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Luong Quoc Tuyen and Ong Van Tuyen

SOME PROPERTIES OF RECTIFIABLE SPACES

Abstract. In this paper, we give some properties of rectifiable spaces and their relationship with P-space, metrizable space. These results are used to generalize some results in [2], [9] and [12]. Moreover, we give the conditions for a rectifiable space to be second-countable.

Key words: Topological group, rectifiable space, G_δ-set, P-space, metrizable space, second-countable space.

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1. Introduction and preliminaries

In 1936, G. Birkhoff introduced topological groups ([3]). After that, M.M. Choban introduced rectifiable spaces ([4]) and V.V. Uspenskij showed that every topological group is a rectifiable space but there exists a rectifiable space which is not a topological group ([16]). Recently, rectifiable spaces had been studied by many authors ([7], [8], [10], [15], for example).

In this paper, we give some properties of rectifiable spaces (see Theorem 1) and their relationship with P-space (see Theorem 2), metrizable space (see Theorem 3). These results are used to generalize results below:

1. Let G be a topological group and F be a non-empty compact G_δ-set in G. Then, there exists a G_δ-set P in G such that $e \in P$ and $FP \subset F$ ([2], Proposition 3.1.19).
2. Let H be a dense subgroup of a topological group G. If H is a P-space, then G is a P-space ([2], Lemma 4.4.1).
3. A κ-Fréchet-Urysohn, biradial topological group G is metrizable ([12], Theorem 3.4).
4. Every bisequential rectifiable space G is metrizable ([9], Theorem 3.3).

Moreover, we give the conditions for a rectifiable space to be second-countable (see Theorem 4).

Throughout this paper, all spaces are T_1, $\mathbb{N}$ denotes the set of all natural numbers, and group G have a unit element e.

Definition 1 ([3]). *A* topological group *G is a group G with a topology such that the product map of $G \times G$ into G is jointly continuous and the inverse map onto itself associating x^{-1} with $x \in G$ is continuous.*

Definition 2 ([4], [5]). *A* rectification *on a space G is a homeomorphism $\varphi : G \times G \to G \times G$ with the following two properties:*

(a) $\varphi(\{x\} \times G) = \{x\} \times G$, for every $x \in G$;

(b) There exists an element $e \in G$ such that $\varphi(x, x) = (x, e)$, for every $x \in G$.

The point $e \in G$ is called a right unit element *of G. A space with a rectification is called a* rectifiable space*. Every rectifiable space is homogenous.*

Remark 1 ([16]). A topological group is a rectifiable space. However, there exists a rectifiable space which is not a topological group.

Definition 3 ([11]). *Let $\mathcal{P}$ be a family of subsets of a space X. For each $x \in X$, $\mathcal{P}$ is a* network *at x in X, if $x \in P$ for every $P \in \mathcal{P}$, and if $x \in U$ with U open in X, then there exists $P \in \mathcal{P}$ such that $x \in P \subset U$.*

Definition 4. *Let X be a topological space. Then,*

1. *X is called a P-*space *([2]) if every G_δ-set in X is open.*
2. *X is called a* cosmic *space ([13]) if X is regular and has a countable network.*

Definition 5 ([1]). *Let ξ and η be any family of non-empty subsets of X.*

1. *The family ξ is called a* prefilter *on a space X if for any $A \in \xi$ and $B \in \xi$ there exists $C \in \xi$ such that $C \subset A \cap B$.*
2. *A prefilter ξ on a space X is said to* converge to a point *$x \in X$ if every open neighborhood of x contains an element of ξ.*
3. *If $x \in X$ belongs to the closure of every element of a prefilter ξ on X, we say that ξ* accumulates to *x or x is a* cluster point of *ξ.*
4. *Two prefilter ξ and η are called to be* synchronous *if for any $A \in \xi$ and for any $B \in \eta$, $A \cap B \neq \emptyset$.*
5. *A* chain *in X is any prefilter ξ in X such that for every $A \in \xi$ and $B \in \eta$ either $A \subset B$ or $B \subset A$. A chain ξ consisting of open subsets of a space X is called a* nest *in X.*

Definition 6. *Let X be a topological space. Then,*

1. *X is called* bisequential *([14]) if for every prefilter ξ in X and every cluster point x of ξ there exitsts a countable prefilter η in X converging to x and synchronous with ξ.*
2. *X is called* biradial *([1]) if for every prefilter ξ in X accumulating to a point $x \in X$ there exitsts a chain in X converging to x and synchronous with ξ.*

3. *X is said to be* π-nested *([1]) at a point $x \in X$ if there exists a nest in X converging to x.*
4. *X is called* nested *([1]) if for every point $x \in X$ if there exists a nest in X which is a base for X at x.*
5. *X is called* κ-Fréchet-Urysohn *([12]) if for every open subset U of X and every $x \in \overline{U}$ there exists a sequence $\{x_n\}$ of points of U converging to x.*

Remark 2. 1. Every bisequential space is biradial. However, there exists a biradial space which is not a bisequential space ([1]).
2. Every bisequential space is κ-Fréchet-Urysohn. However, there exists a κ-Fréchet-Urysohn space which is not a bisequential space ([12]).

Remark 3 ([1])**.** Every nested space is biradial.

Lemma 1 ([1])**.** *Every biradial regular space is π-nested at every point.*

Lemma 2 ([5])**.** *A topological space G is rectifiable if and only if there are two continuous mappings $p : G \times G \to G$, $q : G \times G \to G$ such that for any $x \in G$, $y \in G$, and some $e \in G$ the next identities hold.*

$$p\big(x, q(x, y)\big) = q\big(x, p(x, y)\big) = y \quad \textit{and} \quad q(x, x) = e.$$

Remark 4 ([7])**.** Let G be a rectifiable space and $x \in G$, we have

$$p(x, e) = p\big(x, q(x, x)\big) = x.$$

Moreover, we sometimes write xy instead of $p(x, y)$ for any x, $y \in G$ and AB instead of $p(A, B)$ for any A, $B \subset G$.

Lemma 3 ([8])**.** *Let G be a rectifiable space. Fixed a point $x \in G$, then $f_x, g_x : G \to G$ defined with $f_x(y) = p(x, y)$ and $g_x(y) = q(x, y)$, for each $y \in G$, are homeomorphism, respectively.*

Definition 7 ([8])**.** *Let A be a subset of a rectifiable space G. Then A is called a* rectifiable subspace *of G if we have $p(A, A) \subset A$ and $q(A, A) \subset A$.*

Lemma 4 ([6])**.** *If A is dense in X, then $\overline{U} = \overline{A \cap U}$ for every U open in X.*

Lemma 5 ([7])**.** *If G is a rectifiable space, then G is regular.*

Lemma 6 ([7])**.** *Let X be a rectifiable, first-countable T_0 space. Then X is metrizable.*

2. Main results

Lemma 7. *Let G be a rectifiable space, $A \subset G$ and U be an open subset in G. Then, $p(A, U)$ and $q(A, U)$ are open subsets in G.*

Proof. Let $x \in A$, it follows from Lemma 3 that f_x, g_x are homeomorphism. This implies that they are open maps. Therefore, $f_x(U) = p(x, U)$ and $g_x(U) = q(x, U)$ are open subsets in G. Moreover, since

$$p(A, U) = \bigcup_{x \in A} p(x, U);$$
$$q(A, U) = \bigcup_{x \in A} q(x, U),$$

we have $p(A, U)$ and $q(A, U)$ are open subsets in G. ■

Lemma 8. *Let G be a rectifiable space and $x \in G$. Then, the following statements hold.*

1. *If U is an open neighborhood of x, then there exists an open neighborhood V of e in G such that $xV \subset U$;*
2. *If U is an open neighborhood of e in G, then xU is an open neighborhood of x, and there exists an open neighborhood V of e in G such that $q(xV, x) \subset U$.*

Proof. (1) By Lemma 2 and Remark 4, $p(x, e) = x$ and p is continuous. Moreover, since U is an open neighborhood of x, there exist an open neighborhood W of x and an open neighborhood V of e such that $xV = p(x, V) \subset p(W, V) \subset U$.

(2) Since U is an open neighborhood of e, xU is an open neighborhood of x by Lemma 7. On the other hand, by Lemma 2, we have $q(x, x) = e$ and q is continuous. Thus, there exist two open neighborhoods V_1 and V_2 of x such that $q(V_1, V_2) \subset U$. By (1), there exists an open neighborhood V of e such that $x \in xV \subset V_1 \cap V_2$. Therefore, $q(xV, x) \subset U$. ■

Lemma 9. *Let K be a compact subset and F be a closed subset of a rectifiable space G such that $K \cap F = \emptyset$. Then, there exists an open neighborhood V of e in G such that $KV \cap F = \emptyset$.*

Proof. Since $K \cap F = \emptyset$ and F is closed, it implies that for each $x \in K$, there exists an open neighborhood U_x of x such that

$$U_x \cap F = \emptyset.$$

By Lemma 8, there exists an open neighborhood V_x of e such that $xV_x \subset U_x$, and xV_x is an open neighborhood of x. It follows from Lemma 2 and

Remark 4 that $p(x, e) = x$ and p is continuous. Thus, there exist an open neighborhood W_x of x and an open neighborhood $U_e(x)$ of e such that $p\big(W_x, U_e(x)\big) \subset xV_x$. Since W_x is an open neighborhood of x, it follows from Lemma 8(1) that there exists an open neighborhood $V_e(x)$ of e such that $xV_e(x) \subset W_x$. Thus,

$$p\big(xV_e(x), U_e(x)\big) \subset p\big(W_x, U_e(x)\big) \subset xV_x.$$

It follows from Lemma 8(2) that $x\big(U_e(x) \cap V_e(x)\big)$ is an open neighborhood of x for all $x \in K$. On the other hand, since $\big\{x\big(U_e(x) \cap V_e(x)\big) : x \in K\big\}$ is an open cover of K compact, there exists a finite subset $L \subset K$ such that

$$K \subset \bigcup_{x \in L} x\big(U_e(x) \cap V_e(x)\big).$$

Now, if we put $V = \bigcap\limits_{x \in L} \big(U_e(x) \cap V_e(x)\big)$, then V is an open neighborhood of e. Furthermore, we have $KV \cap F = \emptyset$. In fact, let $y \in K$. Then, there exists $x \in L$ such that $y \in x\big(U_e(x) \cap V_e(x)\big)$. Thus,

$$yV \subset p\Big(x\big(U_e(x) \cap V_e(x)\big), \big(U_e(x) \cap V_e(x)\big)\Big) \subset xV_x \subset G \setminus F.$$

Therefore, $yV \cap F = \emptyset$ for every $y \in K$, it implies that $KV \cap F = \emptyset$. ■

Theorem 1. *Let G be a rectifiable space and F be a non-empty compact G_δ-set in G. Then, there exists a G_δ-set P in G such that $e \in P$ and $FP \subset F$.*

Proof. Let $\{U_n : n \in \mathbb{N}\}$ be a sequence consisting of open subsets of G, and $F = \bigcap\limits_{n \in \mathbb{N}} U_n$. Then, $G \setminus U_n$ closed in G and $F \cap (G \setminus U_n) = \emptyset$ for every $n \in \mathbb{N}$. Moreover, since F is a non-empty compact in G, by Lemma 9, for each $n \in \mathbb{N}$, there exists an open neighborhood V_n of e such that

$$FV_n \cap (G \setminus U_n) = \emptyset.$$

This implies that $FV_n \subset U_n$ for every $n \in \mathbb{N}$. Now, if we put $P = \bigcap\limits_{n \in \mathbb{N}} V_n$, then P is a G_δ-set in G and $e \in P$. Furthermore, we have

$$FP = p\big(F, \bigcap_{n \in \mathbb{N}} V_n\big) \subset \bigcap_{n \in \mathbb{N}} p(F, V_n) \subset \bigcap_{n \in \mathbb{N}} U_n = F.$$

■

By Remark 1 and Theorem 1, we obtained the following corollary.

Corollary 1 ([2], Proposition 3.1.19). *Let G be a topological group and F be a non-empty compact G_δ-set in G. Then, there exists a G_δ-set P in G such that $e \in P$ and $FP \subset F$.*

Theorem 2. *Let H be a dense rectifiable subspace of a rectifiable space G. If H is a P-space, then G is a P-space.*

Proof. Because every rectifiable space is homogenous, we only need to prove that each G_δ-set in G which contains e in G is open. Let $\{U_n : n \in \mathbb{N}\}$ be a sequence consisting of open neighborhoods of e in G, and $U = \bigcap_{n\in\mathbb{N}} U_n$. Now, we only have to prove that U open in G. In fact, let $x \in U$, so $x \in U_n$ for every $n \in \mathbb{N}$. Since U_1 open in G, it follows from Lemma 8(1) that there exists an open neighborhood V_1 of e in G such that $xV_1 \subset U_1$. It follows from Lemma 5 that there exists an open neighborhood W_1 of e in G such that $\overline{W_1} \subset V_1$. On the other hand, since U_2 open in G, it follows from Lemma 8(1) that there exists an neighborhood open W_2 of e in G such that $xW_2 \subset U_2$. Furthermore, because $W_1 \cap W_2$ is an open neighborhood of e in G, it follows from Lemma 5 that there exists an open neighborhood V_2 of e in G such that $\overline{V_2} \subset W_1 \cap W_2$. Continue the process, we can find a sequence $\{V_n : n \in \mathbb{N}\}$ consisting of open neighborhoods of e in G such that $\overline{V_{n+1}} \subset V_n$ and $xV_n \subset U_n$ for every $n \in \mathbb{N}$. Put $V = \bigcap_{n\in\mathbb{N}} V_n$, it implies that V closed in G. Since H is a P-space, we have

$$V \cap H = \bigcap_{n\in\mathbb{N}} (V_n \cap H)$$

open in H. Hence, there exists W open in G such that $V \cap H = W \cap H$. Because H is dense in G, by Lemma 4, we have

$$\overline{W} = \overline{W \cap H} = \overline{V \cap H} \subset \overline{V} = V.$$

Then, we have

$$p(x, V) = p\Big(x, \bigcap_{n\in\mathbb{N}} V_n\Big) \subset \bigcap_{n\in\mathbb{N}} p(x, V_n) = \bigcap_{n\in\mathbb{N}} xV_n \subset \bigcap_{n\in\mathbb{N}} U_n = U.$$

Lastly, since H is a rectifiable subspace of G, it follows from Lemma 2 that for every $y \in H$, we have

$$e = q(y, y) \in q(H, H) \subset H.$$

Thus, $e \in V \cap H \subset W$. Hence, by Remark 4, we get

$$x = p(x, e) \in p(x, W) \subset p(x, \overline{W}) \subset p(x, V) \subset U.$$

Since $p(x, W)$ is an open neighborhood of x in G by Lemma 8(2), it implies that U open in G. ■

By Remark 1 and Theorem 2, we obtained the following corollary.

Corollary 2 ([2], Lemma 4.4.1)**.** *Let H be a dense subgroup of a topological group G. If H is a P-space, then G is a P-space.*

Lemma 10. *If a rectifiable space G is π-nested at some point $a \in G$, then the space G is nested.*

Proof. By homogeneity of the space G we can assume that a is the right unit element e of G. Then, since G is π-nested at e, there exists a nest ξ in G converging to e. Now, if we put

$$\eta = \{q(U, U) : U \in \xi\},$$

then η is a nest which is a base for G at e. In fact,

(1) η is a nest in G. We prove (a), (b) and (c) below.

(a) η is a prefilter in G. Indeed, for every $A \in \eta$ and $B \in \eta$, there exist $U \in \xi$ and $V \in \xi$ such that $A = q(U, U)$ and $B = q(V, V)$. Then,

$$A \cap B = q(U, U) \cap q(V, V).$$

Moreover, since ξ is a prefilter in G, there exists $W \in \xi$ such that $W \subset U \cap V$. It implies that

$$q(W, W) \subset q(U \cap V, U \cap V) \subset q(U, U) \cap q(V, V) = A \cap B.$$

If we put $C = q(W, W)$, then $C \subset A \cap B$ and $C \in \eta$. Therefore, η is a prefilter in G.

(b) For every $A \in \eta$ and $B \in \eta$, we have $A \subset B$ or $B \subset A$. Indeed, since $A \in \eta$ and $B \in \eta$, there exist $U \in \xi$ and $V \in \xi$ such that $A = q(U, U)$ and $B = q(V, V)$. Furthermore, because ξ is a chain in G, we have $U \subset V$ or $V \subset U$. This implies that $A \subset B$ or $B \subset A$.

(c) For every $A \in \eta$, A open in G. Indeed, since $A \in \eta$, there exists $U \in \xi$ such that $A = q(U, U)$. Moreover, because ξ is a nest in G, U open in G. It follows from Lemma 7 that $A = q(U, U)$ open in G.

(2) η is a base for G at e. Indeed, let W be an open neighborhood of e. Then, since $q(e, e) = e$ and q is continuous, there exist two open neighborhoods U_1 and U_2 of e such that $e \in q(U_1, U_2) \subset W$. Now, if we put $U = U_1 \cap U_2$, then $e \in q(U, U) \subset W$ and U is an open neighborhood of e. Moreover, since ξ converges to e, there exists $V \in \xi$ such that $V \subset U$. This implies that $q(V, V) \subset q(U, U)$. On the other hand, for every $x \in V$, we have

$$e = q(x, x) \in q(V, V) \subset q(U, U) \subset W.$$

Next, if we put $A = q(V, V)$ then $A \in \eta$ and $e \in A \subset W$. Therefore, η is a base for G at e.

Hence, η is a nest which is a base for G at e. Lastly, since G is homogenous it follows that G is nested. ■

Theorem 3. *A κ-Fréchet-Urysohn, biradial rectifiable space G is metrizable.*

Proof. Because G is a biradial rectifiable space, we have G is nested by Lemma 1, Lemmas 5 and 10. Let e be the right unit element of G. Then, there is a nest in G which is a base at e.

Case 1: e is isolated. It implies that $\{e\}$ open in G. Hence, G is first-countable because G is homogenous. It follows from Lemma 6 that G is metrizable.

Case 2: e is not isolated. Then, $e \in \overline{G \setminus \{e\}}$ and $G \setminus \{e\}$ open in G. Since G is κ-Fréchet-Urysohn, there is a sequence $\{x_n\}_{n\in\mathbb{N}} \subset G \setminus \{e\}$ such that $x_n \to e$. Let $\{V_\alpha : \alpha \in \Lambda\}$ is a nest in G which is a base at e. If we put $x_{n_1} = x_1$ then $e \in G \setminus \{x_{n_1}\}$ and $G \setminus \{x_{n_1}\}$ open in G. Since G is regular by Lemma 5, there is $\alpha_1 \in \Lambda$ such that

$$e \in V_{\alpha_1} \subset \overline{V_{\alpha_1}} \subset G \setminus \{x_{n_1}\}.$$

Because $\{x_n\}_{n\in\mathbb{N}}$ converges to e and V_{α_1} is an open neighborhood of e, there is $n_2 > n_1$ such that

$$x_{n_2} \in V_{\alpha_1} \quad \text{and} \quad e \in G \setminus \{x_i : i \leq n_2\}.$$

Moreover, since G is regular and $G \setminus \{x_i : i \leq n_2\}$ open in G, there exists $\alpha_2 \in \Lambda$ such that

$$e \in V_{\alpha_2} \subset \overline{V_{\alpha_2}} \subset G \setminus \{x_i : i \leq n_2\}.$$

By induction, we have for every $k \in \mathbb{N}$, there exist $\alpha_k \in \Lambda$ and $n_k \in \mathbb{N}$ such that

$$e \in V_{\alpha_k} \subset \overline{V_{\alpha_k}} \subset G \setminus \{x_i : i \leq n_k\}.$$

Next, we show that $\{V_{\alpha_k} : k \in \mathbb{N}\}$ is a base at e. Indeed, let W be an open neighborhood of e. Then, since $\{V_\alpha : \alpha \in \Lambda\}$ is a base at e, there exists $\beta \in \Lambda$ such that $V_\beta \subset W$ and there exists $m \in \mathbb{N}$ such that $x_m \in V_\beta$. Choose $i \in \mathbb{N}$ such that $n_i > m$, then $x_m \notin V_{\alpha_i}$. Moreover, since $\{V_\alpha : \alpha \in \Lambda\}$ is a chain and $V_\beta \setminus V_{\alpha_i} \neq \emptyset$, then $V_{\alpha_i} \subset V_\beta \subset W$. Hence, $\{V_{\alpha_k} : k \in \mathbb{N}\}$ is a base at e. It follows from G is homogenous that G is first-countable. By Lemma 6, G is metrizable. ■

By Remark 1 and Theorem 3, we obtained the following corollary.

Corollary 3 ([12], Theorem 3.4)**.** *A κ-Fréchet-Urysohn, biradial topological group G is metrizable.*

By Remark 2 and Theorem 3, we obtained the following corollary.

Corollary 4 ([9], Theorem 3.3)**.** *Every bisequential rectifiable space G is metrizable.*

Theorem 4. *Every first-countable cosmic rectifiable space is second-countable.*

Proof. Let G be a first-countable cosmic rectifiable space. Then, it follows from Lemma 6 that G is metrizable. Moreover, because G is a cosmic space, we have G is separable. On the other hand, since a separable metrizable space is second-countable, it implies that G is second-countable. ■

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LUONG QUOC TUYEN
DEPARTMENT OF MATHEMATICS
DA NANG UNIVERSITY
VIETNAM
e-mail: tuyendhdn@gmail.com

ONG VAN TUYEN
ONG ICH KHIEM HIGH SCHOOL
DA NANG CITY, VIETNAM
e-mail: tuyenvan612dn@gmail.com

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