sciendo FASCICULI MATHEMATICI

Nr 61 2018

DOI:10.1515/fascmath-2018-0015

T. Došenović, Z. Hassanzadeh and S. Sedghi

COUPLED FIXED POINT THEOREM IN $b-$FUZZY METRIC SPACES

Abstract. The aim of this paper is to prove a coupled coincidence fixed point theorem in complete $b-$fuzzy metric space using the concept of mixed monotone mappings, which represents a generalization of some recent results.

Key words: $b-$fuzzy metric space, coupled common fixed point theorem, t-norm, Cauchy sequence.

AMS Mathematics Subject Classification: 47H10, 54H25.

1. Introduction and preliminaries

Banach contraction principle [2] is one of the most cited theorem in nonlinear analysis. There are huge number of generalizations of mentioned theorem in different spaces which represent the generalization of metric space see ([1], [7], [11]-[14], [18], [19]).

Czerwik [4] introduced the notion of $b-$metric space, as a generalization of metric space in which the triangular inequality has been replaced by weaker one.

Definition 1. *Let X be a non-empty set, and the mapping $d : X \times X \to [0, \infty)$ satisfies:*

$(b1)$ $d(x, y) = 0$ *if and only if* $x = y$ *for all* $x, y \in X$,

$(b2)$ $d(x, y) = d(y, x)$ *for all* $x, y \in X$,

$(b3)$ *there exists a real number* $s \geq 1$ *such that* $d(x, y) \leq s(d(x, z)+d(y, z))$ *for all* $x, y, z \in X$.

Then d is called a $b-$metric on X and (X, d) is called a $b-$metric space with coefficient $s \geq 1$.

Obviously, each metric space is a $b-$metric space (for $s = 1$). However, Czerwik [4] has shown that a $b-$metric on X need not be a metric on X.

In the same paper Czerwik proved a generalization of Banach contraction principle in $b-$metric space.

As the focus of this paper is $b-$fuzzy metric spaces, first we list definitions related to fuzzy metric spaces, as well as $b-$fuzzy metric spaces.

The concept of fuzzy sets was introduced initially by Zadeh [20]. Using the results of Menger and Zadeh ([10, 20]), Kramosil and Michalek ([8]) introduced the notion of fuzzy metric space. Later, George and Veermani ([6]) modified their definition in way to associate each fuzzy metric to a Hausdorff topology.

Definition 2. *A binary operation* $* : [0,1]\times[0,1] \to [0,1]$ *is a continuous* t*-norm if it satisfies the following conditions:*

1. $*$ *is associative and commutative,*
2. $*$ *is continuous,*
3. $a * 1 = a$ *for all* $a \in [0,1]$,
4. $a * b \leq c * d$ *whenever* $a \leq c$ *and* $b \leq d$, *for each* $a, b, c, d \in [0,1]$.

Two typical examples of continuous t-norm are $a * b = a \cdot b$ and $a * b = \min(a, b)$.

Definition 3. *A 3-tuple* $(X, M, *)$ *is called a fuzzy metric space if* X *is an arbitrary (non-empty) set,* $*$ *is a continuous* t*-norm and* M *is a fuzzy set on* $X^2 \times (0,\infty)$, *satisfying the following conditions for each* $x, y, z \in X$ *and* $t, s > 0$,

1. $M(x, y, t) > 0$,
2. $M(x, y, t) = 1$ *if and only if* $x = y$,
3. $M(x, y, t) = M(y, x, t)$,
4. $M(x, y, t) * M(y, z, s) \leq M(x, z, t + s)$,
5. $M(x, y, .) : (0,\infty) \to [0,1]$ *is continuous.*

Definition 4. *A 3-tuple* $(X, M, *)$ *is called a* $b-$*fuzzy metric space if* X *is an arbitrary (non-empty) set,* $*$ *is a continuous* t*-norm and* M *is a fuzzy set on* $X^2 \times (0,\infty)$, *satisfying the following conditions for each* $x, y, z \in X$, $t, s > 0$ *and* $b \geq 1$ *be a given real number,*

1. $M(x, y, t) > 0$,
2. $M(x, y, t) = 1$ *if and only if* $x = y$,
3. $M(x, y, t) = M(y, x, t)$,
4. $M(x, y, \frac{t}{b}) * M(y, z, \frac{s}{b}) \leq M(x, z, t + s)$,
5. $M(x, y, .) : (0,\infty) \to [0,1]$ *is continuous.*

It should be noted that, the class of $b-$fuzzy metric spaces is effectively larger than that of fuzzy metric spaces, since a $b-$fuzzy metric is a fuzzy metric when $b = 1$.

We present an example shows that a $b-$fuzzy metric on X need not be a fuzzy metric on X.

Example 1. Let $M(x,y,t) = e^{\frac{-|x-y|^p}{t}}$, where $p > 1$ is a real number. We show that M is a $b-$fuzzy metric with $b = 2^{p-1}$.

Obviously conditions (1), (2), (3) and (5) of Definition 4 are satisfied.

If $1 < p < \infty$, then the convexity of the function $f(x) = x^p$ $(x > 0)$ implies

$$\left(\frac{a+c}{2}\right)^p \leq \frac{1}{2}\left(a^p + c^p\right),$$

and hence, $(a+c)^p \leq 2^{p-1}(a^p + c^p)$ holds. Therefore,

$$\begin{aligned}\frac{|x-y|^p}{t+s} &\leq 2^{p-1}\frac{|x-z|^p}{t+s} + 2^{p-1}\frac{|z-y|^p}{t+s}\\ &\leq 2^{p-1}\frac{|x-z|^p}{t} + 2^{p-1}\frac{|z-y|^p}{s}\\ &= \frac{|x-z|^p}{t/2^{p-1}} + \frac{|z-y|^p}{s/2^{p-1}}\end{aligned}$$

Thus for each $x, y, z \in X$ we obtain

$$M(x,y,t+s) = e^{\frac{-|x-y|^p}{t+s}} \geq M(x,z,\frac{t}{2^{p-1}}) * M(z,y,\frac{s}{2^{p-1}}),$$

where $a * b = a \cdot b$. So condition (4) of Definition 4 is hold and M is a $b-$fuzzy metric.

For $p = 2$ and $s = t$ we have

$$\begin{aligned}M(x,y,2t) &= e^{\frac{-(x-y)^2}{2t}}\\ &= e^{\frac{-(x-z+z-y)^2}{2t}}\\ &\geq e^{\frac{-2((x-z)^2+(z-y)^2)}{2t}}\\ &= e^{\frac{-(x-z)^2}{t}} \cdot e^{\frac{-(y-z)^2}{t}}\\ &= *(M(x,z,t), M(z,y,t)),\end{aligned}$$

where $*(a,b) = a \cdot b$. For $s \neq t$, and $p \geq 2$ $(X, M, *)$ is not a fuzzy metric space.

Example 2. Let $M(x,y,t) = e^{\frac{-d(x,y)}{t}}$ or $M(x,y,t) = \frac{t}{t+d(x,y)}$, where d is a b-metric on X and $a * c = a \cdot c$ for all $a, c \in [0,1]$. Then it is easy to show that M is a $b-$fuzzy metric.

Obviously conditions (1), (2), (3) and (5) of Definition 4 are satisfied. For each $x, y, z \in X$ we obtain

$$\begin{aligned}M(x,y,t+s) &= e^{\frac{-d(x,y)}{t+s}}\\ &\geq e^{-b\frac{d(x,z)+d(z,y)}{t+s}}\end{aligned}$$

$$
\begin{aligned}
&= e^{-b\frac{d(x,z)}{t+s}} \cdot e^{-b\frac{d(z,y)}{t+s}} \\
&\geq e^{\frac{-d(x,z)}{t/b}} \cdot e^{\frac{-d(z,y)}{s/b}} \\
&= M(x,z,\frac{t}{b}) * M(z,y,\frac{s}{b}).
\end{aligned}
$$

So condition (4) of Definition 4 is hold and M is a $b-$ fuzzy metric. Similarly, it is easy to see that $M(x,y,t) = \frac{t}{t+d(x,y)}$ is a $b-$ fuzzy metric.

Before stating and proving our results, we present some definition and proposition in $b-$metric space.

Definition 5. *Let $f : \mathbb{R} \to \mathbb{R}$ be a function. Then f is called $b-$nondecreasing, if $x > by$ implies $f(x) \geq f(y)$ for each $x, y \in \mathbb{R}$.*

Lemma 1 ([15]). *Let $(X, M, *)$ be a $b-$fuzzy metric space. Then $M(x,y,t)$ is $b-$nondecreasing with respect to t, for all x, y in X. Also,*

$$M(x,y,b^n t) \geq M(x,y,t), \;\; n \in \mathbb{N}.$$

Let $(X, M, *)$ be a $b-$fuzzy metric space. For $t > 0$, the open ball $B(x,r,t)$ with center $x \in X$ and radius $0 < r < 1$ is defined by

$$B(x,r,t) = \{y \in X : M(x,y,t) > 1 - r\}.$$

We recall the notions of convergence and completeness in a $b-$fuzzy metric space. Let $(X, M, *)$ be a $b-$fuzzy metric space. Let τ be the set of all $A \subset X$ with $x \in A$ if and only if there exists $t > 0$ and $0 < r < 1$ such that $B(x,r,t) \subset A$. Then τ is a topology on X (induced by the $b-$fuzzy metric M). A sequence $\{x_n\}$ in X converges to x if and only if $M(x_n, x, t) \to 1$ as $n \to \infty$, for each $t > 0$. It is called a Cauchy sequence if for each $0 < \varepsilon < 1$ and $t > 0$, there exists $n_0 \in \mathbb{N}$ such that $M(x_n, x_m, t) > 1 - \varepsilon$ for each $n, m \geq n_0$. The $b-$fuzzy metric space $(X, M, *)$ is said to be complete if every Cauchy sequence is convergent. A subset A of X is said to be F-bounded if there exists $t > 0$ and $0 < r < 1$ such that $M(x,y,t) > 1 - r$ for all $x, y \in A$.

Lemma 2 ([15]). *In a $b-$fuzzy metric space $(X, M, *)$ the following assertions hold:*

(i) If sequence $\{x_n\}$ in X converges to x, then x is unique,

(ii) If sequence $\{x_n\}$ in X is converges to x, then sequence $\{x_n\}$ is a Cauchy sequence.

In $b-$fuzzy metric space we have the following proposition.

Proposition 1 ([16], Prop. 1.10). *Let $(X, M, *)$ be a $b-$fuzzy metric space and suppose that $\{x_n\}$ is b-convergent to x then we have*

$$M(x,y,\frac{t}{b}) \leq \limsup_{n\to\infty} M(x_n,y,t) \leq \; M(x,y,bt),$$

$$M(x,y,\frac{t}{b}) \leq \liminf_{n\to\infty} M(x_n,y,t) \ \leq M(x,y,bt).$$

Remark 1. In general, a $b-$fuzzy metric is not continuous.

Definition 6 ([3]). *An element $(x,y) \in X \times X$ is called a coupled fixed point of a mapping $F : X \times X \to X$ if $F(x,y) = x$ and $F(y,x) = y$.*

Definition 7 ([9]). *An element $(x,y) \in X \times X$ is called a coupled coincidence point of the mappings $F : X \times X \to X$ and $g : X \to X$ if $F(x,y) = gx$ and $F(y,x) = gy$.*

Definition 8 ([9]). *Let X be a nonempty set. Then we say that the mappings $F : X \times X \to X$ and $g : X \to X$ are commutative if $gF(x,y) = F(gx, gy)$.*

Theorem 1 ([17]). *Let $(X, M, *)$ be a complete b-fuzzy metric space. Let $F : X \times X \to X$ and $g : X \to X$ be two functions such that*

$$(1) \qquad M(F(x,y),F(u,v),t) \geq \phi(\min\{M(gx,gu,b^2t),M(gy,gv,b^2t)\}),$$

for all $x,y,u,v \in X$ and $t > 0$. Assume that F and g satisfy the following conditions:

1. *$F(X \times X) \subseteq g(X)$,*
2. *$g(X)$ is complete, and*
3. *g is continuous and commutes with F.*

If $\phi \in \Phi$, then there is a unique x in X such that $gx = F(x,x) = x$.

2. The main results

Let Φ denote the class of all functions $\phi : [0,1] \to [0,1]$ such that ϕ is increasing, continuous, $\phi(t) > t$ for all $t \in (0,1)$.

Note that $\phi(0) = 0$ and $\phi(1) = 1$, then $\phi(t) \geq t$ for all $t \in [0,1]$.

We start our work by proving the following crucial lemma.

Lemma 3. *Let $(X, M, *)$ be a b-fuzzy metric space with $b \geq 1$ and let $F : X \times X \to X$ and $g : X \to X$ be two mappings such that*

$$(2) \qquad M(F(x,y),F(u,v),t) \geq \phi(\min\{M(gx,gu,t),M(gy,gv,t)\}),$$

for some $\phi \in \Phi$ and for all $x,y,u,v \in X$ and $t > 0$. Assume that (x,y) is a coupled coincidence point of the mappings F and g. Then $F(x,y) = gx = gy = F(y,x)$.

Proof. Since (x, y) is a coupled coincidence point of the mappings F and g, we have $gx = F(x, y)$ and $gy = F(y, x)$. Assume $gx \neq gy$. Then by (2), we get

$$\begin{aligned} M(gx, gy, t) &= M(F(x, y), F(y, x), t) \\ &\geq \phi(\min\{M(gx, gy, t), M(gy, gx, t)\}) \\ &= \phi(M(gx, gy, t)) \\ &> M(gx, gy, t), \end{aligned}$$

which is a contradiction. So $gx = gy$, and hence $F(x, y) = gx = gy = F(y, x)$. ∎

The following is the main result of this section.

Theorem 2. *Let $(X, M, *)$ be a complete b-fuzzy metric space. Let $F : X \times X \to X$ and $g : X \to X$ be two functions such that*

$$\text{(3)} \quad \begin{aligned} M(F(x, y), F(u, v), t) \geq\ & \phi(\min\{M(gx, gu, b^2t), M(gy, gv, b^2t), \\ & M(gx, F(x, y), b^2t), M(gu, F(u, v), b^2t), \\ & M(gy, F(y, x), b^2t), M(gv, F(v, u), b^2t)\}) \end{aligned}$$

for all $x, y, u, v \in X$ and $t > 0$. Assume that F and g satisfy the following conditions:

(i) $F(X \times X) \subseteq g(X)$,
(ii) $g(X)$ is complete, and
(iii) g is continuous and commutes with F.

If $\phi \in \Phi$, then there is a unique x in X such that $gx = F(x, x) = x$.

Proof. Let $x_0, y_0 \in X$. Since $F(X \times X) \subseteq g(X)$, we can choose $x_1, y_1 \in X$ such that $gx_1 = F(x_0, y_0)$ and $gy_1 = F(y_0, x_0)$. Again since $F(X \times X) \subseteq g(X)$, we can choose $x_2, y_2 \in X$ such that $gx_2 = F(x_1, y_1)$ and $gy_2 = F(y_1, x_1)$. Continuing this process, we can construct two sequences (x_n) and (y_n) in X such that $gx_{n+1} = F(x_n, y_n)$ and $gy_{n+1} = F(y_n, x_n)$. For $n \in \mathbb{N} \cup \{\mathbf{0}\}$, by (3) we have

$$\begin{aligned} M(gx_{n-1}, gx_n, t) &= M(F(x_{n-2}, y_{n-2}), F(x_{n-1}, y_{n-1}), t) \\ &\geq \phi(\min\{M(gx_{n-2}, gx_{n-1}, b^2t), M(gy_{n-2}, gy_{n-1}, b^2t), \\ &\qquad M(gx_{n-2}, gx_{n-1}, b^2t), M(gx_{n-1}, gx_n, b^2t), \\ &\qquad M(gy_{n-2}, gy_{n-1}, b^2t), M(gy_{n-1}, gy_n, b^2t), \}). \end{aligned}$$

Similarly by (3) we have

$$\begin{aligned} M(gy_{n-1}, gy_n, t) &= M(F(y_{n-2}, x_{n-2}), F(y_{n-1}, x_{n-1}), t) \\ &\geq \phi(\min\{M(gy_{n-2}, gy_{n-1}, b^2t), M(gx_{n-2}, gx_{n-1}, b^2t), \end{aligned}$$

$$M(gy_{n-2}, gy_{n-1}, b^2t), M(gy_{n-1}, gy_n, b^2t),$$
$$M(gx_{n-2}, gx_{n-1}, b^2t), M(gx_{n-1}, gx_n, b^2t), \}).$$

Hence, we have

$$\begin{aligned} a_n(t) &= \min\{M(gx_{n-1}, gx_n, t), M(gy_{n-1}, gy_n, t)\} \\ &\geq \phi(\min\{M(gx_{n-2}, gx_{n-1}, b^2t), M(gy_{n-2}, gy_{n-1}, b^2t), \\ &\quad M(gx_{n-1}, gx_n, b^2t), M(gy_{n-1}, gy_n, b^2t)\}) \\ &\geq \phi(\min\{M(gx_{n-2}, gx_{n-1}, b^2t), M(gy_{n-2}, gy_{n-1}, b^2t), \\ &\quad \min\{M(gx_{n-1}, gx_n, b^2t), M(gy_{n-1}, gy_n, b^2t)\}\}). \end{aligned}$$

If min $=$ $\min\{M(gx_{n-1}, gx_n, b^2t), M(gy_{n-1}, gy_n, b^2t)\}$ and using Lemma 1 we have

$$\begin{aligned} &\min\{M(gx_{n-1}, gx_n, t), M(gy_{n-1}, gy_n, t)\} \\ &\qquad \geq \phi(\min\{M(gx_{n-1}, gx_n, b^2t), M(gy_{n-1}, gy_n, b^2t)\}) \\ &\qquad > \min\{M(gx_{n-1}, gx_n, b^2t), M(gy_{n-1}, gy_n, b^2t) \\ &\qquad \geq \min\{M(gx_{n-1}, gx_n, t), M(gy_{n-1}, gy_n, t). \end{aligned}$$

So, we get contraction, and therefore we have

$$a_n(t) \geq \phi(\min\{M(gx_{n-2}, gx_{n-1}, b^2t), M(gy_{n-2}, gy_{n-1}, b^2t)\}).$$

Now, we have

$$a_n(t) \geq \phi(a_{n-1}(b^2t)) > a_{n-1}(b^2t) \geq a_{n-1}(t).$$

Thus $a_n(t)$ is increasing sequence in $[0,1]$ for every $t > 0$. Therefore, tends to a limit $a(t) \leq 1$. We claim that $a(t) = 1$. If $a(t) < 1$ on making $n \to \infty$ in the above inequality we get $a(t) \geq \phi(a(b^2t)) > a(b^2t) \geq a(t)$, a contradiction. Hence $a(t) = 1$, i.e.,

$$\lim_{n\to\infty} \min\{M(gx_{n-1}, gx_n, t), M(gy_{n-1}, gy_n, t)\} = 1,$$

respectively

$$\lim_{n\to\infty} M(gx_n, gx_{n+1}, t) = 1, \qquad \lim_{n\to\infty} M(gy_n, gy_{n+1}, t) = 1.$$

Now, we prove that (gx_n) and (gy_n) are Cauchy sequence in $g(X)$ for $n \in \mathbb{N}$.

First, we prove that for every $\epsilon \in (0,1)$, there exist two numbers $n, m \in \mathbb{N}$ such that

$$M(gx_n, gx_m, t) \wedge M(gy_n, gy_m, t) > 1 - \epsilon,$$

where

$$M(gx_n, gx_m, t) \wedge M(gy_n, gy_m, t) = \min\{M(gx_n, gx_m, t), M(gy_n, gy_m, t)\}.$$

Suppose that this is not true. Then there is an $\epsilon \in (0,1)$ such that for each integer k, there exist integers $m(k)$ and $n(k)$ with $m(k) > n(k) \geq k$ such that

$$\begin{aligned} d_k(t) &= M(gx_{n(k)}, gx_{m(k)}, t) \wedge M(gy_{n(k)}, gy_{m(k)}, t) \\ &\leq 1-\epsilon \quad \text{for} \quad k = 1, 2, \cdots . \end{aligned} \tag{4}$$

We may assume that

$$M(gx_{n(k)}, gx_{m(k)-1}, t) \wedge M(gy_{n(k)}, gy_{m(k)-1}, t) > 1-\epsilon, \tag{5}$$

by choosing $m(k)$ be the smallest number exceeding $n(k)$ for which (4) holds. Using (4), and the fact that $a * b \geq (a \wedge c) * (b \wedge d)$ we have

$$\begin{aligned} 1-\epsilon &\geq d_k(t) \\ &\geq [M(gx_{n(k)}, gx_{m(k)-1}, \frac{t}{2b}) * M(gx_{m(k)-1}, gx_{m(k)}, \frac{t}{2b})] \\ &\wedge [M(gy_{n(k)}, gy_{m(k)-1}, \frac{t}{2b}) * M(gy_{m(k)-1}, gy_{m(k)}, \frac{t}{2b})] \\ &\geq [M(gx_{n(k)}, gx_{m(k)-1}, \frac{t}{2b}) \wedge M(gy_{n(k)}, gy_{m(k)-1}, \frac{t}{2b})] \\ &* [M(gx_{m(k)-1}, gx_{m(k)}, \frac{t}{2b}) \wedge M(gy_{m(k)-1}, gy_{m(k)}, \frac{t}{2b})] \\ &\wedge [M(gx_{n(k)}, gx_{m(k)-1}, \frac{t}{2b}) \wedge M(gy_{n(k)}, gy_{m(k)-1}, \frac{t}{2b})] \\ &* [M(gx_{m(k)-1}, gx_{m(k)}, \frac{t}{2b}) \wedge M(gy_{m(k)-1}, gy_{m(k)}, \frac{t}{2b})] \\ &\geq [M(gx_{m(k)-1}, gx_{m(k)}, \frac{t}{2b}) \wedge M(gy_{m(k)-1}, gy_{m(k)}, \frac{t}{2b})] * a_k(\frac{t}{2b}), \end{aligned}$$

Thus, as $k \to \infty$ in the above inequality we have

$$1-\epsilon \geq \lim_{k\to\infty} d_k(t) \geq (1-\epsilon) * \lim_{k\to\infty} a_k(\frac{t}{2b}) = 1-\epsilon$$

that is

$$\lim_{k\to\infty} d_k(t) = 1-\epsilon,$$

for every $t > 0$.

On the other hand, we have

$$\begin{aligned}
d_k(t) &\geq [M(gx_{n(k)}, gx_{n(k)+1}, \frac{t}{3b}) * M(gx_{n(k)+1}, gx_{m(k)+1}, \frac{t}{3b}) \\
&* M(gx_{m(k)+1}, gx_{m(k)}, \frac{t}{3b})] \\
&\wedge [M(gy_{n(k)}, gy_{n(k)+1}, \frac{t}{3b}) * M(gy_{n(k)+1}, gy_{m(k)+1}, \frac{t}{3b}) \\
&* M(gy_{m(k)+1}, gy_{m(k)}, \frac{t}{3b})] \\
&\geq [M(gx_{n(k)}, gx_{n(k)+1}, \frac{t}{3b}) \wedge M(gy_{n(k)}, gy_{n(k)+1}, \frac{t}{3b})] \\
&* [M(gx_{n(k)+1}, gx_{m(k)+1}, \frac{t}{3b}) \wedge M(gy_{n(k)+1}, gy_{m(k)+1}, \frac{t}{3b})] \\
&* [M(gx_{m(k)+1}, gx_{m(k)}, \frac{t}{3b}) \wedge M(gy_{m(k)+1}, gy_{m(k)}, \frac{t}{3b})] \\
&\wedge [M(gx_{n(k)}, gx_{n(k)+1}, \frac{t}{3b}) \wedge M(gy_{n(k)}, gy_{n(k)+1}, \frac{t}{3b})] \\
&* [M(gx_{n(k)+1}, gx_{m(k)+1}, \frac{t}{3b}) \wedge M(gy_{n(k)+1}, gy_{m(k)+1}, \frac{t}{3b})] \\
&* [M(gx_{m(k)+1}, gx_{m(k)}, \frac{t}{3b}) \wedge M(gy_{m(k)+1}, gy_{m(k)}, \frac{t}{3b})] \\
&= [M(gx_{n(k)}, gx_{n(k)+1}, \frac{t}{3b}) \wedge M(gy_{n(k)}, gy_{n(k)+1}, \frac{t}{3b})] \\
&* [M(gx_{n(k)+1}, gx_{m(k)+1}, \frac{t}{3b}) \wedge M(gy_{n(k)+1}, gy_{m(k)+1}, \frac{t}{3b})] \\
&* [M(gx_{m(k)+1}, gx_{m(k)}, \frac{t}{3b}) \wedge M(gy_{m(k)+1}, gy_{m(k)}, \frac{t}{3b})] \\
&\geq a_k(\frac{t}{3b}) * [M(gx_{n(k)+1}, gx_{m(k)+1}, \frac{t}{3b}) \\
&\wedge M(gy_{n(k)+1}, gy_{m(k)+1}, \frac{t}{3b})] * a_k(\frac{t}{3b}) \\
&= a_k(\frac{t}{3b}) * \min \begin{Bmatrix} M(F(x_{n(k)}, y_{n(k)}), F(x_{m(k)}, y_{m(k)}), \frac{t}{3b}), \\ M(F(y_{n(k)}, x_{n(k)}), F(y_{m(k)}, x_{m(k)}), \frac{t}{3b}) \end{Bmatrix} * a_k(\frac{t}{3b})
\end{aligned}$$

From

$$\begin{aligned}
&M(gx_{n(k)+1}, gx_{m(k)+1}, \frac{t}{3b}) \\
&\quad = M(F(x_{n(k)}, y_{n(k)}), F(x_{m(k)}, y_{m(k)}), \frac{t}{3b}) \\
&\quad \geq \phi(\min\{M(gx_{n(k)}, gx_{m(k)}, \frac{tb}{3}), M(gy_{n(k)}, gy_{m(k)}, \frac{tb}{3}), \\
&\qquad M(gx_{n(k)}, gx_{n(k)+1}, \frac{tb}{3}), M(gx_{m(k)}, gx_{m(k)+1}, \frac{tb}{3}), \\
&\qquad M(gy_{n(k)}, gy_{n(k)+1}, \frac{tb}{3}), M(gy_{m(k)}, gy_{m(k)+1}, \frac{tb}{3})\}),
\end{aligned}$$

and

$$\begin{aligned}
&M(gy_{n(k)+1}, gy_{m(k)+1}, \frac{t}{3b}) \\
&\quad = M(F(y_{n(k)}, x_{n(k)}), F(y_{m(k)}, x_{m(k)}), \frac{t}{3b}) \\
&\quad \geq \phi(\min\{M(gy_{n(k)}, gy_{m(k)}, \frac{tb}{3}), M(gx_{n(k)}, gx_{m(k)}, \frac{tb}{3}), \\
&\qquad M(gy_{n(k)}, gy_{n(k)+1}, \frac{tb}{3}), M(gy_{m(k)}, gy_{m(k)+1}, \frac{tb}{3}), \\
&\qquad M(gx_{n(k)}, gx_{n(k)+1}, \frac{tb}{3}), M(gx_{m(k)}, gx_{m(k)+1}, \frac{tb}{3})\})
\end{aligned}$$

we have

$$\begin{aligned}
&\min\{M(F(x_{n(k)}, y_{n(k)}), F(x_{m(k)}, y_{m(k)}), \frac{t}{3b}), \\
&\qquad M(F(y_{n(k)}, x_{n(k)}), F(y_{m(k)}, x_{m(k)}), \frac{t}{3b})\} \\
&\quad \geq \phi(\min\{M(gx_{n(k)}, gx_{m(k)}, \frac{tb}{3}), M(gy_{n(k)}, gy_{m(k)}, \frac{tb}{3}), \\
&\qquad M(gx_{n(k)}, gx_{n(k)+1}, \frac{tb}{3}), M(gx_{m(k)}, gx_{m(k)+1}, \frac{tb}{3}), \\
&\qquad M(gy_{n(k)}, gy_{n(k)+1}, \frac{tb}{3}), M(gy_{m(k)}, gy_{m(k)+1}, \frac{tb}{3})\}) \\
&\quad \geq \phi(\min\{\min\{M(gx_{n(k)}, gx_{m(k)}, \frac{tb}{3}), M(gy_{n(k)}, gy_{m(k)}, \frac{tb}{3})\}, \\
&\qquad \min\{M(gx_{n(k)}, gx_{n(k)+1}, \frac{tb}{3}), M(gx_{m(k)}, gx_{m(k)+1}, \frac{tb}{3})\}, \\
&\qquad \min\{M(gy_{n(k)}, gy_{n(k)+1}, \frac{tb}{3}), M(gy_{m(k)}, gy_{m(k)+1}, \frac{tb}{3})\}\}) \\
&\quad = \phi(\min\{d_k(\frac{tb}{3}), a_k(\frac{tb}{3}), a_k(\frac{tb}{3})\}) \\
&\quad = \phi(\min\{d_k(\frac{tb}{3}), a_k(\frac{tb}{3})\}).
\end{aligned}$$

Therefore,

$$d_k(t) \geq a_k(\frac{tb}{3}) * \phi(\min\{d_k(\frac{tb}{3}), a_k(\frac{tb}{3})\}) * a_k(\frac{tb}{3}).$$

Thus, as $k \to \infty$ in the above inequality we have

$$1 - \epsilon \geq 1 * \phi(1 - \epsilon) * 1 = \phi(1 - \epsilon) > 1 - \epsilon$$

which is a contradiction.

Thus (gx_n) and (gy_n) are Cauchy in $g(X)$. Since $g(X)$ is complete, we get (gx_n) and (gy_n) are convergent to some $x \in X$ and $y \in X$ respectively.

Since g is continuous, we have (ggx_n) is convergent to gx and (ggy_n) is convergent to gy. Also, since g and F are commute, we have

$$ggx_{n+1} = g(F(x_n, y_n)) = F(gx_n, gy_n),$$

and

$$ggy_{n+1} = g(F(y_n, x_n)) = F(gy_n, gx_n).$$

Thus,

$$\begin{aligned}
&\min\{M(ggx_{n+1}, F(x,y), t), M(ggy_{n+1}, F(y,x), t)\} \\
&\quad = \min\{M(F(gx_n, gy_n), F(x,y), t), M(F(gy_n, gx_n), F(y,x), t)\} \\
&\quad \geq \phi(\min\{M(ggx_n, gx, b^2t), M(ggy_n, gy, b^2t), \\
&\qquad M(ggx_n, ggx_{n+1}, b^2t), M(gx, F(x,y), b^2t), \\
&\qquad M(ggy_n, ggy_{n+1}, b^2t), M(gy, F(y,x), b^2t)\}),
\end{aligned}$$

Letting $n \to \infty$ and using the Proposition 1 we get that

$$\begin{aligned}
&\min\{M(gx, F(x,y), bt), M(gy, F(y,x), bt) \\
&\quad \geq \limsup_{n\to\infty} \min\{M(F(gx_n, gy_n), F(x,y), t), \\
&\qquad M(F(gy_n, gx_n), F(x,y), t)\} \\
&\quad \geq \phi(\min\{M(gx, F(x,y), bt), M(gy, F(y,x), bt)\}).
\end{aligned}$$

This is possible only if $gx = F(x,y)$ and $gy = F(y,x)$.

Hence, by Lemma 1, we have

$$\begin{aligned}
&M(F(x,y), F(y,x), b^2t) = M(gx, gy, b^2t) \\
&\quad \geq M(gx, gy, t) = M(F(x,y), F(y,x), t) \\
&\quad \geq \phi(\min\left\{\begin{array}{l} M(gx, gy, b^2t), M(gy, gx, b^2t), M(gx, F(x,y), b^2t), \\ M(gy, F(y,x), b^2t), M(gy, F(y,x), b^2t), \\ M(gx, F(x,y), b^2t) \end{array}\right\}) \\
&\quad = \phi(\min\{M(gx, gy, b^2t), M(gy, gx, b^2t)\}).
\end{aligned}$$

So, by Lemma 3 we have $gx = F(x,y) = gy = F(y,x)$.

Thus, using Proposition 1 we have

$$\begin{aligned}
M(x, gx, bt) &\geq \limsup_{n\to\infty} M(gx_{n+1}, gx, t) \\
&= \limsup_{n\to\infty} M(F(x_n, y_n), F(x,y), t) \\
&\geq \limsup_{n\to\infty} \phi(\min\{M(gx_n, gx, b^2t), M(gy_n, gy, b^2t), \\
&\quad M(gx_n, gx_{n+1}, b^2t), M(gx, gx, b^2t), \\
&\quad M(gy_n, gy_{n+1}, b^2t), M(gy, gy, b^2t)\}) \\
&\geq \phi(\min\{M(x, gx, bt), M(y, gy, bt)\}).
\end{aligned}$$

Similarly, we may show that

$$M(y, gy, bt) \geq \phi(\min\{M(x, gx, bt), M(y, gy, bt)\}).$$

Thus

$$\begin{aligned}
&\min\{M(x, gx, bt), M(y, gy, bt)\} \\
&\qquad \geq \phi(\min\{M(x, gx, bt), M(y, gy, bt)\}) \\
&\qquad > \min\{M(x, gx, bt), M(y, gy, bt)\}.
\end{aligned}$$

The last inequality happened only if $M(x, gx, t) = 1$ and $M(y, gy, t) = 1$. Hence $x = gx$ and $y = gy$. Thus we get

$$gx = F(x, x) = x.$$

To prove the uniqueness, let $z \in X$ with $z \neq x$ such that

$$z = gz = F(z, z).$$

Then

$$\begin{aligned}
M(x, z, t) &= M(F(x, x), F(z, z), t) \\
&\geq \phi(\min\{M(gx, gz, b^2t), M(gx, gz, b^2t), M(gx, gx, b^2t), \\
&\quad M(gz, gz, b^2t), M(gx, gx, b^2t), M(gz, gz, b^2t)\}) \\
&= \phi(M(gx, gz, b^2t)) \\
&> M(gx, gz, b^2t) = M(x, z, b^2t) \\
&\geq M(x, z, t).
\end{aligned}$$

We get $M(x, z, t) > M(x, z, t)$, which is a contradiction. Thus F and g have a unique common fixed point. ■

Remark 2. Let (x, y) and (u, v) be coupled coincidence point of mapping F and g. Then we get Theorem 1. That is the Theorem 2 is generalization of Theorem 1.

Corollary 1. *Let $(X, M, *)$ be a complete b-fuzzy metric space. Let $F : X \times X \to X$ be function such that*

$$\begin{aligned}
(6) \quad M(F(x, y), F(u, v), t) \geq\ & \phi(\min\{M(x, u, b^2t), M(y, v, b^2t), \\
& M(x, F(x, y), b^2t), M(u, F(u, v), b^2t), \\
& M(y, F(y, x), b^2t), M(v, F(v, u), b^2t)\})
\end{aligned}$$

for all $x, y, u, v \in X$ and $t > 0$, and $\phi \in \Phi$. Then there is a unique x in X such that $F(x, x) = x$.

Proof. Let $g(x) = x$. Then all conditions of previous theorem are satisfied. ■

Corollary 2. *Let $(X, M, *)$ be a complete fuzzy metric space. Let $F : X \times X \to X$ and $g : X \to X$ be two functions such that*

$$(7) \quad M(F(x,y), F(u,v), t) \geq \phi(\min\{M(gx, gu, t), M(gy, gv, t), M(gx, F(x,y), t), M(gu, F(u,v), b^2t), M(gy, F(y,x), b^2t), M(gv, F(v,u), b^2t)\})$$

for all $x, y, u, v \in X$ and $t > 0$. Assume that F and g satisfy the following conditions:

1. *$F(X \times X) \subseteq g(X)$,*
2. *$g(X)$ is complete, and*
3. *g is continuous and commutes with F.*

If $\phi \in \Phi$, then there is a unique x in X such that $gx = F(x,x) = x$.

Proof. Let $b = 1$. Then all conditions of previous theorem are satisfied. ■

Example 3. Let $X = [0,1]$ and $a * c = a \cdot c$ for all $a, c \in [0,1]$ and let M be the $b-$fuzzy set on $X \times X \times (0, +\infty)$ defined as follows:

$$M(x,y,t) = e^{\frac{-(x-y)^2}{t}},$$

for all $t \in \mathbb{R}^+$. Then $(X, M, *)$ is a $b-$fuzzy metric space for $b = 2$. Define $g(x) = \frac{x}{4}$, $F(x,y) = \frac{2x+y}{32\sqrt{2}}$ and $\phi(t) = \sqrt{t}$, , for $t > 0$. It is evident that $F(X \times X) \subseteq g(X)$, g is continuous, $g(X) = [0, \frac{x}{4}]$ is complete and g commutes with F.

Since,

$$\begin{aligned}
(\frac{2x+y}{32\sqrt{2}} - \frac{2u+v}{32\sqrt{2}})^2 &= (\frac{2x-2u}{32\sqrt{2}} + \frac{y-v}{32\sqrt{2}})^2 \\
&\leq \frac{2}{32}[\frac{1}{4}(\frac{2x}{4} - \frac{2u}{4})^2 + \frac{1}{4}(\frac{y}{4} - \frac{v}{4})^2] \\
&\leq \frac{1}{16}[(\frac{x}{4} - \frac{u}{4})^2 + (\frac{y}{4} - \frac{v}{4})^2] \\
&\leq \frac{2}{16}\max\{(\frac{x}{4} - \frac{u}{4})^2, (\frac{y}{4} - \frac{v}{4})^2\} \\
&= \frac{1}{8}\max\{(\frac{x}{4} - \frac{u}{4})^2, (\frac{y}{4} - \frac{v}{4})^2\},
\end{aligned}$$

hence it follows that

$$M(F(x,y), F(u,v), t) = e^{\frac{-(\frac{2x+y}{32\sqrt{2}} - \frac{2u+v}{32\sqrt{2}})^2}{t}} = e^{\frac{-(\frac{2x-2u}{32\sqrt{2}} + \frac{y-v}{32\sqrt{2}})^2}{t}}$$

$$
\begin{aligned}
&\geq e^{\frac{-[(\frac{x}{4}-\frac{u}{4})^2+(\frac{y}{4}-\frac{v}{4})^2]}{8t}} \geq e^{\frac{-\max\{(\frac{x}{4}-\frac{u}{4})^2,(\frac{y}{4}-\frac{v}{4})^2\}}{8t}}\\
&= \sqrt{e^{\frac{-\max\{(\frac{x}{4}-\frac{u}{4})^2,(\frac{y}{4}-\frac{v}{4})^2\}}{4t}}}\\
&= \sqrt{\min\{e^{\frac{-(\frac{x}{4}-\frac{u}{4})^2}{4t}}, e^{\frac{-(\frac{y}{4}-\frac{v}{4})^2}{4t}}\}}\\
&= \sqrt{\min\{M(gx,gu,4t), M(gy,gv,4t)\}}\\
&\geq \sqrt{\min\left\{\begin{array}{l} M(gx,gu,4t), M(gy,gv,4t), M(gx,F(x,y),4t),\\ M(gu,F(u,v),4t), M(gy,F(y,x),4t), M(gv,F(v,u),4t)\end{array}\right\}}
\end{aligned}
$$

for all x, y, u, v in X. Thus all the conditions of Theorem 2 are satisfied and 0 is a unique point in X such that $g0 = F(0,0) = 0$.

Acknowledgment. The first author is thankful to MNTRRS-174009.

References

[1] Aghajani A., Abbas M., Common fixed point of generalized weak contractive mappings in partially ordered b-metric spaces, *Mathematica Slovaca,* 64(4)(2014), 941-960.

[2] Banach S., Sur les operations dans les ensembles abstraits el leur application aux equations integrals, *Fundam. Math.,* 3(1922), 133-181.

[3] Bhaskar T.G., Lakshmikantham V., Fixed point theorems in partially ordered metric spaces and applications, *Nonlinear Analysis,* 65(2006), 1379-1393.

[4] Czerwik S., Contraction mappings in b-metric spaces, *Acta Mathematica et Informatica Universitatis Ostraviensis,* 1(1993), 5-11.

[5] Fang J.X., On fixed point theorems in fuzzy metric spaces, *Fuzzy Sets Sys.,* 46(1992), 107-113.

[6] George A., Veeramani P., On some result in fuzzy metric space, *Fuzzy Sets Syst.,* 64(1994), 395-399.

[7] Jovanovic M., Kadelburg Z., Radenović S., Common fixed point results in metric-type spaces, *Fixed Point Theory Appl.,* (2010), Article ID 978121, 15 pages, doi:10.1155/2010/978121.

[8] Kramosil I., Michalek J., Fuzzy metric and statistical metric spaces, *Kybernetica,* 11(1975), 326-334.

[9] Lakshmikantham V., Ćirić B.Lj., Coupled fixed point theorems for nonlinear contractions in partially ordered metric spaces, *Nonlinear Analysis,* 70(2009), 4341-4349.

[10] Menger K., Statistical metrics, *Proc. Nat. Acad. Sci. U.S.A.,* 28(1942), 535-537.

[11] Miheţ D., A Banach contraction theorem in fuzzy metric spaces, *Fuzzy Sets Syst.,* 144(2004), 431-439.

[12] Moradi S., Omid M., A fixed point theorem for integral type inequality depending on another function, *Int. J. Math. Analysis,* 4(30)(2010), 1491-1499.

[13] SCHWEIZER B., SHERWOOD H., TARDIFF R.M., Contractions on PM-space examples and counterexamples, *Stochastica,* 1(1988), 5-17.
[14] SINGH S.L., CZERWIK S., KROL K., SINGH A., Coincidences and fixed points of hybrid contractions, *Tamsui Oxf. J. Math. Sci.,* 24(2008), 401-416.
[15] SEDGHI S., SHOBE N., Common fixed point theorem in b-fuzzy metric space, *Nonlinear Functional Analysis and Applications,* 17(3)(2012), 349-359.
[16] SEDGHI S., SHOBE N., Common fixed point theorem for R-weakly commuting maps in b-fuzzy metric space, *Nonlinear Functional Analysis and Applications,* 19(2)(2014), 285-295.
[17] SEDGHI S., SHOBE N., DOŠENOVIĆ T., JAVAHERI A., Coupled fixed point theorem in b-fuzzy metric spaces, *Novi Sad J. Math.,* 47(1)(2017), 77-88.
[18] SEDGHI S., SHOBE N., SELAHSHOOR M.A., A common fixed point theorem for four mappings in two complete fuzzy metric spaces, *Advances in Fuzzy Mathematics,* 1(1)(2006).
[19] SEDGHI S., TURKOGLU D., SHOBE N., Generalization common fixed point theorem in complete fuzzy metric spaces, *Journal of Computational Analysis and Applictions,* 3(9)(2007), 337-348.
[20] ZADEH L.A., Fuzzy sets, *Inform and Control,* 8(1965), 338-353.

TATJANA DOŠENOVIĆ
FACULTY OF TECHNOLOGY
BULEVAR CARA LAZARA 1
UNIVERSITY OF NOVI SAD, SERBIA
tatjanad@tf.uns.ac.rs

ZEINAB HASSANZADEH
DEPARTMENT OF MATHEMATICS
QAEMSHAHR BRANCH
ISLAMIC AZAD UNIVERSITY
QAEMSHAHR, IRAN
e-mail: Z.hassanzadeh1368@yahoo.com

SHABAN SEDGHI
DEPARTMENT OF MATHEMATICS
QAEMSHAHR BRANCH
ISLAMIC AZAD UNIVERSITY
QAEMSHAHR, IRAN
e-mail: sedghi_gh@yahoo.com

Received on 03.05.2018 and, in revised form, on 19.12.2018.