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ON SOME RECENT FIXED POINT RESULTS FOR SINGLE AND MULTI-VALUED MAPPINGS IN b-METRIC SPACES

Abstract. The main purpose of this paper is to improve and correct some results in b-metric spaces. Moreover, we prove that some results can be slightly relaxed and also we explore some proof techniques which provide short proofs of the results.

Key words: b-metric space, b-complete, b-Cauchy, b-continuous, Picard sequence, multi-valued mapping.

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1. Definitions, notations and preliminaries

We start our exposition with the next result which will prove extremely useful in the sequel.

Lemma 1 ([15]). *Let (X, d, s) be a b-metric space and $\{x_n\}_{n\in\mathbb{N}}$ a sequence in X. If there exists $\gamma \in [0, 1)$ such that*

$$d(x_{n+1}, x_n) \leq \gamma d(x_n, x_{n-1})$$

for all $n \in \mathbb{N}$, then $\{x_n\}_{n\in\mathbb{N}}$ is a b-Cauchy sequence.

Otherwise, for more details on b-metric spaces we refer the reader to ([1]–[5], [8]–[11], [13]–[15], [17]–[21]).

In [6], authors proved some fixed point theorems in b-metric spaces. We will restrict our attention to the following two results.

Theorem 1 ([6] Theorem 1). *Let $(X, d, s \geq 1)$ be a complete b-metric space and define the sequence $\{x_n\}$ in X by the recursion*

$$x_n = Tx_{n-1} = T^n x_0.$$

Let $T : X \to X$ be a mapping such that

$$d(Tx,Ty) \leq \lambda_1 d(x,y) + \lambda_2 d(x,Tx) + \lambda_3 d(y,Ty) + \lambda_4 [d(y,Tx) + d(x,Ty)] \tag{1}$$

for all $x, y \in X$, where $\lambda_1 + 2s\lambda_2 + \lambda_3 + 2s\lambda_4 \leq 1$.

Then there exists $x^* \in X$ such that $x_n \to x^*$ and x^* is a unique fixed point.

Remark 1. The condition $\lambda_1 + 2s\lambda_2 + \lambda_3 + 2s\lambda_4 \leq 1$ should be replaced by

$$\lambda_i \geq 0,\ i = \overline{1,4}, \qquad \lambda_1 + 2s\lambda_2 + \lambda_3 + 2s\lambda_4 < 1.$$

Indeed, for $\lambda_2 = \lambda_3 = \lambda_4 = 0$, $\lambda_1 = 1$ and $s = 1$, we have

$$d(Tx,Ty) \leq d(x,y), \quad x,y \in X.$$

For $T = I_X$, Theorem 1 is not valid, since the fixed point of T is not unique.

If $s = 1$, then (X,d) is a metric space and the condition (1) implies

$$d(Tx,Ty) \leq k \max\left\{d(x,y), d(x,Tx), d(y,Ty), \frac{d(y,Tx)+d(x,Ty)}{2}\right\}, \tag{2}$$

where $k = \lambda_1 + \lambda_2 + \lambda_3 + 2\lambda_4 < 1$. Note that

$$\lambda_1 + \lambda_2 + \lambda_3 + 2\lambda_4 \leq \lambda_1 + 2\lambda_2 + \lambda_3 + 2\lambda_4 < 1.$$

With (2) we recover the well known result for generalized Ćirić contraction map in the metric space and obtain a unique fixed point.

Remark 2. Let us note that the condition $\lambda_1 + 2s\lambda_2 + \lambda_3 + 2s\lambda_4 < 1$ from [6] implies the condition $\lambda_1 + \lambda_2 + \lambda_3 + 2s\lambda_4 < 1$. Now, according to the new condition, we can improve the proof of Theorem 1 from [6]. Firstly, the proof that $\{x_n\}$ is a b-Cauchy sequence can be shorter than that in [6]. Indeed, if $x_n \neq x_{n-1}$, for all $n \in \mathbb{N}$, we have

$$\begin{aligned} d(Tx_{n-1},Tx_n) \leq\ & \lambda_1 d(x_{n-1},x_n) + \lambda_2 d(x_{n-1},x_n) + \lambda_3 d(x_n,x_{n+1}) \\ & + \lambda_4 s d(x_{n-1},x_n) + \lambda_4 s d(x_n,x_{n+1}), \end{aligned}$$

and also

$$\begin{aligned} d(Tx_n,Tx_{n-1}) \leq\ & \lambda_1 d(x_n,x_{n-1}) + \lambda_2 d(x_n,x_{n+1}) + \lambda_3 d(x_{n-1},x_n) \\ & + \lambda_4 s d(x_{n-1},x_n) + \lambda_4 s d(x_n,x_{n+1}). \end{aligned}$$

It follows easily that $d(x_{n+1}, x_n) \leq kd(x_n, x_{n-1})$ where

$$k = \max\left\{\frac{\lambda_1 + \lambda_3 + s\lambda_4}{1 - \lambda_2 - s\lambda_4}, \frac{\lambda_1 + \lambda_2 + s\lambda_4}{1 - \lambda_3 - s\lambda_4}\right\} < 1,$$

and according to Lemma 1, we have that $\{x_n\}$ is a b-Cauchy sequence.

Remark 3. It is not hard to check that the proof of Theorem 1 in [6] is not correct (see pages 3 and 4). Really, the fact that $x^* = \lim\limits_{n\to\infty} x_n$ is the fixed point of T is not clear enough since it was not shown that

$$\frac{s + s^2\lambda_2 + s\lambda_4}{1 - s\lambda_3 - s^2\lambda_4} \quad \text{and} \quad \frac{s\lambda_1 + s^2\lambda_2 + s^2\lambda_4}{1 - s\lambda_3 - s^2\lambda_4}$$

are both positive numbers. In fact, for $\lambda_3 = \frac{1}{s}$ we get that both expressions are negative.

We prove this with the new condition $\lambda_1 + \lambda_2 + \lambda_3 + 2s\lambda_4 < 1$.

$$\begin{aligned}
\frac{1}{s}d(x^*, Tx^*) &\leq d(x^*, x_{n+1}) + d(Tx_n, Tx^*) \\
&\leq d(x^*, x_{n+1}) + \lambda_1 d(x_n, x^*) + \lambda_2 d(x_n, x_{n+1}) + \lambda_3 d(x^*, Tx^*) \\
&\quad + \lambda_4 (d(x^*, x_{n+1}) + d(x_n, Tx^*)) \\
&\leq d(x^*, x_{n+1}) + \lambda_1 d(x_n, x^*) + \lambda_2 d(x_n, x_{n+1}) + \lambda_3 d(x^*, Tx^*) \\
&\quad + \lambda_4 d(x^*, x_{n+1}) + \lambda_4 s d(x_n, x^*) + \lambda_4 s d(x^*, Tx^*) \\
&\leq (1 + \lambda_4) d(x^*, x_{n+1}) + (\lambda_1 + \lambda_4 s) d(x_n, x^*) \\
&\quad + \lambda_2 d(x_n, x_{n+1}) + (\lambda_3 + \lambda_4 s) d(x^*, Tx^*),
\end{aligned}$$

or

$$\begin{aligned}
(3) \qquad \left(\frac{1}{s} - \lambda_3 - \lambda_4 s\right) d(x^*, Tx^*) &\leq (1 + \lambda_4) d(x^*, x_{n+1}) \\
&\quad + (\lambda_1 + \lambda_4 s) d(x_n, x^*) + \lambda_2 d(x_n, x_{n+1}).
\end{aligned}$$

Similarly,

$$\begin{aligned}
\frac{1}{s}d(x^*, Tx^*) &\leq d(x^*, x_{n+1}) + d(Tx^*, Tx_n) \\
&\leq (1 + \lambda_4) d(x^*, x_{n+1}) + (\lambda_1 + \lambda_4 s) d(x_n, x^*) \\
&\quad + \lambda_3 d(x_n, x_{n+1}) + (\lambda_2 + \lambda_4 s) d(x^*, Tx^*),
\end{aligned}$$

that is.,

$$\begin{aligned}
(4) \qquad \left(\frac{1}{s} - \lambda_2 - \lambda_4 s\right) d(x^*, Tx^*) &\leq (1 + \lambda_4) d(x^*, x_{n+1}) \\
&\quad + (\lambda_1 + \lambda_4 s) d(x_n, x^*) + \lambda_3 d(x_n, x_{n+1}).
\end{aligned}$$

Adding (3) and (4) we obtain

$$\begin{aligned}&\left(\frac{2}{s}-\lambda_2-\lambda_3-2\lambda_4 s\right)d\left(x^*,Tx^*\right)\\&\qquad\le 2\left(1+\lambda_4\right)d\left(x^*,x_{n+1}\right)+2\left(\lambda_1+\lambda_4 s\right)d\left(x_n,x^*\right)\\&\qquad\quad+\ \left(\lambda_2+\lambda_3\right)d\left(x_n,x_{n+1}\right)\to 0,\ \text{ as } \ n\to\infty.\end{aligned}$$

Hence, we can conclude the following.

Conclusion. Theorem 1 from [6] holds if the coefficients $\lambda_i\ge 0$, $i=\overline{1,4}$, satisfy at least one of the following conditions:
1. $\lambda_1+\lambda_2+\lambda_3+2\lambda_4<1$ for $s\in[1,2]$;
2. $\frac{2}{s}<\lambda_1+\lambda_2+\lambda_3+2\lambda_4<1$ for $s\in(2,+\infty)$.

Our approach with the new condition $\lambda_1+\lambda_2+\lambda_3+2s\lambda_4<1$ provides the generalization and improves Theorem 3.7 from [10], that is., Theorem 2.19 from [18].

Remark 4. Note that condition (2) for $s>1$ implies

$$d(Tx,Ty)\le k\max\left\{d(x,y),d(x,Tx),d(y,Ty),\frac{d(x,Ty)+d(y,Tx)}{2s}\right\}$$

where $k=\lambda_1+\lambda_2+\lambda_3+2s\lambda_4$.

Theorem 2 ([6] Theorem 2)**.** *Let $(X,d,s\ge 1)$ be a complete b-metric space. Let $T:X\to X$ be a mapping for which there exist $\lambda_1,\lambda_2\in[0,\frac{1}{3})$ such that*

$$d\left(Tx,Ty\right)\le\lambda_1 d\left(x,y\right)+\lambda_2\left[d\left(x,Tx\right)+d\left(y,Ty\right)\right],\tag{5}$$

for all $x,y\in X$.

Then there exists $x^*\in X$ such that $x_n\to x^*$ and x^* is the unique fixed point.

Remark 5. The proof of Theorem 2 is not correct since $\dfrac{\lambda_1+\lambda_2}{1-\lambda_2}\le 1$. We give the improved version of this theorem.

If $s=1$, then (X,d) is a metric space and the condition $\lambda_1+2\lambda_2<1$ is appropriate for metric spaces.

Let $s>1$. By the same method as in [6], we have

$$d(x_n,x_{n+1})\le\frac{\lambda_1+\lambda_2}{1-\lambda_2}d(x_{n-1},x_n)=kd(x_{n-1},x_n).$$

Since $\lambda_1,\lambda_2\in[0,\frac{1}{3})$, it follows that $\dfrac{\lambda_1+\lambda_2}{1-\lambda_2}=k<1$, and using Lemma 1, we can conclude that the sequence $\{x_n\}$ is a b-Cauchy sequence.

The proof falls naturally into three parts.

Case 1°**.** If T is continuous, then $Tx_n \to Tx^*$ as $n \to \infty$, and $x^* = \lim\limits_{n\to\infty} x_n$ is the fixed point of T.

Case 2°**.** If d is continuous, then substituting $x = x_n$ and $y = \lim\limits_{n\to\infty} x_n$ in (5), we obtain

$$d(x_{n+1}, Tx^*) = d(Tx_n, Tx^*) \le \lambda_1 d(x_n, x^*) + \lambda_2[d(x_n, x_{n+1}) + d(x^*, Tx^*)].$$

Letting $n \to \infty$, it follows that

$$d(x^*, Tx^*) \le \lambda_1 \cdot 0 + \lambda_2 \cdot 0 + \lambda_2 d(x^*, Tx^*),$$

i.e. $(1-\lambda_2)d(x^*, Tx^*) \le 0$. Using the fact that $\lambda_2 \in [0, \frac{1}{3})$, we have $Tx^* = x^*$.

Case 3°**.** Neither 1° nor 2° is satisfied. Then

$$\begin{aligned} \frac{1}{s} d(x^*, Tx^*) &\le d(x^*, x_{n+1}) + d(Tx_n, Tx^*) \\ &\le d(x^*, x_{n+1}) + \lambda_1 d(x_n, x^*) + \lambda_2[d(x_n, x_{n+1}) + d(x^*, Tx^*)], \end{aligned}$$

i.e. $(\frac{1}{s} - \lambda_2)d(x^*, Tx^*) \le 0$. We conclude that T has a fixed point $x^* = \lim\limits_{n\to\infty} x_n$ if $\lambda_2 < \frac{1}{s}$.

From what has already been proved, we deduce that T has a fixed point if $\lambda_2 < \min\{\frac{1}{3}, \frac{1}{s}\}$.

Now, we will show that our viewpoint sheds some new light on an interesting new result proved in [7].

Theorem 3 ([7] Theorem 2.2)**.** *Let $(X, d, s \ge 1)$ be a complete b-metric space and T, S self-mappings on X which satisfy*

$$\begin{aligned} d(Sx, Ty) \le a_1 d(x, Sx) + a_2 d(y, Ty) + a_3 d(x, Ty) \\ + a_4 d(y, Sx) + a_5 d(x, y), \end{aligned} \tag{6}$$

for all $x, y \in X$, where a_1, a_2, a_3, a_4, a_5 are nonnegative real numbers satisfying:

(*i*) $s^2 a_1 + s^2 a_2 + s^3 a_3 + s^3 a_4 + s^2 a_5 < 1$,
(*ii*) $a_1 = a_2$ *or* $a_3 = a_4$.

Then S and T have a unique common fixed point.

Remark 6. If $s = 1$, then (X, d) is a metric space with the assumptions

$$a_1 + a_2 + a_3 + a_4 + a_5 < 1, \quad a_1 = a_2 \quad \text{or} \quad a_3 = a_4.$$

It follows immediately that the condition (*ii*) is superfluous.

We will also prove that [[7] Theorem 2.4] is still true if we drop the assumption of function φ. We repeat the relevant material from [7].

Definition 1. *A function $\psi : [0,\infty) \to [0,\infty)$ is said to be an altering distance function if ψ is continuous and strictly increasing and if $\psi(t) = 0$ if and only if $t = 0$.*

Follow the notation used in [7], Φ denotes the next set.

$$\Phi = \Big\{\varphi : [0,\infty)^2 \to [0,\infty) \mid \varphi(0,0) \geq 0, \ \ \varphi(x,y) > 0 \text{ if } (x,y) \neq (0,0), \\ \varphi\left(\liminf_{n\to\infty} a_n, \liminf_{n\to\infty} b_n\right) \leq \liminf_{n\to\infty} \varphi(a_n, b_n)\Big\}$$

The next result is stated and proved in [7].

Theorem 4. *Let $(X, d, s \geq 1)$ be a complete b-metric space and T, f self-mappings on X which satisfy*

$$\psi(sd(Tx, fy)) \leq \frac{\psi\left(\frac{d(x,fy)+\frac{d(y,Tx)}{s^3}}{s+1}\right)}{1+\varphi(d(x,fy), d(y,Tx))}, \tag{7}$$

for all $x, y \in X$, where ψ is an altering distance function, $\varphi \in \Phi$ and T is continuous. Then T and f have a unique common fixed point.

Remark 7. We will now show how to dispense with the assumption on function φ. Indeed, the condition (7) implies

$$sd(Tx, fy) \leq \frac{d(x, fy)}{s+1} + \frac{d(y, Tx)}{s^3(s+1)},$$

i.e.

$$\begin{aligned} d(Tx, fy) &\leq \frac{d(x, fy)}{s(s+1)} + \frac{d(y, Tx)}{s^4(s+1)} \\ &\leq \frac{1}{s(s+1)}[d(x, fy) + d(y, Tx)]. \end{aligned}$$

Let $x_0 \in X$, $x_1 = Tx_0$ and $x_2 = fx_1$. Define the sequence $\{x_n\}$ by $x_{2n+1} = Tx_{2n}$ and $x_{2n+2} = fx_{2n+1}$, for every $n \geq 0$. It follows that

$$\begin{aligned} d(x_{2n+1}, x_{2n}) &= d(Tx_{2n}, fx_{2n-1}) \\ &\leq \frac{1}{s(s+1)}[d(x_{2n}, fx_{2n-1}) + d(x_{2n-1}, Tx_{2n})] \\ &\leq \frac{1}{s(s+1)}[d(x_{2n}, x_{2n}) + d(x_{2n-1}, x_{2n+1})] \\ &\leq \frac{1}{s(s+1)} d(x_{2n-1}, x_{2n+1}) \end{aligned}$$

$$\leq \frac{1}{s+1}[d(x_{2n-1}, x_{2n}) + d(x_{2n}, x_{2n+1})],$$

and we obtain

$$\left(1 - \frac{1}{s+1}\right) d(x_{2n+1}, x_{2n}) \leq \frac{1}{s+1} d(x_{2n-1}, x_{2n}),$$

This clearly forces

$$d(x_{2n+1}, x_{2n}) \leq \frac{1}{s} d(x_{2n-1}, x_{2n}).$$

According to Lemma 1, we conclude that $\{x_n\}$ is a b-Cauchy sequence.

In the notation of [21], Ψ stands for the family of all functions $\psi, \varphi : [0, \infty) \to [0, \infty)$ with the properties:
(a) $\varphi(t) < \psi(t)$ for each $t > 0, \varphi(0) = \psi(0) = 0$;
(b) φ and ψ are continuous functions;
(c) ψ is increasing,
and Θ denotes the set of all functions $\theta : [0, \infty)^4 \to [0, \infty)$ satisfying the following conditions:
(a) θ is continuous,
(b) $\theta(p, q, r, s) = 0$ if and only if $pqrs = 0$.

Example 1. The following functions belong to Θ:
1) $\theta(p, q, r, s) = k \min\{p, q, r, s\} + p \cdot q \cdot r \cdot s, \quad k > 0$,
2) $\theta(p, q, r, s) = \ln(1 + p \cdot q \cdot r \cdot s)$.

Also in [21], a partially ordered set in a b-metric space and a regular space were introduced.

Definition 2 ([21] Definition 2.1)**.** *Let X be a nonempty set. Then $(X, d, \preceq)$ is called a partially ordered b-metric space if d is a b-metric on a partially ordered set $(X, \preceq)$. The space $(X, d, \preceq)$ is called regular if the following condition holds: if a non-decreasing sequence $\{x_n\}$ tends to x, then $x_n \preceq x$ for all n.*

The next theorem is the main result in [21].

Theorem 5. *Suppose that $(X, d, s \geq 1, \preceq)$ is a partially ordered complete b-metric space and $\{T_n\}$ a nondecreasing sequence of self maps on X. If there exists a continuous function $\alpha : X \times X \to [0, 1)$ such that for all $x, y \in X$*

$$\alpha(T_i x, T_j y) \leq a_{i,j} \alpha(x, y)$$

and

$$\psi\left(s^3 d(T_i x, T_j y)\right) \leq \alpha(x, y) \varphi(M_{i,j}(x, y)) + \theta(d(x, T_i x), d(y, T_j y), d(x, T_j y), d(y, T_i x)),$$

for all $x, y \in X$ *with* $x \preceq y$*, where* $(\psi, \varphi) \in \Psi, \theta \in \Theta$ *and*

$$M_{i,j}(x,y) = \max\left\{d(x,y), d(x,T_i x), d(y,T_j y), \frac{d(x,T_j y) + d(y,T_i x)}{2s}\right\},$$

and $0 \le a_{i,j}$ $(i, j \in \mathbb{N})$*, satisfy*

(*i*) $A_n = \prod_{i=1}^{n} a_{i,i+1} < 1$*, for all* n*,*

(*ii*) $\overline{\lim_{i\to\infty}} a_{i,j} < 1$*, for each* j*.*

Suppose that:

(*i*) T *is continuous, or*

(*ii*) $(X, d, \preceq)$ *is regular.*

If there exists $x_0 \in X$ *such that* $x_0 \preceq Tx_0$*, then all* $T_n's$ *have a common fixed point in* X*.*

Remark 8. The proof of Theorem 5 can be much shorter using Lemma 1. Indeed, on page 59, the proof should start with $\psi(s^3 d(T_n(x_{n-1}), T_{n+1}(x_n))$, and it follows that

$$\begin{aligned}
&\psi(s^3 d(T_n(x_{n-1}), T_{n+1}(x_n))) \\
&\qquad \le \alpha(x_{n-1}, x_n)\varphi(M_{n,n+1}(x_{n-1}, x_n)) \\
&\qquad\quad + \theta(d(x_{n-1}, x_n), d(x_n, x_{n+1}), d(x_{n-1}, x_{n+1}), d(x_n, x_n)) \\
&\qquad = \alpha(x_{n-1}, x_n)\varphi(M_{n,n+1}(x_{n-1}, x_n)) \\
&\qquad \le A_{n-1}\alpha(x_0, x_1)\varphi(\max\{d(x_{n-1}, x_n), d(x_n, x_{n+1})\}).
\end{aligned}$$

If $\max\{d(x_{n-1}, x_n), d(x_n, x_{n+1}) = d(x_n, x_{n+1})$, then we have

$$\begin{aligned}
\psi(d(x_n, x_{n+1})) &\le \psi(s^3 d(T_n(x_{n-1}), T_{n+1}(x_n))) \\
&\le A_{n-1}\alpha(x_0, x_1)\varphi(d(x_n, x_{n+1})) \\
&\le A_{n-1}\alpha(x_0, x_1)\psi(d(x_n, x_{n+1})) \\
&< \psi(d(x_n, x_{n+1})),
\end{aligned}$$

which is impossible.

We can conclude that $\psi(s^3 d(T_n(x_{n-1}), T_{n+1}(x_n))) \le \psi(d(x_n, x_{n+1}))$. If $x_n \neq x_{n+1}$ for all $n \in \mathbb{N}$, then

$$d(x_n, x_{n+1}) \le \lambda d(x_{n-1}, x_n), \quad \lambda \le \frac{1}{s^3}.$$

By Lemma 1, the sequence $\{x_n\}$ is a b-Cauchy sequence and $x_n \to x$ as $n \to +\infty$.

In [13] the authors defined Chatterjea's type contraction in the context of b-metric spaces and proved the following result.

Theorem 6 ([13] Theorem 2.1)**.** *Let $(X, d, s \geq 1)$ be a complete b-metric space, d a continuous function, $T : X \to X$ a Chatterjea's map such that the inequality $\sup_{n\in\mathbb{N}} d(T^n x, x) < \infty$ holds for all $x \in X$. Then*

(i) there exists a unique fixed point (say ξ) of T;

(ii) for any $x_0 \in X$ the sequence $\{x_n\}$ converges to ξ, where $x_{n+1} = Tx_n$, $n = 0, 1, 2, \ldots$;

(iii) there holds the a priori error estimate

$$d(\xi, T^m x) \leq \alpha^m \sup_{j\in\mathbb{N}} d\left(T^j x, x\right). \tag{8}$$

Recently, C. Chifu and G. Petrusel ([17], Theorem 2.1, Theorem 2.2.) considered the existence of fixed points for some multi-valued mappings in the context of b-metric spaces.

Theorem 7 ([17] Theorem 2.1)**.** *Let $(X, d, s > 1)$ be a complete b-metric space and $T : X \to P(X)$ a multivalued operator such that:*

(i) there exist $a, b, c \in \mathbb{R}_+$, $a + b + 2cs < \frac{s-1}{s^2}$ and $b + cs < \frac{1}{s}$ such that

$$\begin{aligned} H(T(x), T(y)) \leq\ & ad(x, y) + b[D(x, T(x)) + D(y, T(y))] \\ & + c[D(x, T(y)) + D(y, T(x))], \end{aligned}$$

for all $x, y \in X$;

(ii) T is closed.

In these conditions $\mathrm{Fix}(T) \neq \emptyset$.

Theorem 8 ([17] Theorem 2.2)**.** *Let $(X, d, s > 1)$ be a complete b-metric space and $T : X \to P(X)$ a multi-valued operator such that:*

(i) there exist $a, b, c \in \mathbb{R}_+$, $a + b + 2cs < \frac{s-1}{s^2}$ and $b + cs < \frac{1}{s}$ such that

$$\begin{aligned} H(T(x), T(y)) \leq\ & ad(x, y) + b[D(x, T(x)) + D(y, T(y))] \\ & + c[D(x, T(y)) + D(y, T(x))], \end{aligned}$$

for all $x, y \in X$;

(ii) T is closed.

If $S\mathrm{Fix}(T) \neq \emptyset$, then $S\mathrm{Fix}(T) = \mathrm{Fix}(T) = \{x\}$.

Remark 9. Note that we did not really have to use the condition $b+cs < \frac{1}{s}$. Indeed, since $a+b+2cs < \frac{s-1}{s^2} = \frac{1}{s} - \frac{1}{s^2} < \frac{1}{s}$, then $b+cs < a+b+2cs < \frac{1}{s}$.

Remark 10. In Theorem 7 and 8, the contractive condition

$$\begin{aligned} H(T(x), T(y)) \leq\ & ad(x, y) + b[D(x, T(x)) + D(y, T(y))] \\ & + c[D(x, T(y)) + D(y, T(x))], \end{aligned} \tag{9}$$

where $a+b+2cs<\frac{s-1}{s^2}$, can be replaced by the next two conditions:

$$H\left(T\left(x\right),T\left(y\right)\right)\leq a_1 d\left(x,y\right)+b_1 D\left(x,T\left(x\right)\right)+c_1 D\left(y,T\left(y\right)\right)+d_1\left[D\left(x,T\left(y\right)\right)+D\left(y,T\left(x\right)\right)\right],\tag{10}$$

where $a_1+\frac{b_1+c_1}{2}+2d_1 s<\frac{s-1}{s^2}$ and

$$H\left(T\left(x\right),T\left(y\right)\right)\leq \lambda_1 d\left(x,y\right)+\lambda_2 D\left(x,T\left(x\right)\right)+\lambda_3 D\left(y,T\left(y\right)\right)+\lambda_4 D\left(x,T\left(y\right)\right)+\lambda_5 D\left(y,T\left(x\right)\right),\tag{11}$$

with $\lambda_1+\frac{\lambda_2+\lambda_3}{2}+s\left(\lambda_4+\lambda_5\right)<\frac{s-1}{s^2}$.

We will now show that all the above contractive conditions are equivalent to each other.

It is easily seen that (9) implies (10) and from (10) we have (11). Substituting $H\left(T\left(y\right),T\left(x\right)\right)$ into (1.11) and combining with (11), we obtain

$$H\left(T\left(x\right),T\left(y\right)\right)\leq \lambda_1 d\left(x,y\right)+\frac{\lambda_2+\lambda_3}{2}\left[D\left(x,T\left(x\right)\right)+D\left(y,T\left(y\right)\right)\right]+\frac{\lambda_4+\lambda_5}{2}\left[D\left(x,T\left(y\right)\right)+D\left(y,T\left(x\right)\right)\right].$$

When $a=\lambda_1$, $b=\frac{\lambda_2+\lambda_3}{2}$, $c=\frac{\lambda_4+\lambda_5}{2}$, we have (9).

Each of them can be associated with the general conditions which are considered in the metric spaces:

$$H\left(T\left(x\right),T\left(y\right)\right)\leq k_1\max\left\{d\left(x,y\right),\frac{D\left(x,Tx\right)+D\left(y,Ty\right)}{2s},\frac{D\left(x,Ty\right)+D\left(y,Tx\right)}{2s}\right\},\tag{12}$$

where $k_1=a+2bs+2cs$, and

$$H\left(T\left(x\right),T\left(y\right)\right)\leq k_2\max\left\{d\left(x,y\right),D\left(x,Tx\right),D\left(y,Ty\right),\frac{D\left(x,Ty\right)+D\left(y,Tx\right)}{2s}\right\},\tag{13}$$

with $k_2=a_1+b_1+c_1+2sd_1$, and also

$$H\left(T\left(x\right),T\left(y\right)\right)\leq k_3\max\left\{d\left(x,y\right),D\left(x,Tx\right),D\left(y,Ty\right),D\left(x,Ty\right),D\left(y,Tx\right)\right\},\tag{14}$$

where $k_3=\lambda_1+\lambda_2+\lambda_3+\lambda_4+\lambda_5$.

It follows easily that (12) implies (13) and (13) implies (14).

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