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DIAMETER OF REDUCED SPHERICAL CONVEX BODIES

Abstract. The intersection L of two different non-opposite hemispheres of the unit sphere S^2 is called a lune. By $\Delta(L)$ we denote the distance of the centers of the semicircles bounding L. By the thickness $\Delta(C)$ of a convex body $C \subset S^2$ we mean the minimal value of $\Delta(L)$ over all lunes $L \supset C$. We call a convex body $R \subset S^2$ reduced provided $\Delta(Z) < \Delta(R)$ for every convex body Z being a proper subset of R. Our aim is to estimate the diameter of R, where $\Delta(R) < \frac{\pi}{2}$, in terms of its thickness.

Key words: spherical convex body, spherical geometry, hemisphere, lune, width, constant width, thickness, diameter.

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1. Introduction

Let S^2 be the unit sphere of the 3-dimensional Euclidean space E^3. A *great circle* of S^2 is the intersection of S^2 with any two-dimensional linear subspace of E^3. By a pair of *antipodes* of S^2 we mean any pair of points being the intersection of S^2 with a one-dimensional subspace of E^3. Observe that if two different points $a, b \in S^2$ are not antipodes, there is exactly one great circle passing through them. By the *arc ab* connecting them we understand the shorter part of the great circle through a and b. By the *distance* $|ab|$ of a and b we mean the length of the arc ab. The notion of the *diameter* of any set $A \subset S^2$ not containing antipodes is taken with respect to this distance and denoted by $\mathrm{diam}(A)$.

A subset of S^2 is called *convex* if it does not contain any pair of antipodes of S^2 and if together with every two points it contains the arc connecting them. A closed convex set $C \subset S^2$ with non-empty interior is called a convex body. Its boundary is denoted by $\mathrm{bd}(C)$. If no arc is in $\mathrm{bd}(C)$, we say that the body is *strictly convex*. Convexity on S^2 is considered in very many papers and monographs. For instance in [1], [2], [3], [4], [5] and [14].

The set of points of S^2 in the distance at most ρ, where $0 < \rho \leq \frac{\pi}{2}$, from a point $c \in S^2$ is called a *spherical disk*, or shorter *a disk*, of *radius* ρ and *center* c. Disks of radius $\frac{\pi}{2}$ are called *hemispheres*. Two hemispheres whose centers are antipodes are called *opposite hemispheres*. The set of points of a great circle of S^2 which are at distance at most $\frac{\pi}{2}$ from a fixed point p of this great circle is called a *semicircle*. We call p the *center* of this semicircle.

Let $C \subset S^2$ be a convex body and $p \in \mathrm{bd}(C)$. We say that a hemisphere K *supports* C *at* p provided $C \subset K$ and p is in the great circle bounding K.

Since the intersection of every family of convex sets is also convex, for every set $A \subset S^2$ contained in an open hemisphere of S^2 there is the smallest convex set $\mathrm{conv}(A)$ containing A. We call it *the convex hull of* A.

If non-opposite hemispheres G and H are different, then $L = G \cap H$ is called a *lune*. The semicircles bounding L and contained in G and H, respectively, are denoted by G/H and H/G. The *thickness* $\Delta(L)$ of L is defined as the distance of the centers of G/H and H/G.

After [8] recall a few notions. For any hemisphere K supporting a convex body $C \subset S^2$ we find a hemisphere K^* supporting C such that the lune $K \cap K^*$ is of the minimum thickness (by compactness arguments at least one such a hemisphere K^* exists). The thickness of the lune $K \cap K^*$ is called *the width of* C *determined by* K and it is denoted by $\mathrm{width}_K(C)$. By the *thickness* $\Delta(C)$ of a convex body $C \subset S^2$ we understand the minimum of $\mathrm{width}_K(C)$ over all supporting hemispheres K of C. We say that a convex body $R \subset S^2$ is *reduced* if for every convex body $Z \subset R$ different from R we have $\Delta(Z) < \Delta(R)$. This definition is analogous to the definition of a reduced body in normed spaces (for a survey of results on reduced bodies see [10]). If for all hemispheres K supporting C the numbers $\mathrm{width}_K(C)$ are equal, we say that C is *of constant width*. Spherical bodies of constant width are discussed in [12] and applied in [6].

Just bodies of constant width, and in particular disks, are simple examples of reduced bodies on S^2. Also each of the four parts of a disk dissected by two orthogonal great circles through the center of this disk is a reduced body. It is called a *quarter of a disk*. There is a wide class of reduced odd-gons on S^2 (see [9]). In particular, the regular odd-gons of thickness at most $\frac{\pi}{2}$ are reduced.

2. Two lemmas

Lemma 1. *Let $L \subset S^2$ be a lune of thickness at most $\frac{\pi}{2}$ whose bounding semicircles are Q and Q'. For every u, v, z in Q such that $v \in uz$ and for every $q \in L$ we have $|qv| \leq \max\{|qu|, |qz|\}$.*

Proof. If $\Delta(L) = \frac{\pi}{2}$ and q is the center of Q', then the distance between

q and any point of Q is the same, and thus the assertion is obvious. Consider the opposite case when $\Delta(L) < \frac{\pi}{2}$, or $\Delta(L) = \frac{\pi}{2}$ but q is not the center of Q'. Clearly, the closest point $p \in Q$ to q is unique. Observe that for $x \in Q$ the distance $|qx|$ increases as the distance $|px|$ increases. This easily implies the assertion of our lemma. ■

In a standard way, an extreme point of a convex body $C \subset S^2$ is defined as a point for which the set $C \setminus \{e\}$ is convex (see [8]). The set of extreme points of C is denoted by $E(C)$.

Lemma 2. *For every convex body $C \subset S^2$ of diameter at most $\frac{\pi}{2}$ we have* $\mathrm{diam}(E(C)) = \mathrm{diam}(C)$.

Proof. Clearly, $\mathrm{diam}(E(C)) \leq \mathrm{diam}(C)$.

In order to show the opposite inequality $\mathrm{diam}(C) \leq \mathrm{diam}(E(C))$, thanks to $\mathrm{diam}(C) = \mathrm{diam}(\mathrm{bd}(C))$, it is sufficient to show that $|cd| \leq \mathrm{diam}(E(C))$ for every $c, d \in \mathrm{bd}(C)$. If $c, d \in E(C)$, this is trivial. In the opposite case, at least one of these points does not belong to $E(C)$. If, say $d \notin E(C)$, then having in mind that C is a convex body and $d \in \mathrm{bd}(C)$ we see that there are $e, f \in E(C)$ different from d such that $d \in ef$. From $E(C) \subset \mathrm{bd}(C)$, we see that $e, f \in \mathrm{bd}(C)$. Since also $d \in \mathrm{bd}(C)$, the arc ef is a subset of $\mathrm{bd}(C)$.

Recall that by Theorem 3 of [8] we have $\mathrm{width}_K(C) \leq \mathrm{diam}(C)$ for every hemisphere K supporting C. In particular, for the hemisphere K supporting C at all points of the arc ef. Thus by the assumption that $\mathrm{diam}(C) \leq \frac{\pi}{2}$ we obtain $\mathrm{width}_K(C) \leq \frac{\pi}{2}$ for our particular K. Hence we may apply Lemma 1 taking this K/K^* in the part of Q there. We obtain $|cd| \leq \max\{|ce|, |cf|\}$.

If $c \in E(C)$, from $e, f \in E(C)$ we conclude that $|cd| \leq \mathrm{diam}(E(C))$. If $c \notin E(C)$, from $c \in \mathrm{bd}(C)$ we see that there are $g, h \in E(C)$ such that $c \in gh$. Similarly to the preceding paragraph we show that $|ec| \leq \max\{|eg|, |eh|\}$ and $|fc| \leq \max\{|fg|, |fh|\}$. By these two inequalities and by the preceding paragraph we get $|cd| \leq \max\{|eg|, |eh|, |fg|, |fh|\} \leq \mathrm{diam}(E(C))$, which ends the proof. ■

The assumption that $\mathrm{diam}(C) \leq \frac{\pi}{2}$ is substantial in Lemma 2, as it follows from the example of a regular triangle of any diameter greater than $\frac{\pi}{2}$. The weaker assumption that $\Delta(C) \leq \frac{\pi}{2}$ is not sufficient, which we see taking in the part of C any isosceles triangle T with $\Delta(T) \leq \frac{\pi}{2}$ and the arms longer than $\frac{\pi}{2}$ (so with the base shorter than $\frac{\pi}{2}$). The diameter of T equals to the distance between the midpoint of the base and the opposite vertex of T. Hence $\mathrm{diam}(T)$ is over the length of each of the sides.

3. Diameter of reduced spherical bodies

The following theorem is analogous to the first part of Theorem 9 from [7] and confirms the conjecture from [9], p. 214. By the way, the much weaker estimate $2\arctan\left(\sqrt{2}\tan\frac{\Delta(R)}{2}\right)$ results from Theorem 2 in [13].

Theorem 1. *For every reduced spherical body* $R \subset S^2$ *with* $\Delta(R) < \frac{\pi}{2}$ *we have* $\mathrm{diam}(R) \le \arccos(\cos^2\Delta(R))$. *This value is attained if and only if* R *is the quarter of disk of radius* $\Delta(R)$. *If* $\Delta(R) \ge \frac{\pi}{2}$, *then* $\mathrm{diam}(R) = \Delta(R)$.

Proof. Assume that $\Delta(R) < \frac{\pi}{2}$. In order to show the first statement, by Lemma 2 it is sufficient to show that the distance between any two different points e_1, e_2 of $E(R)$ is at most $\arccos(\cos^2\Delta(R))$. Since R is reduced, according to the statement of Theorem 4 in [8] there exist lunes $L_j \supset R$, where $j \in \{1, 2\}$, of thickness $\Delta(R)$ with e_j as the center of one of the two semicircles bounding L_j (see Figure). Denote by b_j the center of the other semicircle bounding L_j.

If $e_1 = b_2$ or $e_2 = b_1$, then $|e_1e_2| = \Delta(R)$, which by $\Delta(R) \in (0, \frac{\pi}{2})$ is at most $\arccos(\cos^2\Delta(R))$. Otherwise $L_1 \cap L_2$ is a non-degenerate spherical quadrangle with points e_1, b_2, b_1, e_2 in its consecutive sides. Therefore, since $e_1 \ne e_2$, arcs e_1b_1 and e_2b_2 intersect at exactly one point. Denote it by g. Observe that it may happen $b_1 = b_2 = g$.

Let F be the great circle orthogonal to the great circle containing e_1b_1 and passing through e_2. Since $e_2 \in L_1$, we see that F intersects e_1b_1. Let f

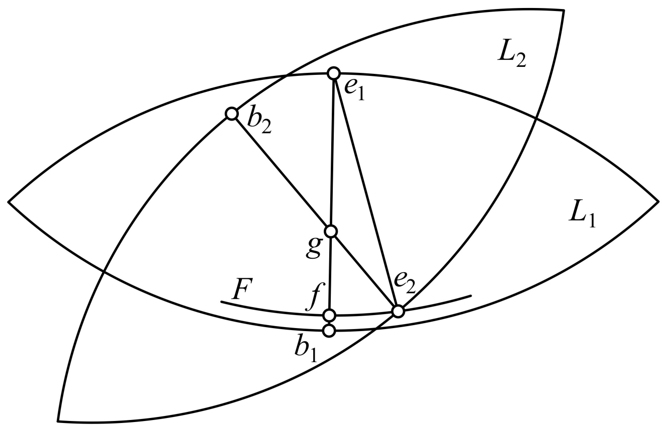

Figure. *Illustration to the proof of Theorem 1.*

be the intersection point of them. Note that we do not exclude the case $f = b_1$. From $|e_2b_2| = \Delta(R)$ we see that $|ge_2| \le \Delta(R) < \frac{\pi}{2}$. Thus from the right spherical triangle gfe_2 we conclude that $|fe_2| \le \Delta(R)$. Moreover, from $|e_1b_1| = \Delta(R)$ and $f \in e_1b_1$ we obtain $|fe_1| \le \Delta(R)$. Consequently, from the formula $\cos k = \cos l_1 \cos l_2$ for the right spherical triangle with hypotenuse k and legs l_1, l_2 applied to the triangle $e_1 f e_2$ (again see Figure) we obtain $|e_1e_2| \leq \arccos(\cos^2 \Delta(R))$.

Observe that thanks to $\Delta(R) < \frac{\pi}{2}$, the shown inequality becomes the equality only if $g = f = b_1 = b_2$. In this case, by Proposition 3.2 of [11], our body R is a quarter of disk of radius $\Delta(R)$.

Finally, we show the last statement of the theorem. Assume that $\Delta(R) \geq \frac{\pi}{2}$. By Theorem 4.3 of [11], the body R is of constant width $\Delta(R)$.

There is an arc $pq \subset R$ of length equal to $\mathrm{diam}(R)$. Clearly, $p \in \mathrm{bd}(R)$. Take a lune L from Theorem 5.2 of [11] such that p is the center of a semicircle bounding L. Denote by s the center of the other semicircle S bounding L. By the third part of Lemma 3 in [8], we have $|px| < |ps|$ for every $x \in S$ different from s. Hence $|px| \leq |ps|$ for every $x \in S$. So also $|pz| \leq |ps|$ for every $z \in L$. In particular, $|pq| \leq |ps|$. Since $\mathrm{diam}(R) = |pq|$ and $\Delta(R) = \Delta(L) = |ps|$, we obtain $\mathrm{diam}(R) \leq \Delta(R)$.

Let us show the opposite inequality.

If $\mathrm{diam}(R) \leq \frac{\pi}{2}$, by Theorem 3 of [8] we get $\Delta(R) = \mathrm{diam}(R)$. If $\mathrm{diam}(R) > \frac{\pi}{2}$, then by Proposition 1 of [8] we get $\Delta(R) \leq \mathrm{diam}(W)$. Consequently, always we have $\Delta(R) \leq \mathrm{diam}(R)$.

From the inequalities obtained in the two preceding paragraphs we get the equality $\mathrm{diam}(R) = \Delta(R)$. Hence the last assertion of the theorem is proved. ■

Proposition 3.5 of [11] implies that *for every reduced spherical body R with $\Delta(R) \leq \frac{\pi}{2}$ on S^2 we have* $\mathrm{diam}(R) \leq \frac{\pi}{2}$. Here is a more precise form of this statement.

Proposition 1. *Let $R \subset S^2$ be a reduced body. Then* $\mathrm{diam}(R) < \frac{\pi}{2}$ *if and only if $\Delta(R) < \frac{\pi}{2}$. Moreover,* $\mathrm{diam}(R) = \frac{\pi}{2}$ *if and only if $\Delta(R) = \frac{\pi}{2}$.*

Proof. We start with proving the first statement of our proposition. The function $f(x) = \arccos(\cos^2 x)$ is increasing in the interval $[0, \frac{\pi}{2}]$ as a composition of the decreasing functions $\arccos x$ and $\cos^2 x$. From $f(\frac{\pi}{2}) = \frac{\pi}{2}$ we conclude that in the interval $[0, \frac{\pi}{2})$ the function $f(x)$ accepts only the values below $\frac{\pi}{2}$. Thus by Theorem 1, if $\Delta(R) < \frac{\pi}{2}$, then $\mathrm{diam}(R) < \frac{\pi}{2}$. The opposite implication results from the inequality $\Delta(C) \leq \mathrm{diam}(C)$ for every spherical convex body C, which follows from Theorem 3 and Proposition 1 of [8].

Let us show the second part of our proposition.

Assume that $\mathrm{diam}(R) = \frac{\pi}{2}$. The inequality $\Delta(R) < \frac{\pi}{2}$ is impossible, by the first statement of our proposition. Also the inequality $\Delta(R) > \frac{\pi}{2}$ is impossible, because by $\Delta(R) \leq \mathrm{diam}(R)$ (see Proposition 1 of [8]), it would imply $\operatorname{diam}(R) > \frac{\pi}{2}$ in contradiction to the assumption at the beginning of this paragraph. So $\Delta(R) = \frac{\pi}{2}$.

Now assume that $\Delta(R) = \frac{\pi}{2}$. By the second statement of Theorem 1 we get $\operatorname{diam}(R) = \frac{\pi}{2}$. ■

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