega(n)) \quad \textit{as} \quad n \to \infty.$$

Corollary 1. *We can observe that the matrices A and B considered by Xh. Z. Krasniqi or M. L. Mittal and M. V. Singh in Theorem can be changed by any infinite lower triangular matrices with nonnegative entries and since, for* $s \geq 1$,

$$\begin{aligned}\left|T_{n,A,B}^{(\alpha)} f(0) - f(0)\right| &\leq H_{n,A,B}^{s,\alpha} f(0) \\ &\leq \max_{0\leq\nu\leq n} \left|S_{\nu}^{(\alpha)} f(0) - f(0)\right| = o(\omega(n))\end{aligned}$$

Theorem 1 reduces to the results from [2], [3] and many other papers.

Corollary 2. *Under the assumption of Theorem 2 we have the relation*

$$H_{n,A,B}^{s,\alpha} f(0) = o(1) \left\{ \sum_{r=0}^{n} \sum_{k=0}^{r} a_{n,r} b_{r,k} [\omega(k)]^s \right\}^{1/s}$$

for $s > 0$ *and for not necessary monotonic function* ω.

Remark 1. We note that in the proofs of the Theorem cited above from [2], [3] and theorems from many other papers (see e.g. [1], [4], [5]) there is used the following property

$$\sum_{s=0}^{r} c_{r,s} (s+1)^{\beta} = O\left((r+1)^{\beta}\right),$$

with $\beta > 0$, but it should be used for $\beta > -1$. Our Lemma 3 shows that this property also holds when $\beta > -1$ for sequences $(c_{r,s})$ generating the Euler or Cesàro methods.

3. Auxiliary results

We begin this section by some notations from [6]. We have.

$$L_k^{(\alpha+1)}(y) = \sum_{\nu=0}^{k} L_{\nu}^{(\alpha)}(y), \quad L_{\nu}^{(\alpha)}(0) = \binom{\nu+\alpha}{\nu}$$

and therefore

$$S_k^{(\alpha)} f(0) = \frac{1}{\Gamma(\alpha+1)} \int_0^{\infty} e^{-y} y^{\alpha} L_{\nu}^{(\alpha+1)}(y) f(y)\, dy.$$

Hence, by evidence equality

$$\frac{1}{\Gamma(\alpha+1)} \int_0^{\infty} e^{-y} y^{\alpha} L_{\nu}^{(\alpha+1)}(y)\, dy = \begin{cases} 1 & \text{if} \quad \nu = 0, \\ 0 & \text{if} \quad \nu \neq 0, \end{cases}$$

we have

$$S_k^{(\alpha)} f(0) - f(0) = \frac{1}{\Gamma(\alpha+1)} \int_0^\infty e^{-y} y^\alpha L_\nu^{(\alpha+1)}(y)\, \Delta_0 f(y)\, dy.$$

Next, we present the known estimates:

Lemma 1 ([6], p. 172)**.** *Let α be an arbitrary real number, c and δ be fixed positive constants. Then*

$$\left| L_n^{(\alpha)}(x) \right| = \begin{cases} O(n^\alpha) & \text{if } 0 \le x \le \frac{c}{n}, \\ O\left(x^{-(2\alpha+1)/4} n^{(2\alpha-1)/4}\right) & \text{if } \frac{c}{n} \le x \le \delta. \end{cases}$$

Lemma 2 ([6], p. 235)**.** *Let α and λ be arbitrary real numbers, $\delta > 0$ and $0 < \eta < 4$. Then*

$$\max_x e^{-x/2} x^\lambda \left| L_n^{(\alpha)}(x) \right| = \begin{cases} O\left(n^{\max\left(\lambda - \frac{1}{2}, \frac{\alpha}{2} - \frac{1}{4}\right)}\right) & \text{if } \delta \le x \le (4-\eta)\, n, \\ O\left(n^{\max\left(\lambda - \frac{1}{3}, \frac{\alpha}{2} - \frac{1}{4}\right)}\right) & \text{if } x \ge \delta. \end{cases}$$

We will need additionally the following estimates:

Lemma 3. *Let $\beta > -1$. If $q > 0$, then*

$$\frac{1}{(1+q)^n} \sum_{k=0}^{n} \binom{n}{k} q^k (1+k)^\beta \le \left(1 + \frac{1}{q}\right)(1+n)^\beta$$

and if $\gamma > -1$, then

$$\frac{1}{A_n^{(\gamma)}} \sum_{k=0}^{n} A_{n-k}^{(\gamma-1)} (1+k)^\beta = O\left((1+n)^\beta\right).$$

Proof. Since

$$\begin{aligned} \frac{(1+q)^{n+1}}{n+1} &= \int_{-1}^{q} (1+z)^n\, dz \ge \int_0^q (1+z)^n\, dz \\ &= \int_0^q \sum_{k=0}^{n} \binom{n}{k} z^k dz = \sum_{k=0}^{n} \binom{n}{k} \frac{q^{k+1}}{k+1}, \end{aligned}$$

therefore

$$\begin{aligned} \frac{1}{(1+q)^n} \sum_{k=0}^{n} \binom{n}{k} q^k (1+k)^\beta &= \frac{1}{(1+q)^n} \sum_{k=0}^{n} \binom{n}{k} q^k \frac{(1+k)^{\beta+1}}{1+k} \\ &\le \frac{(1+n)^{\beta+1}}{q(1+q)^n} \sum_{k=0}^{n} \binom{n}{k} q^{k+1} \frac{1}{1+k} \\ &\le \frac{(1+n)^{\beta+1}}{q(1+q)^n} \frac{(1+q)^{n+1}}{n+1} = \left(1 + \frac{1}{q}\right)(1+n)^\beta \end{aligned}$$

and our first result is evident.

For the second one we know follow by A. Zygmund [7, Vol. I, (1.15) and Theorem 1.17] that

$$A_n^{(\gamma)} = \binom{n+\gamma}{n} \simeq O\left((n+1)^{\gamma}\right)$$

is positive for $\gamma > -1$. Moreover, $A_n^{(\gamma)}$ is increasing (as a function of n) for $\gamma > 0$ and decreasing for $-1 < \gamma < 0$. Hence, for $\beta < 0$,

$$\begin{aligned}
&\frac{1}{A_n^{(\gamma)}} \sum_{k=0}^{n} A_{n-k}^{(\gamma-1)} (1+k)^{\beta} \\
&= \frac{1}{A_n^{(\gamma)}} \sum_{k=0}^{[n/2]-1} A_{n-k}^{(\gamma-1)} (1+k)^{\beta} + \frac{1}{A_n^{(\gamma)}} \sum_{k=[n/2]}^{n} A_{n-k}^{(\gamma-1)} (1+k)^{\beta} \\
&= O\left(\frac{(n+1)^{\gamma-1}}{(n+1)^{\gamma}}\right) \sum_{k=0}^{[n/2]-1} (1+k)^{\beta} + O\left((1+n)^{\beta}\right) \frac{1}{A_n^{(\gamma)}} \sum_{k=[n/2]}^{n} A_{n-k}^{(\gamma-1)} \\
&\leq O\left((n+1)^{-1}\right) \sum_{k=0}^{n} (1+k)^{\beta} \int_k^{k+1} dz + O\left((1+n)^{\beta}\right) \frac{1}{A_n^{(\gamma)}} \sum_{k=0}^{n} A_{n-k}^{(\gamma-1)} \\
&\leq O\left((n+1)^{-1}\right) \sum_{k=0}^{n} \int_k^{k+1} z^{\beta} dz + O\left((1+n)^{\beta}\right) \\
&= O\left((n+1)^{-1}\right) \int_0^{n+1} z^{\beta} dz + O\left((1+n)^{\beta}\right) \\
&= O\left((n+1)^{-1}\right) \frac{(n+1)^{\beta+1}}{\beta+1} + O\left((1+n)^{\beta}\right) = O\left((1+n)^{\beta}\right).
\end{aligned}$$

If $\beta \geq 0$, then the result is evident. Thus our proof is complete. ■

4. Proofs of theorems

Proof of Theorem 1. It is clear that

$$\begin{aligned}
S_n^{(\alpha)} f(0) - f(0) &= \frac{1}{\Gamma(\alpha+1)} \int_0^{\infty} e^{-y} y^{\alpha} L_n^{(\alpha+1)}(y)\, \Delta_0 f(y)\, dy \\
&= \left(\int_0^{1/n} + \int_{1/n}^{\delta} + \int_{\delta}^{n} + \int_n^{\infty}\right) = J_1 + J_2 + J_3 + J_4,
\end{aligned}$$

then

$$\left|S_n^{(\alpha)} f(0) - f(0)\right| \leq |J_1| + |J_2| + |J_3| + |J_4|$$

and by Lemma 1 and (1)

$$|J_1| = \frac{O\left(n^{\alpha+1}\right)}{\Gamma(\alpha+1)} \int_0^{1/n} e^{-y} y^{\alpha} |\Delta_0 f(y)| \, dy = o(\omega(n)).$$

Next, by Lemma 1 and integrating by parts with $\alpha \in \left(-1, -\frac{1}{2}\right)$, we obtain

$$\begin{aligned}
|J_2| &\leq \frac{1}{\Gamma(\alpha+1)} \int_{1/n}^{\delta} e^{-y} y^{\alpha} |\Delta_0 f(y)| \left| L_n^{(\alpha+1)}(y) \right| dy \\
&= \frac{O\left(n^{(2\alpha+1)/4}\right)}{\Gamma(\alpha+1)} \int_{1/n}^{\delta} y^{-(2\alpha+3)/4} \frac{d}{dy} \left(\int_0^y e^{-t} t^{\alpha} |\Delta_0 f(t)| \, dt \right) dy \\
&= \frac{O\left(n^{(2\alpha+1)/4}\right)}{\Gamma(\alpha+1)} \left\{ \left[y^{-(2\alpha+3)/4} \left(\int_0^y e^{-t} t^{\alpha} |\Delta_0 f(t)| \, dt \right) \right]_{1/n}^{\delta} \right. \\
&\quad \left. + \int_{1/n}^{\delta} \frac{2\alpha+3}{4} y^{-(2\alpha+7)/4} \left(\int_0^y e^{-t} t^{\alpha} |\Delta_0 f(t)| \, dt \right) dy \right\} \\
&= \frac{O\left(n^{(2\alpha+1)/4}\right)}{\Gamma(\alpha+1)} \left\{ \delta^{-(2\alpha+3)/4} \left(\int_0^{\delta} e^{-t} t^{\alpha} |\Delta_0 f(t)| \, dt \right) \right. \\
&\quad - n^{(2\alpha+3)/4} \left(\int_0^{1/n} e^{-t} t^{\alpha} |\Delta_0 f(t)| \, dt \right) \\
&\quad \left. + \int_{1/n}^{\delta} \frac{2\alpha+3}{4} y^{-(2\alpha+7)/4} \left(\int_0^y e^{-t} t^{\alpha} |\Delta_0 f(t)| \, dt \right) dy \right\} \\
&= \frac{O\left(n^{(2\alpha+1)/4}\right)}{\Gamma(\alpha+1)} \left\{ \delta^{-(2\alpha+3)/4} \left(\int_0^{\delta} e^{-t} t^{\alpha} |\Delta_0 f(t)| \, dt \right) \right. \\
&\quad \left. + \int_{1/n}^{\delta} \frac{2\alpha+3}{4} y^{-(2\alpha+7)/4} \left(\int_0^y e^{-t} t^{\alpha} |\Delta_0 f(t)| \, dt \right) dy \right\}.
\end{aligned}$$

Using (1) and the monotonicity of ω we get

$$\begin{aligned}
|J_2| &\leq O\left(n^{(2\alpha+1)/4}\right) \left\{ \delta^{(2\alpha+1)/4} o\left(\omega\left(\frac{1}{\delta} \right) \right) \right. \\
&\quad \left. + \int_{1/n}^{\delta} \left(\frac{2\alpha+3}{4} y^{-(2\alpha+7)/4} \right) y^{\alpha+1} o\left(\omega\left(\frac{1}{y} \right) \right) dy \right\} \\
&= o\left(n^{(2\alpha+1)/4} \omega(n)\right) \left\{ \delta^{(2\alpha+1)/4} + \frac{2\alpha+3}{4} \int_{1/n}^{\delta} y^{(2\alpha-3)/4} dy \right\} \\
&= o\left(n^{(2\alpha+1)/4} \omega(n)\right) \left\{ \delta^{(2\alpha+1)/4} + \frac{(2\alpha+3)/4}{(2\alpha+1)/4} \left[y^{(2\alpha+1)/4} \right]_{1/n}^{\delta} \right\}
\end{aligned}$$

$$
\begin{aligned}
&= o\left(n^{(2\alpha+1)/4}\omega(n)\right)\left\{\delta^{(2\alpha+1)/4}+\frac{2\alpha+3}{2\alpha+1}\delta^{(2\alpha+1)/4}\right.\\
&\qquad\left.-\frac{2\alpha+3}{2\alpha+1}n^{-(2\alpha+1)/4}\right\}\\
&= o\left(n^{(2\alpha+1)/4}\omega(n)\right)\left\{\frac{4\alpha+4}{2\alpha+1}\delta^{(2\alpha+1)/4}-\frac{2\alpha+3}{2\alpha+1}n^{-(2\alpha+1)/4}\right\}\\
&\le o\left(n^{(2\alpha+1)/4}\omega(n)\right)\left\{-\frac{2\alpha+3}{2\alpha+1}n^{-(2\alpha+1)/4}\right\}\le o(\omega(n)).
\end{aligned}
$$

Applying Lemma 2 with $\alpha+1$ instead of $\alpha, \lambda=\frac{2\alpha-3}{4}$ (since $\max\left\{\lambda-\frac{1}{2}, \frac{\alpha+1}{2}-\frac{1}{4}\right\}=\frac{2\alpha+1}{4}$) and (2) we obtain

$$
\begin{aligned}
|J_3| &\le \frac{1}{\Gamma(\alpha+1)}\int_\delta^n e^{-y/2}y^{(2\alpha-3)/4}\left|\Delta_0 f(y)\right| e^{-y/2}y^{(2\alpha+3)/4}\left|L_n^{(\alpha+1)}(y)\right|dy\\
&= \frac{O\left(n^{(2\alpha+1)/4}\right)}{\Gamma(\alpha+1)}\int_\delta^n e^{-y/2}y^{(2\alpha-3)/4}\left|\Delta_0 f(y)\right|dy = o(\omega(n)).
\end{aligned}
$$

Further, by Lemma 2 with $\alpha+1$ instead of α and $\lambda=\frac{1}{3}$ (since $\max\left\{\lambda-\frac{1}{3}, \frac{\alpha+1}{2}-\frac{1}{4}\right\}=\lambda-\frac{1}{3}=0$) and (3) we get

$$
\begin{aligned}
|J_4| &\le \frac{1}{\Gamma(\alpha+1)}\int_n^\infty e^{-y/2}y^{(3\alpha-1)/3}\left|\Delta_0 f(y)\right| e^{-y/2}y^{1/3}\left|L_n^{(\alpha+1)}(y)\right|dy\\
&= \frac{O(1)}{\Gamma(\alpha+1)}\int_n^\infty e^{-y/2}y^{(3\alpha-1)/3}\left|\Delta_0 f(y)\right|dy = o(\omega(n)).
\end{aligned}
$$

Finally, collecting the above estimates we have

$$
\left|S_n^{(\alpha)}f(0)-f(0)\right| = o(\omega(n))
$$

and thus our proof is complete. ■

Proof of Theorem 2. Let $\delta>0$, and as above

$$
S_n^{(\alpha)}f(0)-f(0)=J_1+J_2+J_3+J_4.
$$

For the proof we note that taking $\delta=1/n$, we have $J_2=0$ and by the condition (4) we obtain

$$
|J_3|\le o(\omega(n)).
$$

Moreover, the conditions (1) and (3) imply

$$
|J_1|\le o(\omega(n)) \quad \text{and} \quad |J_4|\le o(\omega(n)),
$$

similarly as above, and thus our proof is complete. ■

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