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NORMAL FAMILIES AND SHARED FUNCTIONS

Abstract. Let $k \in \mathbb{N}, m \in \mathbb{N} \cup \{0\}$, and let $a(z)(\not\equiv 0)$ be a holomorphic function, all zeros of $a(z)$ have multiplicities at most m. Let $\mathcal{F}$ be a family of meromorphic functions in D. If for each $f \in \mathcal{F}$, the zeros of f have multiplicities at least $k+m+1$ and all poles of f are of multiplicity at least $m+1$, and for $f, g \in \mathcal{F}$, $ff^{(k)} - a(z)$ and $gg^{(k)} - a(z)$ share 0, then $\mathcal{F}$ is normal in D. Some examples are given to show that the conditions are best, and the result removes the condition "m is an even integer" in the result due to Sun [Kragujevac Journal of Math 38(2), 173-282, 2014].

Key words: meromorphic function, normal criterion, Shared function.

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1. Introduction and main results

Let $D \subset \mathbb{C}$ be a domain, and let $\mathcal{F}$ be a family of meromorphic functions defined in D. Then $\mathcal{F}$ is said to be normal in D, if for every sequence $f_n \in \mathcal{F}$ there exists a subsequence f_{n_k} converges spherically locally uniformly to a meromorphic function or ∞.

Let f and g be two meromorphic functions in D, and let $\phi(z)$ be a function. If the functions $f(z) - \phi(z)$ and $g(z) - \phi(z)$ have the same zeros (ignoring multiplicity) in D, then we say that f and g share a function $\phi(z)$ IM.

Chen and Fang [1] proved the following theorem.

Theorem A. *If f is a transcendental meromorphic function, then ff' takes any non-zero finite complex number infinitely times.*

Lu and Gu [4] considered the general order derivative in Theorem A. They proved the following result.

Theorem B. *Let $k \in \mathbb{N}$. If f is a transcendental meromorphic function, all of whose zeros have multiplicity $k+2$ at least, then $ff^{(k)}$ takes any non-zero finite complex number infinitely times.*

Theorem C. *Let* $k \in \mathbb{N}$, $a \in \mathbb{C}\backslash\{0\}$ *and let* $\mathcal{F}$ *be a family of meromorphic function in* D. *If* $ff^{(k)} \neq a$ *for each function* $f \in \mathcal{F}$, *and if the zeros of* f *have multiplicities at least* $k+2$, *then* $\mathcal{F}$ *is normal in* D.

This result has undergone various improvements in [8], [5], [6], [9], Meng and Hu proved the following result.

Theorem D. *Let* $k \in \mathbb{N}$, $a \in \mathbb{C} \setminus \{0\}$. *Let* $\mathcal{F}$ *be a family of meromorphic functions in* D. *If for each* $f \in \mathcal{F}$, *the zeros of* f *have multiplicities at least* $k+1$, *and for* $f, g \in \mathcal{F}$, $ff^{(k)} - a$ *and* $gg^{(k)} - a$ *share* 0, *then* $\mathcal{F}$ *is normal in* D.

Recently, Sun [6] considered the case of sharing a holomorphic function and obtained the following theorem.

Theorem E. *Let* $k \in \mathbb{N}$, m *is an even integer, and let* $a(z)(\equiv 0)$ *be a holomorphic function, all zeros of* $a(z)$ *have multiplicities at most* m. *Let* $\mathcal{F}$ *be a family of meromorphic functions in* D. *If for each* $f \in \mathcal{F}$, *the zeros of* f *have multiplicities at least* $k+m+1$ *and all poles of* f *are of multiplicity at least* $m+1$, *and for* $f, g \in \mathcal{F}$, $ff^{(k)} - a(z)$ *and* $gg^{(k)} - a(z)$ *share* 0, *then* $\mathcal{F}$ *is normal in* D.

The following problem was posed by the author in [6].

What happens to Theorem E if the condition "m is an even integer" is removed.

In this paper, we answer this question and prove the following theorems.

Theorem 1. *Let* $k \in \mathbb{N}$, $m \in \mathbb{N}\cup\{0\}$, *and let* $a(z)(\not\equiv 0)$ *be a holomorphic function, all zeros of* $a(z)$ *have multiplicities at most* m. *Let* $\mathcal{F}$ *be a family of meromorphic functions in* D. *If for each* $f \in \mathcal{F}$, *the zeros of* f *have multiplicities at least* $k+m+1$ *and all poles of* f *are of multiplicity at least* $m+1$, *and for* $f, g \in \mathcal{F}$, $ff^{(k)} - a(z)$ *and* $gg^{(k)} - a(z)$ *share* 0, *then* $\mathcal{F}$ *is normal in* D.

Theorem 2. *Let* $k \in \mathbb{N}$, $m \in \mathbb{N}\cup\{0\}$, *and let* $a(z)(\not\equiv 0)$ *be a holomorphic function, all zeros of* $a(z)$ *have multiplicities at most* m. *Let* $\mathcal{F}$ *be a family of meromorphic functions in* D. *If for each* $f \in \mathcal{F}$, *the zeros of* f *have multiplicities at least* $k+m+1$ *and all poles of* f *are of multiplicity at least* $m+1$, *and for* $f \in \mathcal{F}$, $ff^{(k)} - a(z)$ *has at most one zero in* D, *then* $\mathcal{F}$ *is normal in* D.

Example 1. Let $D = \{z : |z| < 1\}$ and $a(z) \equiv 0$. Let $\mathcal{F} = \{f_n(z)\}$ where

$$f_n(z) = e^{nz}, \quad z \in D, \quad n = 1, 2 \cdots .$$

Then $f_n f_n^{(k)} - a(z)$ does not have zero in D for each positive integer n, however $\mathcal{F}$ is not normal at $z = 0$. This shows that $a(z) \not\equiv 0$ is necessary in Theorem 1-2.

Example 2. Let $D = \{z : |z| < 1\}$ and $a(z) = \frac{1}{z^{k+2}}$. Let $\mathcal{F} = \{f_n(z)\}$ where

$$f_n(z) = \frac{1}{nz}, \quad z \in D, \quad n = 1, 2 \cdots, \quad n^2 \neq (-1)^k k!.$$

Then $f_n f_n^{(k)} - a(z)$ does not have zero in D for each positive integer n, however $\mathcal{F}$ is not normal at $z = 0$. This shows that Theorem 1-2 are not valid if $a(z)$ is a meromorphic function in D.

Example 3. Let $D = \{z : |z| < 1\}$, $a(z) = 1$. Let $\mathcal{F} = \{f_n(z)\}$ where

$$f_n(z) = nz - \frac{n}{4} + \frac{1}{n}, \quad z \in D, \quad n = 1, 2 \cdots.$$

Then

$$f_n f_n^{'} - a(z) = f_n f_n^{'} - 1 = n^2 z - \frac{n^2}{4},$$

which has exactly one zero in D for each positive integer n, however $\mathcal{F}$ is not normal at $z = \frac{1}{4}$. This shows that the condition "all zeros of f have multiplicity at least $k + m + 1$" in Theorem 1-2 is necessary.

2. Some lemmas

Let us set some notations. we use $\longrightarrow$ to stand for convergence, $\Rightarrow$ to stand for spherical local uniform convergence in $D \subset \mathbb{C}$.

To prove our Theorems, we need the following lemmas.

Lemma 1 ([7]). *Let $\mathcal{F}$ be a family of functions meromorphic in the unit disk Δ such that all zeros of functions in $\mathcal{F}$ have multiplicity $\geq q$. Let α be a real number satisfying $-q < \alpha < 1$. Then $\mathcal{F}$ is not normal in any neighborhood of $z_0 \in \Delta$ if and only if there exist*

(a) points z_n, $z_n \to z_0$, $z_0 \in \Delta$,

(b) functions $f_n \in \mathcal{F}$, and

(c) positive numbers $\rho_n \to 0$

such that $\rho_n^\alpha f_n(z_n + \rho_n \xi) = g_n(\xi) \Rightarrow g(\xi)$ spherically uniformly on compact subsets of $\mathbb{C}$, where $g(\xi)$ is a non-constant meromorphic function in $\mathbb{C}$ satisfying that all zeros of $g(\xi)$ have multiplicity at least q.

Lemma 2 ([9]). *Let $k \in \mathbb{N}$, $a \in \mathbb{C} \setminus \{0\}$, and let $f(z)$ be a non-constant meromorphic in $\mathbb{C}$ with all zeros that have multiplicity at least $k + 1$. Then $f(z)f^{(k)}(z) - a$ have at least two distinct zeros.*

Lemma 3. *Let $k, m \in \mathbb{N}$, let $p(z)$ be a polynomial with $deg(p) = m$, and let $f(z)$ be a non-constant rational function in $\mathbb{C}$ with $f(z) \neq 0$. Then $f(z)f^{(k)}(z) - p(z)$ has at least $k+2$ distinct zeros.*

The proof of Lemma 3 is almost the same with Chang [2] and Lemma 11 in Deng etc. [3], we omit the detail.

Lemma 4 ([6]). *Let $k, m \in \mathbb{N}$, let $p(z)$ be a polynomial with $deg(p) = m$, and let $f(z)$ be a non-constant meromorphic in $\mathbb{C}$, the zeros of f have multiplicities at least $k+m+1$ and all poles of f are of multiplicity at least $m+1$. Then $f(z)f^{(k)}(z) - p(z)$ has at least two distinct zeros.*

Lemma 5. *Let $k \in \mathbb{N}$, and let $\{f_n\}$ be a sequence of meromorphic functions in D, $g_n(z)$ be a sequence of holomorphic functions in D such that $g_n(z) \Rightarrow g(z)$, where $g(z)(\neq 0)$ be a holomorphic function. If, for each $n \in \mathbb{N}$, all zeros of function $f_n(z)$ have multiplicity at least $k+1$, and $f_n(z)f_n^{(k)}(z) - g_n(z)$ has at most one zero in D, then $\{f_n\}$ is normal in D.*

Proof. Suppose that $\{f_n\}$ is not normal at $z_0 \in D$. By Lemma 1, there exists a sequence z_n of complex numbers $z_n \to z_0$, a sequence ρ_n of positive numbers $\rho_n \to 0$, and a subsequence of $\{f_n\}$ (we may still denote by $\{f_n\}$)such that

$$h_n(\xi) = \frac{f_n(z_n + \rho_n \xi)}{\rho_n^{\frac{k}{2}}} \Rightarrow h(\xi)$$

locally uniformly on compact subsets of $\mathbb{C}$, where $h(\xi)$ is a non-constant meromorphic function in $\mathbb{C}$. By Hurwitz's theorem, all zeros of $h(\xi)$ have multiplicity at least $k+1$. Then

$$\begin{aligned} h_n(\xi)h_n^{(k)}(\xi) - g_n(z_n + \rho_n\xi) \\ &= f_n(z_n + \rho_n\xi)f_n^{(k)}(z_n + \rho_n\xi) - g_n(z_n + \rho_n\xi) \\ &\Rightarrow h(\xi)h^{(k)}(\xi) - g(z_0) \end{aligned}$$

for all $\xi \in \mathbb{C}/\{h^{-1}(\infty)\}$.

Obviously, $h(\xi)h^{(k)}(\xi) - g(z_0) \not\equiv 0$.

In fact, suppose that $h(\xi)h^{(k)}(\xi) - g(z_0) \equiv 0$,then $h(\xi) \neq 0$ since $g(z_0) \neq 0$. It follows that

$$\frac{1}{h^2(\xi)} \equiv \frac{h^{(k)}(\xi)}{g(z_0)h(\xi)}.$$

Hence

$$2m(r, \frac{1}{h}) = m(r, \frac{h^{(k)}}{g(z_0)h}) = S(r, h).$$

Then $T(r, h) = S(r, h)$ since $h \neq 0$. So h is a constant, a contradiction.

Next, we claim that $h(\xi)h^{(k)}(\xi) - g(z_0)$ has at most one zero.

Otherwise, suppose that ξ_1, ξ_2 are two distinct zeros of $h(\xi)h^{(k)}(\xi)-g(z_0)$. We choose a positive number δ small enough such that $D_1 \cap D_2 = \emptyset$ and $h(\xi)h^{(k)}(\xi)-g(z_0)$ has no other zeros in $D_1 \cup D_2$ except for ξ_1 and ξ_2,where $D_1 = \{\xi : |\xi - \xi_1| < \delta\}$ and $D_2 = \{\xi : |\xi - \xi_2| < \delta\}$.

By Hurwitz's theorem,for sufficiently large n, there exist points $\xi_{1,n} \to \xi_1$ and $\xi_{2,n} \to \xi_2$ such that

$$\begin{aligned} f_n(z_n+\rho_n\xi_{1,n})f_n^{(k)}(z_n+\rho_n\xi_{1,n}) - g_n(z_n+\rho_n\xi_{1,n}) &= 0, \\ f_n(z_n+\rho_n\xi_{2,n})f_n^{(k)}(z_n+\rho_n\xi_{2,n}) - g_n(z_n+\rho_n\xi_{2,n}) &= 0. \end{aligned}$$

Since $f_n(z)f_n^{(k)}(z) - g_n(z)$ has at most one zero in D, then $z_n + \rho_n\xi_{1,n} = z_n+\rho_n\xi_{2,n}$, this is $\xi_{1,n} = \xi_{2,n} = \frac{z_0-z_n}{\rho_n}$, which contradicts the fact $D_1 \cap D_2 = \emptyset$. The claim is proved. ■

It follows from Lemma 2 that $h(z)h^{(k)}(z)-g(z_0)$ has at least two distinct zeros, a contradiction. Thus $\{f_n\}$ is normal in D.

3. Proof of theorem

Proof of Theorem 2. Suppose that $\mathcal{F}$ is not normal at z_0. From Lemma 5, we have $a(z_0) = 0$. Without loss of generality,we assume that $z_0 = 0$ and $a(z) = z^t b(z)$, where $1 \le t \le m$, $b(0) = 1$. Then by Lemma 1, there exists a sequence of complex numbers $z_n \longrightarrow 0$, a sequence of functions $f_n \in \mathcal{F}$ and a sequence of positive numbers $\rho_n \longrightarrow 0$ such that

$$g_n(\xi) = \frac{f_n(z_n+\rho_n\xi)}{\rho_n^{\frac{k+t}{2}}} \Rightarrow g(\xi)$$

locally uniformly on compact subsets of $\mathbb{C}$, where $g(\xi)$ is a non-constant meromorphic functions in $\mathbb{C}$. By Hurwitz's theorem, all zeros of $g(\xi)$ have multiplicity at least $k+m+1$ and all of poles of $g(\xi)$ have multiplicity at least $m+1$.

Next, we consider two cases.

Case 1. $\frac{z_n}{\rho_n} \to \infty$.Set

$$F_n(\xi) = \frac{f_n(z_n+\rho_n\xi)}{\rho_n^{\frac{k+t}{2}}}.$$

It follows that

$$\begin{aligned} &F_n(\xi)F_n^{(k)}(\xi) - (1+\xi)^t b(z_n+z_n\xi) \\ &\qquad = \frac{f_n(z_n+z_n\xi)f_n^{(k)}(z_n+z_n\xi) - a(z_n+z_n\xi)}{z_n^t}. \end{aligned}$$

As the same argument as in Lemma 5, we can deduce that $F_n(\xi)F_n^{(k)}(\xi)-(1+\xi)^t b(z_n+z_n\xi)$ has at most one zero in $\Delta=\{\xi:|\xi|<1\}$.

Since all zeros of F_n have multiplicity at least $k+m+1$, and $(1+\xi)^t b(z_n+z_n\xi)\to(1+\xi)^t\neq 0$ for $\xi\in\Delta$. Then by Lemma 5, $\{F_n\}$ is normal in Δ.

Therefore, there exists a subsequence of $\{F_n(z)\}$(we still express it as $\{F_n(z)\}$) such that $\{F_n(z)\}$ converges spherically locally uniformly to a meromorphic function $F(z)$ or ∞.

If $F(0)\neq\infty$, then

$$\begin{aligned} g^{(k+m)}(\xi) &= \lim_{n\to\infty} g_n^{(k+m)}(\xi) \\ &= \lim_{n\to\infty}\frac{f_n^{(k+m)}(z_n+\rho_n\xi)}{\rho_n^{\frac{k+t}{2}-(k+m)}} \\ &= \lim_{n\to\infty}\left(\frac{\rho_n}{z_n}\right)^{k+m-\frac{k+t}{2}} F_n^{(k+m)}\left(\frac{\rho_n}{z_n}\xi\right)=0, \end{aligned}$$

for all $\xi\in\mathbb{C}/\{g^{-1}(\infty)\}$.

Hence $g^{(k+m)}\equiv 0$. It follows that g is a polynomial with $deg(g)\leq k+m$. Since all zeros of g have multiplicity at least $k+m+1$, then we deduce that g is a constant, which is a contradiction.

If $F(0)=\infty$, then

$$\frac{1}{F_n\left(\frac{\rho_n}{z_n}\xi\right)}=\frac{z_n^{\frac{k+t}{2}}}{f_n(z_n+\rho_n\xi)}\to\frac{1}{F(0)}=0,$$

when $\xi\in\mathbb{C}/\{g^{-1}(0)\}$, we have

$$\begin{aligned} \frac{1}{g(\xi)} &= \lim_{n\to\infty}\frac{\rho_n^{\frac{k+t}{2}}}{f_n(z_n+\rho_n\xi)} \\ &= \lim_{n\to\infty}\left(\frac{\rho_n}{z_n}\right)^{\frac{k+t}{2}}\frac{z_n^{\frac{k+t}{2}}}{f_n(z_n+\rho_n\xi)}=0. \end{aligned}$$

Hence $g(\xi)=\infty$, which contradicts that $g(\xi)$ is a non-constant meromorphic function.

Case 2. $\frac{z_n}{\rho_n}\to\alpha,\alpha\in\mathbb{C}$. Then we obtain

$$\begin{aligned} & g_n(\xi)\,g_n^{(k)}(\xi)-\left(\xi+\frac{z_n}{\rho_n}\right)^t b(z_n+\rho_n\xi) \\ &\quad=\frac{f_n(z_n+\rho_n\xi)\,f_n^{(k)}(z_n+\rho_n\xi)-a(z_n+\rho_n\xi)}{\rho_n^t} \\ &\quad\Rightarrow g(\xi)\,g^{(k)}(\xi)-(\xi+\alpha)^t, \end{aligned}$$

for all $\xi \in \mathbb{C}/\{g^{-1}(\infty)\}$.

Since for sufficiently large n, $f_n(z_n+\rho_n\xi) f_n^{(k)}(z_n+\rho_n\xi) - a(z_n+\rho_n\xi)$ has one distinct zero,from the proof Lemma 5, we can deduce that $g(\xi) g^{(k)}(\xi) - (\xi+\alpha)^t$ has at most one distinct zero.

By Lemma 4, $g(\xi) g^{(k)}(\xi) - (\xi+\alpha)^t$ have at least two distinct zeros. Thus $g(\xi)$ is a constant, we can get a contradiction. Thus $\mathcal{F}$ is normal at $z_0 = 0$.

Hence $\mathcal{F}$ is normal in D. ∎

Proof of Theorem 1. Let $z_0 \in D$, we show that $\mathcal{F}$ is normal at z_0, let $f \in \mathcal{F}$.

We consider two cases.

Case 1. $f(z_0) f^{(k)}(z_0) \neq a(z_0)$.

Then there exists a disk $D_\sigma(z_0) = \{z : |z-z_0| < \sigma\}$ such that $f(z) f^{(k)}(z) \neq a(z)$ in $D_\sigma(z_0)$.

Since $f, g \in \mathcal{F}$, $f(z)f^{(k)}(z) - a(z)$ and $g(z)g^{(k)}(z) - a(z)$ share 0 in D. So, for each $g \in \mathcal{F}$, $g(z)g^{(k)}(z) \neq a(z)$ in $D_\sigma(z_0)$. By Theorem 2, $\mathcal{F}$ is normal in $D_\sigma(z_0)$. Hence $\mathcal{F}$ is normal at z_0.

Case 2. $f(z_0) f^{(k)}(z_0) = a(z_0)$.

Then there exists a disk $D_\sigma(z_0) = \{z : |z-z_0| < \sigma\}$ such that $f(z) f^{(k)}(z) \neq a(z)$ in $D^0_\sigma(z_0) = \{z : 0 < |z-z_0| < \sigma\}$.

Since $f, g \in \mathcal{F}$, $f(z)f^{(k)}(z) - a(z)$ and $g(z)g^{(k)}(z) - a(z)$ share 0 in D. Thus, for each $g \in \mathcal{F}$,$g(z)g^{(k)}(z) \neq a(z)$ in $D^0_\sigma(z_0)$ and $g(z_0)g^{(k)}(z_0) = a(z_0)$. Therefore, $g(z)g^{(k)}(z) - a(z)$ have only distinct zero in $D_\sigma(z_0)$. By Theorem 2, $\mathcal{F}$ is normal in $D_\sigma(z_0)$. Thus $\mathcal{F}$ is normal at z_0.

Hence $\mathcal{F}$ is normal in D. ∎

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