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SOME INEQUALITIES FOR WEIGHTED HARMONIC AND ARITHMETIC OPERATOR MEANS

Abstract. In this paper we establish some upper and lower bounds for the difference between the weighted arithmetic and harmonic operator means under various assumption for the positive invertible operators A, B. Some applications when A, B are bounded above and below by positive constants are given as well.

Key words: Young's inequality, convex functions, arithmetic mean-harmonic mean inequality, operator means, operator inequalities.

AMS Mathematics Subject Classification: 47A63, 47A30, 15A60, 26D15, 26D10.

1. Introduction

Throughout this paper A, B are positive invertible operators on a complex Hilbert space $(H, \langle \cdot, \cdot \rangle)$. We use the following notations for operators

$$A\nabla_\nu B := (1-\nu) A + \nu B,$$

the *weighted operator arithmetic mean,*

$$A\sharp_\nu B := A^{1/2} \left(A^{-1/2} B A^{-1/2} \right)^\nu A^{1/2},$$

the *weighted operator geometric mean* and

$$A!_\nu B := \left((1-\nu) A^{-1} + \nu B^{-1} \right)^{-1}$$

the *weighted operator harmonic mean*, where $\nu \in [0,1]$.

When $\nu = \frac{1}{2}$, we write $A\nabla B$, $A\sharp B$ and $A!B$ for brevity, respectively.

The following fundamental inequality between the weighted arithmetic, geometric and harmonic operator means holds

$$A!_\nu B \le A\sharp_\nu B \le A\nabla_\nu B \tag{1}$$

for any $\nu \in [0,1]$.

For various recent inequalities between these means we recommend the recent papers [2]-[5], [7]-[10] and the references therein.

In this paper we establish some upper and lower bounds for the difference $A\nabla_\nu B - A!_\nu B$ for $\nu \in [0,1]$ under various assumption for the positive invertible operators A, B. Some applications when A, B are bounded above and below by positive constants are given as well.

2. Main results

We have the following result:

Theorem 1. *Let A, B be positive invertible operators. Then for any $\nu \in [0,1]$ we have*

$$\begin{aligned} rA(B-A)A^{-1}(B-A)(B+A)^{-1}A &\\ \le A\nabla_\nu B - A!_\nu B &\\ \le RA(B-A)A^{-1}(B-A)(B+A)^{-1}A, & \end{aligned} \tag{2}$$

where $r = \min\{\nu, 1-\nu\}$ and $R = \max\{\nu, 1-\nu\}$.

Proof. Recall the following result obtained by Dragomir in 2006 [6] that provides a refinement and a reverse for the weighted Jensen's discrete inequality:

$$\begin{aligned} n \min_{j\in\{1,2,\dots,n\}} \{p_j\} \left[\frac{1}{n}\sum_{j=1}^{n}\Phi(x_j) - \Phi\left(\frac{1}{n}\sum_{j=1}^{n}x_j\right)\right] &\\ \le \frac{1}{P_n}\sum_{j=1}^{n}p_j\Phi(x_j) - \Phi\left(\frac{1}{P_n}\sum_{j=1}^{n}p_j x_j\right) &\\ \le n \max_{j\in\{1,2,\dots,n\}} \{p_j\} \left[\frac{1}{n}\sum_{j=1}^{n}\Phi(x_j) - \Phi\left(\frac{1}{n}\sum_{j=1}^{n}x_j\right)\right], & \end{aligned} \tag{3}$$

where $\Phi : C \to \mathbb{R}$ is a convex function defined on convex subset C of the linear space X, $\{x_j\}_{j\in\{1,2,\dots,n\}}$ are vectors in C and $\{p_j\}_{j\in\{1,2,\dots,n\}}$ are nonnegative numbers with $P_n = \sum_{j=1}^{n} p_j > 0$.

For $n = 2$, we deduce from (3) that

$$\begin{aligned} 2\min\{\nu, 1-\nu\}\left[\frac{\Phi(x)+\Phi(y)}{2} - \Phi\left(\frac{x+y}{2}\right)\right] &\\ \le \nu\Phi(x) + (1-\nu)\Phi(y) - \Phi[\nu x + (1-\nu)y] &\\ \le 2\max\{\nu, 1-\nu\}\left[\frac{\Phi(x)+\Phi(y)}{2} - \Phi\left(\frac{x+y}{2}\right)\right] & \end{aligned} \tag{4}$$

for any $x, y \in C$ and $\nu \in [0,1]$.

If we write the inequality (4) for the convex function $\Phi(x) = \frac{1}{x}$, $x > 0$, then we have

$$2r\left(\frac{\frac{1}{x}+\frac{1}{y}}{2} - \frac{2}{x+y}\right) \leq \frac{\nu}{x} + \frac{1-\nu}{y} - \frac{1}{\nu x + (1-\nu)\, y} \leq 2R\left(\frac{\frac{1}{x}+\frac{1}{y}}{2} - \frac{2}{x+y}\right) \tag{5}$$

for any $x,\ y > 0$ where $r = \min\{\nu, 1-\nu\}$ and $R = \max\{\nu, 1-\nu\}$.

If we take $y = \frac{1}{a}$, $x = \frac{1}{b}$ in (5), then we have

$$2r\left(\frac{b+a}{2} - \frac{2}{\frac{1}{b}+\frac{1}{a}}\right) \leq \nu b + (1-\nu)\, a - \left(\nu b^{-1} + (1-\nu)\, a^{-1}\right)^{-1} \leq 2R\left(\frac{b+a}{2} - \frac{2}{\frac{1}{b}+\frac{1}{a}}\right) \tag{6}$$

for any $a, b > 0$ and $\nu \in [0,1]$ where $r = \min\{\nu, 1-\nu\}$ and $R = \max\{\nu, 1-\nu\}$.

Since

$$\frac{b+a}{2} - \frac{2}{\frac{1}{b}+\frac{1}{a}} = \frac{b+a}{2} - \frac{2ab}{b+a} = \frac{1}{2}\frac{(b-a)^2}{a+b}$$

hence by (6) we have

$$r\frac{(b-a)^2}{a+b} \leq \nu b + (1-\nu)\, a - \left(\nu b^{-1} + (1-\nu)\, a^{-1}\right)^{-1} \leq R\frac{(b-a)^2}{a+b} \tag{7}$$

for any $a, b > 0$ and $\nu \in [0,1]$.

This is an inequality of interest in itself.

If we take $a = 1$ and $b = t$ in (7), then we get

$$r\,(t-1)^2\,(t+1)^{-1} \leq \nu t + 1 - \nu - \left(\nu t^{-1} + 1 - \nu\right)^{-1} \leq R\,(t-1)^2\,(t+1)^{-1} \tag{8}$$

for any $t > 0$.

If we use the continuous functional calculus for the positive invertible operator X we get

$$r\,(X-I)^2\,(X+I)^{-1} \leq \nu X + (1-\nu)\, I - \left(\nu X^{-1} + (1-\nu)\, I\right)^{-1} \leq R\,(X-I)^2\,(X+I)^{-1}. \tag{9}$$

If we write the inequality (9) for $X = A^{-1/2}BA^{-1/2}$, then we get

$$\begin{aligned}
(10) \qquad & r\left(A^{-1/2}BA^{-1/2}-I\right)^{2}\left(A^{-1/2}BA^{-1/2}+I\right)^{-1} \\
&\leq \nu A^{-1/2}BA^{-1/2}+(1-\nu)I \\
&\quad -\left(\nu\left(A^{-1/2}BA^{-1/2}\right)^{-1}+(1-\nu)I\right)^{-1} \\
&\leq R\left(A^{-1/2}BA^{-1/2}-I\right)^{2}\left(A^{-1/2}BA^{-1/2}+I\right)^{-1}.
\end{aligned}$$

If we multiply the inequality (10) both sides with $A^{1/2}$, then we get

$$\begin{aligned}
(11) \qquad & rA^{1/2}\left(A^{-1/2}BA^{-1/2}-I\right)^{2}\left(A^{-1/2}BA^{-1/2}+I\right)^{-1}A^{1/2} \\
&\leq \nu B+(1-\nu)A \\
&\quad - A^{1/2}\left(\nu\left(A^{-1/2}BA^{-1/2}\right)^{-1}+(1-\nu)I\right)^{-1}A^{1/2} \\
&\leq RA^{1/2}\left(A^{-1/2}BA^{-1/2}-I\right)^{2}\left(A^{-1/2}BA^{-1/2}+I\right)^{-1}A^{1/2}.
\end{aligned}$$

Since

$$\begin{aligned}
& A^{1/2}\left(\nu\left(A^{-1/2}BA^{-1/2}\right)^{-1}+(1-\nu)I\right)^{-1}A^{1/2} \\
&= A^{1/2}\left(\nu A^{1/2}B^{-1}A^{1/2}+(1-\nu)I\right)^{-1}A^{1/2} \\
&= A^{1/2}\left(A^{1/2}\left(\nu B^{-1}+(1-\nu)A^{-1}\right)A^{1/2}\right)^{-1}A^{1/2} \\
&= A^{1/2}\left(A^{1/2}\left(\nu B^{-1}+(1-\nu)A^{-1}\right)A^{1/2}\right)^{-1}A^{1/2} \\
&= A^{1/2}A^{-1/2}\left(\nu B^{-1}+(1-\nu)A^{-1}\right)^{-1}A^{-1/2}A^{1/2} = A!_{\nu}B
\end{aligned}$$

and

$$\begin{aligned}
& A^{1/2}\left(A^{-1/2}BA^{-1/2}-I\right)^{2}\left(A^{-1/2}BA^{-1/2}+I\right)^{-1}A^{1/2} \\
&= A^{1/2}\left(A^{-1/2}(B-A)A^{-1/2}\right)^{2}\left(A^{-1/2}(B+A)A^{-1/2}\right)^{-1}A^{1/2} \\
&= A^{1/2}A^{-1/2}(B-A)A^{-1/2}A^{-1/2}(B-A) \\
&\quad \times A^{-1/2}A^{1/2}(B+A)^{-1}A^{1/2}A^{1/2} \\
&= A(B-A)A^{-1}(B-A)(B+A)^{-1}A,
\end{aligned}$$

then by (11) we get the desired result (2). ■

Remark 1. Since, as above,

$$2\left(A\nabla B-A!B\right)=A\left(B-A\right)A^{-1}\left(B-A\right)\left(B+A\right)^{-1}A$$

then (2) can be written as

$$2r\left(A\nabla B-A!B\right)\leq A\nabla_{\nu}B-A!_{\nu}B\leq 2R\left(A\nabla B-A!B\right) \tag{12}$$

The first inequality in (12) was obtained in [10].

We observe that, if $\nu=\frac{1}{2}$, (2) becomes equality.

When some boundedness conditions are known, then we have the following result as well.

Theorem 2. *Let A, B be positive invertible operators and $M>m>0$ such that*

$$MA\geq B\geq mA. \tag{13}$$

Then for any $\nu\in[0,1]$ we have

$$rk\left(m,M\right)A\leq A\nabla_{\nu}B-A!_{\nu}B\leq RK\left(m,M\right)A \tag{14}$$

where $r=\min\{\nu,1-\nu\}$, $R=\max\{\nu,1-\nu\}$ and the bounds $K\left(m,M\right)$ and $k\left(m,M\right)$ are given by

$$K\left(m,M\right):=\begin{cases}\left(m-1\right)^{2}\left(m+1\right)^{-1} & \text{if } M<1,\\ \max\left\{\left(m-1\right)^{2}\left(m+1\right)^{-1},\right. & \\ \qquad\left.\left(M-1\right)^{2}\left(M+1\right)^{-1}\right\} & \text{if } m\leq 1\leq M,\\ \left(M-1\right)^{2}\left(M+1\right)^{-1} & \text{if } 1<m\end{cases} \tag{15}$$

and

$$k\left(m,M\right):=\begin{cases}\left(M-1\right)^{2}\left(M+1\right)^{-1} & \text{if } M<1,\\ 0 & \text{if } m\leq 1\leq M,\\ \left(m-1\right)^{2}\left(m+1\right)^{-1} & \text{if } 1<m.\end{cases} \tag{16}$$

In particular,

$$\frac{1}{2}k\left(m,M\right)A\leq A\nabla B-A!B\leq\frac{1}{2}K\left(m,M\right)A. \tag{17}$$

Proof. As in the proof of Theorem 1 we have

$$r\varphi\left(t\right)\leq\nu t+1-\nu-\left(\nu t^{-1}+1-\nu\right)^{-1}\leq R\varphi\left(t\right) \tag{18}$$

for any $t>0$, where $\varphi(t)=(t-1)^2(t+1)^{-1}$.

If we take the derivative of φ, we have

$$\begin{aligned}\varphi'(t) &= 2(t-1)(t+1)^{-1}-(t+1)^{-2}(t-1)^2\\ &= (t-1)(t+1)^{-2}[2(t+1)-(t-1)]\\ &= (t-1)(t+1)^{-2}(2t+3)\end{aligned}$$

for any $t>0$.

We observe that the function φ is decreasing on $(0,1)$ and increasing on $(1,\infty)$. We have $\varphi(0)=1$, $\varphi(1)=0$ and $\lim_{t\to\infty}\varphi(t)=\infty$.

Using the properties of the function φ we have

$$\max_{t\in[m,M]}\varphi(t)=\begin{cases}\varphi(m) \text{ if } M<1,\\ \max\{\varphi(m),\varphi(M)\} \text{ if } m\le 1\le M, \\ \varphi(M) \text{ if } 1<m,\end{cases}=K(m,M)$$

and

$$\min_{t\in[m,M]}\varphi(t)=\begin{cases}\varphi(M) \text{ if } M<1,\\ 0 \text{ if } m\le 1\le M, \\ \varphi(m) \text{ if } 1<m,\end{cases}=k(m,M).$$

From (18) we have

$$rk(m,M)\le \nu t+1-\nu-\left(\nu t^{-1}+1-\nu\right)^{-1}\le RK(m,M) \tag{19}$$

for all $t\in[m,M]$.

If we use the continuous functional calculus for the positive invertible operator X with $mI\le X\le MI$, then we have

$$\begin{aligned}rk(m,M)I &\le \nu X+(1-\nu)I-\left(\nu X^{-1}+(1-\nu)I\right)^{-1}\\ &\le RK(m,M)I.\end{aligned} \tag{20}$$

If we multiply (13) both sides by $A^{-1/2}$ we get $MI\ge A^{-1/2}BA^{-1/2}\ge mI$.

By writing the inequality (20) for $X=A^{-1/2}BA^{-1/2}$ we obtain

$$\begin{aligned}rk(m,M)I &\le \nu A^{-1/2}BA^{-1/2}+(1-\nu)I\\ &\quad-\left(\nu\left(A^{-1/2}BA^{-1/2}\right)^{-1}+(1-\nu)I\right)^{-1}\\ &\le RK(m,M)I.\end{aligned} \tag{21}$$

Finally, if we multiply both sides of (21) by $A^{1/2}$ we get the desired result (14). ■

Remark 2. Since $\varphi(t) \in [0,1]$ for $t \in [0,1]$, then $B \le A$ implies that

$$(0 \le)\, A\nabla_\nu B - A!_\nu B \le RA$$

for any $\nu \in [0,1]$. In particular,

$$(0 \le)\, A\nabla B - A!B \le \frac{1}{2}A.$$

We also have:

Theorem 3. *Let A, B be positive invertible operators. Then for any $\nu \in [0,1]$ we have*

$$\begin{aligned}(0 \le)\, & A\nabla_\nu B - A!_\nu B \\ & \le \nu(1-\nu)(B-A)A^{-1}(B-A)\left(B^{-1}+A^{-1}\right)A \\ & \le \frac{1}{4}(B-A)A^{-1}(B-A)\left(B^{-1}+A^{-1}\right)A.\end{aligned} \tag{22}$$

Proof. In [1] we obtained the following reverse of Jensen's inequality:

$$\begin{aligned}0 & \le (1-\nu)f(x) + \nu f(x) - f((1-\nu)x + \nu x) \\ & \le \nu(1-\nu)(y-x)\left[f'(y) - f'(x)\right].\end{aligned} \tag{23}$$

for any x, $y \in \mathring{I}$ and $\nu \in [0,1]$, provided the function $f : I \subset \mathbb{R} \to \mathbb{R}$ is a differentiable convex function on $\mathring{I}$, the interior of the interval I.

If we write the inequality (23) for the convex function $\Phi(x) = \frac{1}{x}$, $x > 0$, then we have

$$\frac{\nu}{y} + \frac{1-\nu}{x} - \frac{1}{\nu y + (1-\nu)x} \le \nu(1-\nu)(y-x)\left(\frac{1}{x^2} - \frac{1}{y^2}\right) \tag{24}$$

for any x, $y > 0$.

If we take $y = \frac{1}{b}$ and $x = \frac{1}{a}$ with $a, b > 0$ in (24), then we get

$$\nu b + (1-\nu)a - \left(\nu b^{-1} + (1-\nu)a^{-1}\right)^{-1} \le \nu(1-\nu)\left(\frac{1}{b} - \frac{1}{a}\right)\left(a^2 - b^2\right)$$

namely,

$$\nu b + (1-\nu)a - \left(\nu b^{-1} + (1-\nu)a^{-1}\right)^{-1} \le \nu(1-\nu)\frac{a+b}{ab}(b-a)^2 \tag{25}$$

for any a, $b > 0$ and $\nu \in [0,1]$.

This is an inequality of interest in itself.

If we take $a = 1$ and $b = t$ in (25), then we get

$$\nu t + 1 - \nu - \left(\nu t^{-1} + 1 - \nu\right)^{-1} \le \nu(1-\nu)(t-1)^2\left(1 + t^{-1}\right)$$

for any $t > 0$.

If we use the continuous functional calculus for the positive invertible operator X we get

$$\nu X + (1-\nu) I - \left(\nu X^{-1} + (1-\nu) I\right)^{-1} \leq \nu (1-\nu) (X - I)^2 \left(X^{-1} + I\right). \tag{26}$$

If we write the inequality (9) for $X = A^{-1/2} B A^{-1/2}$, then we get

$$\nu A^{-1/2} B A^{-1/2} + (1-\nu) I - \left(\nu \left(A^{-1/2} B A^{-1/2}\right)^{-1} + (1-\nu) I\right)^{-1} \leq \nu (1-\nu) \left(A^{-1/2} B A^{-1/2} - I\right)^2 \left(\left(A^{-1/2} B A^{-1/2}\right)^{-1} + I\right), \tag{27}$$

and $\nu \in [0,1]$.

If we multiply the inequality (10) both sides with $A^{1/2}$, then we get

$$A \nabla_\nu B - A !_\nu B \leq \nu (1-\nu) A^{1/2} \left(A^{-1/2} B A^{-1/2} - I\right)^2 \times \left(\left(A^{-1/2} B A^{-1/2}\right)^{-1} + I\right) A^{1/2}, \tag{28}$$

and since

$$\begin{aligned} & A^{1/2} \left(A^{-1/2} B A^{-1/2} - I\right)^2 \left(\left(A^{-1/2} B A^{-1/2}\right)^{-1} + I\right) A^{1/2} \\ & = A^{1/2} A^{-1/2} (B - A) \\ & \quad \times A^{-1/2} A^{-1/2} (B - A) A^{-1/2} A^{1/2} \left(B^{-1} + A^{-1}\right) A^{1/2} A^{1/2} \\ & = (B - A) A^{-1} (B - A) \left(B^{-1} + A^{-1}\right) A \end{aligned}$$

hence from (28) we get the desired result (22).

The last part is obvious from the fact that $\nu (1-\nu) \leq \frac{1}{4}$, $\nu \in [0,1]$. ∎

We also have:

Theorem 4. *Let A, B be positive invertible operators and $M > m > 0$ such that the condition (13) is valid. Then for any $\nu \in [0,1]$ we have*

$$(0 \leq) A \nabla_\nu B - A !_\nu B \leq \nu (1-\nu) L (m, M) A \tag{29}$$

where

$$L(m, M) := \begin{cases} (m-1)^2 \left(1 + m^{-1}\right) & \text{if } M < 1, \\ \max\left\{(m-1)^2 \left(1 + m^{-1}\right), (M-1)^2 \left(1 + M^{-1}\right)\right\} & \text{if } m \leq 1 \leq M, \\ (M-1)^2 \left(1 + M^{-1}\right) & \text{if } 1 < m. \end{cases} \tag{30}$$

In particular,

$$(0 \le)\, A \nabla B - A!B \le \frac{1}{4} L\left(m, M\right) A. \tag{31}$$

Proof. As in the proof of Theorem 3 we have

$$\nu t + 1 - \nu - \left(\nu t^{-1} + 1 - \nu\right)^{-1} \le \nu\left(1 - \nu\right) \psi\left(t\right) \tag{32}$$

for any $t > 0$ and $\nu \in [0, 1]$, where $\psi(t) = (t-1)^2 \left(1 + t^{-1}\right)$.

If we take the derivative of ψ, we have

$$\begin{aligned} \psi'(t) &= 2(t-1)\left(1 + t^{-1}\right) - (t-1)^2 t^{-2} \\ &= (t-1)\left(2 + 2t^{-1} - t^{-1} + t^{-2}\right) \\ &= (t-1)\left(2 + t^{-1} + t^{-2}\right) \end{aligned}$$

for any $t > 0$.

We observe that the function ψ is decreasing on $(0, 1)$ and increasing on $(1, \infty)$. We have $\lim_{t \to 0+} \psi(t) = \infty$, $\varphi(1) = 0$ and $\lim_{t \to \infty} \varphi(t) = \infty$.

Using the properties of the function ψ we have

$$\max_{t \in [m, M]} \psi(t) = \begin{cases} \psi(m) & \text{if} \quad M < 1, \\ \max\left\{\psi(m), \psi(M)\right\} & \text{if} \quad m \le 1 \le M, \\ \psi(M) & \text{if} \quad 1 < m, \end{cases} = L\left(m, M\right).$$

Therefore, by (32) we have

$$\nu t + 1 - \nu - \left(\nu t^{-1} + 1 - \nu\right)^{-1} \le \nu\left(1 - \nu\right) L\left(m, M\right)$$

for all $t \in [m, M]$ and $\nu \in [0, 1]$.

By utilizing a similar argument to the one in the proof of Theorem 2 we deduce the desired result (30). ■

3. Applications

For two positive invertible operators A, B and positive real numbers m, m', M, M' assume that one of the following conditions (i) $0 < mI \le A \le m'I < M'I \le B \le MI$ and (ii) $0 < mI \le B \le m'I < M'I \le A \le MI$, holds. Put $h := \frac{M}{m}$ and $h' := \frac{M'}{m'}$. We observe that h, $h' > 1$ and if either of the condition (i) or (ii) holds, then $h \ge h'$.

If (i) is valid, then we have

$$A < h'A = \frac{M'}{m'} A \le B \le \frac{M}{m} A = hA, \tag{33}$$

while, if (ii) is valid, then we have

$$\frac{1}{h}A \le B \le \frac{1}{h'}A < A. \tag{34}$$

Proposition 1. *Let A, B positive invertible operators and positive real numbers m, m', M, M' such that the condition (i) holds. Then for any $\nu \in [0,1]$ we have*

$$\begin{aligned} r\left(h'-1\right)^2\left(h'+1\right)^{-1}A &\le A\nabla_\nu B - A!_\nu B \\ &\le R\,(h-1)^2\,(h+1)^{-1}\,A, \end{aligned} \tag{35}$$

where $r = \min\{\nu, 1-\nu\}$, $R = \max\{\nu, 1-\nu\}$ and

$$A\nabla_\nu B - A!_\nu B \le \nu\,(1-\nu)\,(h-1)^2\left(1+h^{-1}\right)A. \tag{36}$$

In particular, we have

$$\begin{aligned} \frac{1}{2}\left(h'-1\right)^2\left(h'+1\right)^{-1}A &\le A\nabla B - A!B \\ &\le \frac{1}{2}\,(h-1)^2\,(h+1)^{-1}\,A, \end{aligned} \tag{37}$$

and

$$A\nabla B - A!B \le \frac{1}{4}\,(h-1)^2\left(1+h^{-1}\right)A. \tag{38}$$

The proof follows by utilizing the inequality (33), Theorem 2 and Theorem 4.

Proposition 2. *Let A, B positive invertible operators and positive real numbers m, m', M, M' such that the condition (ii) holds. Then for any $\nu \in [0,1]$ we have*

$$\begin{aligned} r\left(h'-1\right)^2\left(h'+1\right)^{-1}\left(h'\right)^{-1}A &\le A\nabla_\nu B - A!_\nu B \\ &\le R\,(h-1)^2\,(h+1)^{-1}\,h^{-1}A, \end{aligned} \tag{39}$$

and

$$A\nabla_\nu B - A!_\nu B \le \nu\,(1-\nu)\,(h-1)^2\left(1+h^{-1}\right)h^{-1}A. \tag{40}$$

In particular, we have

$$\begin{aligned} \frac{1}{2}\left(h'-1\right)^2\left(h'+1\right)^{-1}\left(h'\right)^{-1}A &\le A\nabla B - A!B \\ &\le \frac{1}{2}\,(h-1)^2\,(h+1)^{-1}\,h^{-1}A, \end{aligned} \tag{41}$$

and

$$A\nabla B - A!B \le \frac{1}{4}\,(h-1)^2\left(1+h^{-1}\right)h^{-1}A. \tag{42}$$

The proof follows by utilizing the inequality (34), Theorem 2 and Theorem 4.

If we consider the function $D(x,y) : [1,10] \times [0,1] \to \mathbb{R}$,

$$D(x,y) = y(1-y)\left(1+x^{-1}\right) - \max\{y, 1-y\}(x+1)^{-1}$$

then the plot of this function in Figure 1 shows that it take both positive and negative values, meaning that some time the upper bound for the quantity $A\nabla_\nu B - A!_\nu B$ provided by (35) is better and other time worse than the one from (39).

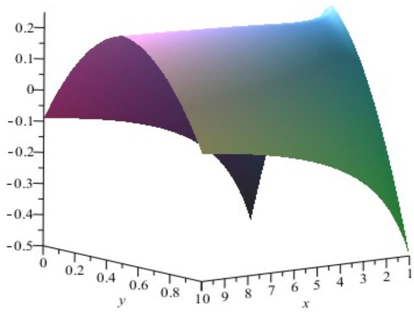

Figure 1. *Plot of the difference function $D(x,y)$*

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