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THE SOLVABILITY OF FUNCTIONAL QUADRATIC VOLTERRA-URYSOHN INTEGRAL EQUATIONS ON THE HALF LINE

Abstract. We study the solvability of general quadratic Volterra integral equations in the space of Lebesgue integrable functions on the half line. Using the conjunction of the technique of measures of weak noncompactness with modified Schauder fixed point principle we show that the integral equation, under certain conditions, has at least one solution. Moreover, that result generalizes several ones obtained earlier in many research papers and monographs.

Key words: quadratic integral equations, integro-differential equations, Schauder fixed point theorem, superposition operator.

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1. Introduction

We are going to study the solvability of the following integral equation

$$x(t) = g\left(t,\ (T_1x)(t),\ (T_2x)(t)\int_0^t k(t,s)f(s,x(s))ds\ ,\right. \qquad (1)$$

$$\left.(T_3x)(t)\int_0^t u(t,s,x(s))ds\right), \quad t \in \mathbb{R}^+,$$

where T_i, $i = 1,2,3$ are operators which map $L_1(\mathbb{R}^+)$, i.e. the space of Lebesgue integrable functions on $\mathbb{R}^+$ into itself continuously.

Many authors have studied different particular cases of the integral equations (1) on noncompact intervals (cf. [8, 9, 13]).

Developing a modified new method of the descriptive theory [11], we obtain a new generalization of the Scorza-Dragoni theorem for general operator $T : \mathcal{N} \times X \to Y$ defined on the product of a topological space $\mathcal{N}$ with σ-finite Borel regular measure and a metrizable separable locally compact space X.

Let us mention that the functional quadratic integral equations are often applicable in the theory of radiative transfer, kinetic theory of gases, in the theory of neutron transport, in the traffic theory, in plasma physics and in numerous branches of mathematical physics (cf. [6, 7, 12]).

In this paper we generalize a lot of variants of the Scorza-Dragoni theorem. We will unify some known results, for particular cases of equation (1) in one proof and will extend some of them from compact interval to noncompact one in the space $L_1(\mathbb{R}^+)$. Our main tools are the measure of noncompactness and Schauder's fixed point theorem.

2. Notation and auxiliary facts

Let $\mathbb{R}$ be the field of real numbers and $\mathbb{R}^+$ be the interval $[0,\infty)$. If Z is a Lebesgue measurable subset of $\mathbb{R}$, then the symbol $meas(Z)$ stands for the Lebesgue measure of Z. Further, denote by $L_1(Z)$ the space of all real functions defined and Lebesgue measurable on the set Z. When $Z = \mathbb{R}^+$, we will write L_1 and L_∞ instead of $L_1(\mathbb{R}^+)$ and $L_\infty(\mathbb{R}^+)$ respectively.

Denote by $C(D)$ the Banach space of real functions defined and continuous on a nonempty bounded and closed subset D of $\mathbb{R}$. The space $C(D)$ will be considered with the standard maximum norm. Let us fix a nonempty and bounded subset X of $C(D)$ and a positive number τ. For $x \in X$ and $\varepsilon \geq 0$ let us denote by $\omega^\tau(x,\varepsilon)$ the modulus of continuity of the function x, on the closed and bounded interval $[0,\tau]$ defined by

$$\omega^\tau(x,\varepsilon) = \sup\{|x(t_2) - x(t_1)| : t_1, t_2 \in [0,\tau],\ |t_2 - t_1| \leq \varepsilon\}.$$

A measure on a σ-algebra $\mathbb{A}$ of subsets of a certain set $\mathcal{N}$ is defined as a σ-additive function of a set $\gamma : \mathbb{A} \to [0,\infty]$ such that $\gamma(\phi) = 0$. A measure defined on the σ-algebra $\mathbb{B}$ of all Borel subsets of a certain topological space is called a Borel measure. A Borel measure $\gamma : \mathbb{A} \to [0,\infty]$, where $\mathbb{A}$ is a σ-algebra of measurable sets, is called regular if, for every set $\mathcal{H} \in \mathbb{A}$ and every $\epsilon > 0$, there exist a closed set $\mathcal{F} \subseteq \mathcal{H}$ and open set $\mathcal{G} \supseteq \mathcal{H}$ such that $\gamma(\mathcal{G}\backslash\mathcal{F}) < \epsilon$.

Now we present the concept of measure of weak noncompactness. Assume that $(E, \|\cdot\|)$ is an arbitrary Banach space with zero element θ. Denote by $B(x,r)$ the closed ball centered at x and with radius r. The symbol B_r stands for the ball $B(\theta,r)$. Denote by $\mathcal{M}_E$ the family of all nonempty and bounded subsets of E and by $\mathcal{N}_E^W$ its subfamily consisting of all relatively weakly compact sets. The symbol $\bar{X}^W$ stands for the weak closure of a set X and the symbol $\mathrm{Conv}X$ will denote the convex closed hull of a set X.

Definition 1 ([5]). *A mapping $\mu : \mathcal{M}_E \to [0,\ \infty)$ is said to be a measure of weak noncompactness in E if it satisfies the following conditions:*

(a) *The family $Ker\ \mu = \{X \in \mathcal{M}_E : \mu(X) = 0\}$ is nonempty and $Ker\ \mu \subset \mathcal{N}_E^W$.*

(*b*) $X \subset Y \Rightarrow \mu(X) \leq \mu(Y)$.
(*c*) $\mu(ConvX) = \mu(X)$.
(*d*) $\mu(\lambda X + (1-\lambda)Y) \leq \lambda\mu(X) + (1-\lambda)\mu(Y)$ *for* $\lambda \in [0,1]$.
(*e*) *If* $X_n \in \mathcal{M}_E$, $X_n = \bar{X_n}^W$ *and* $X_{n+1} \subset X_n$ *for* $n = 1, 2, \cdots$ *and if* $\lim_{n\to\infty} \mu(X_n) = 0$, *then the intersection* $X_\infty = \bigcap_{n=1}^{\infty} X_n$ *is nonempty.*

The family Ker μ described in (1) is said to be the kernel of the measure of weak noncompactness μ. Let us observe that the intersection set X_∞ from (5) belongs to Ker μ. Indeed, since $\mu(X_\infty) \leq \mu(X_n)$ for every n, then we have that $\mu(X_\infty) = 0$. This simple observation will be important in our further considerations.

Now, for a nonempty and bounded subset X of the space L_1 let us define:

$$c(X) = \lim_{\epsilon\to 0}\left\{\sup_{x\in X}\left\{\sup\left[\int_D |x(t)|dt, D \subset \mathbb{R}^+, meas(D) \leq \varepsilon\right]\right\}\right\} \tag{2}$$

and

$$d(X) = \lim_{T\to\infty}\left\{\sup\left[\int_T^\infty |x(t)|dt : x \in X\right]\right\}. \tag{3}$$

Put

$$\mu(X) = c(X) + d(X). \tag{4}$$

It can be shown [4] that the function μ is a measure of weak noncompactness in the space L_1.

Now, we will investigate many properties of operators acting on different function spaces. Let us recall some basic lemmas.

Definition 2 ([1]). *Assume that a function* $f : \mathbb{R}^+ \times \mathbb{R} \to \mathbb{R}$ *satisfies Carathéodory conditions i.e. it is measurable in* t *for any* $x \in \mathbb{R}$ *and continuous in* x *for almost all* $t \in \mathbb{R}^+$. *Then to every measurable function* x, *we may assign the function*

$$F(x)(t) = f(t, x(t)), \quad t \in \mathbb{R}^+.$$

The operator F *defined in such a way is called the superposition* (*Nemytskii*) *operator generated by the function* f.

Theorem 1 ([1]). *Suppose that* f *satisfies Carathéodory conditions. The superposition operator* F *maps the space* L_1 *into* L_1 *if and only if*

$$|f(t,x)| \leq a(t) + b\ |x|, \tag{5}$$

for all $t \in \mathbb{R}^+$ *and* $x \in \mathbb{R}$, *where* $a \in L_1$ *and* $b \geq 0$. *Moreover, this operator is continuous.*

Assume that $I \subseteq \mathbb{R}^+$ is an interval. The following Lusin-Dragoni theorem explains the structure of measurable functions and functions satisfying Carathéodory conditions. Below and subsequently by D^c we will denote the complement of the set D.

Theorem 2 ([14]). *Let $f : I \times \mathbb{R} \to \mathbb{R}$ be a function satisfying Carathéodory conditions. Then for each $\epsilon > 0$ there exists a closed subset D_ϵ of the interval I such that $meas(D_\epsilon^c) \leq \epsilon$ and $f|_{D_\epsilon \times \mathbb{R}}$ is continuous.*

Consider a general operator $T : \mathcal{N} \times X \to Y$, where $\mathcal{N}$ is a topological space with Borel measure γ and X and Y are topological spaces. We will denote $T^t(x) = T_x(t) = (Tx)(t)$ for $(t, x) \in \mathcal{N} \times X$.

Definition 3. *A mapping T is called a Carathéodory function if the mapping $T^t : X \to Y$ is continuous for every $t \in \mathcal{N}$, and the mapping $T_x : \mathcal{N} \to Y$ is measurable for every $x \in X$. We say that a mapping T possesses the Scorza-Dragoni property if, for every $\epsilon > 0$, there exists a closed set $\mathcal{N}_\epsilon \subseteq \mathcal{N}$ such that $\gamma(\mathcal{N} \backslash \mathcal{N}_\epsilon) < \epsilon$ and the restriction $T|_{\mathcal{N}_\epsilon \times X}$ is continuous.*

Recall that the Scorza-Dragoni property plays a similar role for functions defined on $I \times \mathbb{R}$ as the Lusin property for functions defined on I. We will extend such a result for noncompact intervals.

The following generalization of Theorem 2 is obtained by direct generalization of the method used in [11] in topological spaces to general operator T. The measurability of a function is considered with respect to a σ-algebra of γ-measurable sets (cf. [11, Theorem 3]).

Lemma 1. *Let $\mathcal{N}$ is a topological space with σ-finite regular measure γ, X is a metrizable separable locally compact space, Y is a metrizable separable space, and $T : \mathcal{N} \times X \to Y$ is a Carathéodory function. Then T possesses the Scorza-Dragoni property.*

Proof. Let d_X and d_Y be the metrics that generate the topologies of the spaces X and Y, respectively, and let $\tilde{X} = \{x_1, x_2, \cdots\}$ be a set dense in X. We fix an arbitrary $\epsilon > 0$. The space Y has at most countable base. Hence, for every $k \in \mathbb{N}$ one can always find a closed set $\mathcal{H}_k$ in $\mathcal{N}$ for which $\gamma(\mathcal{N} \backslash \mathcal{H}_k) < \frac{\epsilon}{2^{k+2}}$ and the restriction $T_{x_k}|_{\mathcal{H}_k}$ is continuous (cf. [11, Theorem 1]). The set $\mathcal{E} = \bigcap_{k=1}^{\infty} \mathcal{H}_k$ is closed and $\gamma(\mathcal{N} \backslash \mathcal{E}_k) < \frac{\epsilon}{4}$.

Let $(\mathcal{G}_m)_{m=1}^{\infty}$ be an increasing sequence of open sets from X such that their closures $\overline{\mathcal{G}}_m$ are compact and $\bigcup_{m=1}^{\infty} \mathcal{G}_m = X$. We set

$$\mathcal{H}_{m,n} = \left\{ t \in \mathcal{E} : (\forall x', x'' \in \mathcal{G}_m) \left(d_X(x', x'') < \frac{1}{n} \right.\right.$$
$$\left.\left. \Rightarrow d_Y\left(T^t(x'), T^t(x'')\right) \leq \frac{1}{m} \right) \right\}.$$

Let us show that $\bigcup_{n=1}^{\infty} \mathcal{H}_{m,n} = \mathcal{E}$ for any fixed number m. Consider arbitrary $t \in \mathcal{E}$ and $m \in \mathbb{N}$. The function $T^t : X \to Y$ is continuous on X and, therefore, by virtue of the Cantor theorem, it is uniformly continuous on the compact set $\overline{\mathcal{G}}_m$ and, hence, on $\mathcal{G}_m$. This implies that there exists a number $n_0 \in \mathbb{N}$ such that, for all $x', x'' \in \mathcal{G}_m$, the relation $d_X(x', x'') < \frac{1}{n_0}$ yields $d_Y(T^t(x'), (T^t(x'')) \le \frac{1}{m}$. Therefore, $t \in \mathcal{H}_{m,n_0}$ and, hence, $t \in \bigcup_{n=1}^{\infty} \mathcal{H}_{m,n}$. We now show that the sets $\mathcal{H}_{m,n}$ are measurable. For this purpose, we put $X_m = \widetilde{X} \bigcap \mathcal{G}_m$ and consider the set

$$\widetilde{\mathcal{H}}_{m,n} = \left\{ t \in \mathcal{E} : (\forall x', x'' \in X_m) \left(d_X(x', x'') < \frac{1}{n} \right.\right.$$
$$\left.\left. \Rightarrow d_Y\left(T^t(x'), T^t(x'')\right) \le \frac{1}{m} \right) \right\}.$$

It is obvious that $\mathcal{H}_{m,n} \subseteq \widetilde{\mathcal{H}}_{m,n}$. Let us show that $\widetilde{\mathcal{H}}_{m,n} \subseteq \mathcal{H}_{m,n}$. Assume that $t \in \widetilde{\mathcal{H}}_{m,n}$ and the points $x', x'' \in \mathcal{G}_m$ are such that $d_X(x', x'') < \frac{1}{n}$. Since $\overline{X}_m \supseteq \mathcal{G}_m$, there exist sequences of points (x'_k) and (x''_k) in X_m such that $x'_k \to x$ and $x''_k \to x$ as $k \to \infty$, and $d_X(x'_k, x''_k) < \frac{1}{n}$ for all numbers k. We have $d_Y(T^t(x'_k), T^t(x''_k)) \le \frac{1}{m}$ for every k. Since the function T^t is continuous, passing to the limit as $k \to \infty$ in the last inequality we obtain $d_Y(T^t(x'_k), T^t(x''_k)) \le \frac{1}{m}$, whence $t \in \mathcal{H}_{m,n}$. Therefore, $\mathcal{H}_{m,n} = \widetilde{\mathcal{H}}_{m,n}$. Consider the at most countable set

$$S_{m,n} = \left\{ (x', x'') \in X_m^2 : d_X(x', x'') < \frac{1}{n} \right\}.$$

For $(x', x'') \in S_{m,n}$, we set

$$\mathcal{Q}_{x',x''} = \left\{ t \in \mathcal{E} : d_Y\left(T^t(x'), T^t(x'')\right) \le \frac{1}{m} \right\}.$$

Since the function $(T_{x'}, T_{x''}) : \mathcal{N} \to Y^2$, where $(T_{x'}, T_{x''})(t) = (T_{x'}(t), T_{x''}(t))$, is measurable and the function $d_Y : Y^2 \to \mathbb{R}$ is continuous, their composition $h = d_Y \circ (T_x, T_x)$ is a measurable function. Hence, the sets $\mathcal{Q}_{x',x''} = h^{-1}([0, \frac{1}{m}])$ are measurable for all $(x', x'') \in S_{m,n}$. On the other hand, it is clear that $\overline{\mathcal{H}}_{m,n} = \bigcap_{(x',x'') \in S_{m,n}} \mathcal{Q}_{x',x''}$. Therefore, the set $\mathcal{H}_{m,n} = \widetilde{\mathcal{H}}_{m,n}$ is also measurable as an at most countable intersection of measurable sets.

It is clear that the sequence $(\mathcal{H}_{m,n})_{n=1}^{\infty}$ increases for every m. Therefore, by virtue of the property of continuity from below, we have $\gamma(\mathcal{H}_{m,n}) \to \gamma(\mathcal{E})$ as $t \to \infty$. Hence, for every $m \in \mathbb{N}$, there exists a number n_m such that $\gamma(\mathcal{E} \backslash \mathcal{H}_{m,n_m}) < \frac{\epsilon}{2^{m+2}}$. The set $\mathcal{Q} = \bigcap_{m=1}^{\infty} \mathcal{H}_{m,n_m}$ is measurable and

$$\gamma(\mathcal{N} \backslash \mathcal{Q}) = \gamma((\mathcal{N} \backslash \mathcal{E}) \bigcup (\mathcal{E} \backslash \mathcal{Q})) < \gamma(\mathcal{N} \backslash \mathcal{E}) + \gamma(\mathcal{E} \backslash \mathcal{Q}) < \frac{\epsilon}{2}.$$

Let us show that the restriction $T|_{\mathcal{Q}\times X}$ is a continuous function. We take an arbitrary point $z_0 = (x_0, y_0) \in \mathcal{Q}\times X$ and fix an arbitrary $\delta > 0$. One can always find numbers m_1 and m_2 for which $\frac{1}{m_1} < \frac{\delta}{3}$ and $x_0 \in \mathcal{G}_{m_2}$. Denote $m_0 = max\{m_1, m_2\}$. Since $x_0 \in \mathcal{G}_{m_0}$ and $\overline{X}_{m_0} \supseteq \mathcal{G}_{m_0}$, there exists a number k_0 such that $x_{k_0} \in G_{m_0}$ and $d_X(x_{k_0}, x_0) < \frac{1}{2n_{m_0}}$. The restriction $T_{x_{k_0}}|_{\mathcal{Q}}$ is continuous because $\mathcal{Q} \subseteq \mathcal{H}_{k_0}$. Therefore, there exists a neighborhood Γ of a point t_0 in $\mathcal{Q}$ such that, for all $t \in \Gamma$, the inequality $d_Y(T_{x_{k_0}}(t), T_{x_{k_0}}(t_0)) < \frac{\delta}{3}$ is true. Let

$$\mathcal{V} = \left\{x \in X : d_X(x, x_0) < \frac{1}{2n_{m_0}}\right\}.$$

Then $\mathcal{W} = \Gamma \times \mathcal{V}$ is a neighborhood of a point z_0 in the space $\mathcal{Q} \times X$. For an arbitrary point $z = (t, x) \in \mathcal{W}$, we have

$$d_X(x, x_{k_0}) \leq d_X(x, x_0) + d_X(x_0, x_{k_0}) < \frac{1}{n_{m_0}}, \quad d_X(x_{k_0}, x_0) < \frac{1}{n_{m_0}}$$

and $t, t_0 \in \mathcal{H}_{m_0, n_{m_0}}$. Therefore,

$$d_Y((Tx)(t), (Tx_{k_0})(t)) \leq \frac{1}{m_0} \leq \frac{1}{m_1} < \frac{\delta}{3}$$

and, similarly, $d_Y((Tx_{k_0})(t_0), (Tx_0)(t_0)) \leq \frac{\delta}{3}$. In this case, we have

$$\begin{aligned} d_Y((Tx)(t), (Tx_0)(t_0)) \leq\ & d_Y((Tx)(t), (Tx_{k_0})(t)) \\ & + d_Y((Tx_{k_0})(t), (Tx_{k_0})(t_0)) \\ & + d_Y((Tx_{k_0})(t_0), (Tx_0)(t_0)) \\ <\ & \frac{\delta}{3} + \frac{\delta}{3} + \frac{\delta}{3} = \delta. \end{aligned}$$

Thus, the continuity of the restriction $T|_{\mathcal{Q}}$ is proved. It follows from the regularity of the measure γ that there exists a closed set $\mathcal{N}_\epsilon \subseteq \mathcal{Q}$ such that $\gamma(\mathcal{Q}\backslash\mathcal{N}_\epsilon) < \frac{\epsilon}{2}$. Then $\gamma(\mathcal{N}\backslash\mathcal{N}_\epsilon) < \epsilon$. It is clear that the restriction of the operator T to the set $\mathcal{N}_\epsilon \times X$ is continuous. Thus, the set N_ϵ is the required one. ∎

Theorem 3 ([15]). *Let $u : \mathbb{R}^+ \times \mathbb{R}^+ \times \mathbb{R} \to \mathbb{R}$ satisfies Carathéodory conditions, i.e. it is measurable in (t, s) for any $x \in \mathbb{R}$ and continuous in x for almost all (t, s). Assume that*

$$|u(t, s, x)| \leq k_1(t, s),$$

where the nonnegative function k_1 is measurable in (t, s) such that the linear integral operator K_1 with the kernel $k_1(t, s)$ maps L_1 into L_∞. Then the *operator* $(Ux)(t) = \int_0^t u(t,s,x(s))\, ds$ *maps* L_1 *into* L_∞. *Moreover, if for arbitrary* $h > 0$ *and* $x_i \in \mathbb{R}$ $(i = 1, 2)$

$$\lim_{\delta\to 0} \| \int_\Omega \max_{|x_i|\le h, |x_1 - x_2| \le \delta} |u(t,s,x_1) - u(t,s,x_2)|\, ds\|_{L_\infty} = 0,$$

then U *is a continuous operator.*

Remark 1. Observe that if Ω is a nonempty and measurable subset of $\mathbb{R}^+$, then we can also consider the linear Volterra integral operator $(Kx)(t) = \int_0^t k(t,s)x(s)\, ds$ associated with the Lebesgue space $L_p(\Omega)$, $1 \le p \le \infty$. Namely, if $x \in L_p(\Omega)$, $1 \le p \le \infty$, then we can extend x to the whole half axis $\mathbb{R}^+$ by putting $x(t) = 0$ for $t \in (\mathbb{R}^+ \setminus \Omega)$. Then we can treat the operator K in the usual way.

3. Main result

Rewrite (1) as $x = Gx$, where

$$(Gx)(t) = g(t, (T_1x)(t), (Ax)(t), (Bx)(t)),$$

$$(Ax) = (T_2x)(KFx), \quad (Bx) = (T_3x)(Ux),$$

$$(Kx)(t) = \int_0^t k(t,s)x(s)ds, \quad (Ux)(t) = \int_0^t u(t,s,x(s))ds, \quad Fx = f(t,x),$$

and $T_i(x), i = 1, 2, 3$ are operators which map the space L_1 into itself continuously.

Consider (1) and the following assumptions:

(*i*) $g(t,x,y,z) : \mathbb{R}^+ \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is measurable in t and continuous in x, y and z for almost all t. There exist positive constants $b_i, i = 4, 5, 6$ and a positive function $a_4 \in L_1$ such that the function

$$|g(t,x,y,z)| \le a_4(t) + b_4|x| + b_5|y| + b_5|z|$$

for $t \in \mathbb{R}^+$ and $x, y, z \in \mathbb{R}$.

(*ii*) $f : \mathbb{R}^+ \times \mathbb{R} \to \mathbb{R}$ satisfies Carathéodory conditions and there are a positive function $a \in L_1$ and a constant $b \ge 0$ such that

$$|f(t,x)| \le a(t) + b\, |x|$$

for all $t \in \mathbb{R}^+$ and $x \in \mathbb{R}$.

(*iii*) $u(t,s,x) : \mathbb{R}^+ \times \mathbb{R}^+ \times \mathbb{R} \to \mathbb{R}$ is measurable in t, s and continuous in x for almost all t. Moreover, for arbitrary fixed $(s,x) \in \mathbb{R}^+ \times \mathbb{R}$ the function $t \to u(t,s,x)$ is integrable.

(iv) There are functions $k, k_1 : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}$ satisfying Carathéodory conditions such that:

$$|u(t,s,x)| \le k_1(t,s)$$

for all $t, s \ge 0$ and $x \in \mathbb{R}$. Assume that the linear integral operator $(K_1x)(t) = \int_0^t k_1(t,s)x(s)ds$ maps L_1 into L_∞. The linear integral operator $(Kx)(t) = \int_0^t k(t,s)x(s)ds$ maps L_1 into L_∞ and is continuous. Moreover, assume that for arbitrary $h > 0$ and $x_i \in \mathbb{R}^+$ $(i = 1,2)$

$$\lim_{\delta\to 0} \| \int_\Omega \max_{|x_i|\le h, |x_1-x_2|\le\delta} |u(t,s,x_1) - u(t,s,x_2)|\, ds\|_{L_\infty} = 0.$$

(v) The operators $(T_ix)(t) : \mathbb{R}^+ \times \mathbb{R} \to \mathbb{R}$, $i = 1,2,3$ satisfy Carathéodory conditions and map continuously the space L_1 into itself. Moreover, there are positive functions $a_i \in L_1$ and positive constants b_i, such that

$$|(T_ix)(t)| \le a_i(t) + b_i|x(t)|, \quad i = 1,2,3$$

for each $t \in \mathbb{R}^+$ and $x \in \mathbb{R}$.

(vi) Let

$$W > \sqrt{\begin{array}{l} 4bb_2b_5\|K\|_{L_\infty}\big[\|a_4\|_{L_1} + b_4\|a_1\|_{L_1} \\ \quad + b_5\|a_2\|_{L_1}\|K\|_{L_\infty}\|a\|_{L_1} + b_6\|K_1\|_{L_\infty}\|a_3\|_{L_1}\big]\end{array}},$$

where

$$W = 1 - (b_1b_4 + b_3b_6\|K_1\|_{L_\infty} + b_5\|K\|_{L_\infty}(b\|a_2\|_{L_1} + b_2\|a\|_{L_1})$$

and let r denotes a positive solution of the quadratic equation

$$\begin{aligned}\big[\, &\|a_4\|_{L_1} + b_4\|a_1\|_{L_1} + b_5\|a_2\|_{L_1}\|K\|_{L_\infty}\|a\|_{L_1} \\ &+ b_6\|K_1\|_{L_\infty}\|a_3\|_{L_1}\big] - Wr + bb_2b_5\|K\|_{L_\infty}r^2 = 0.\end{aligned}$$

Then we can prove the following theorem.

Theorem 4. *Let the assumptions (i)–(vi) be satisfied. If*

$$q = (b_1b_4 + b_2b_5\|K\|_{L_\infty}\big[\|a\|_{L_1} + br\big] + b_3b_6\|K_1\|_{L_\infty}) < 1,$$

then equation (1) *has at least one integrable solution on* $\mathbb{R}^+$.

Proof. The proof will be given in six steps.

— **Step 1.** The operator $G : L_1 \to L_1$ and is continuous.
— **Step 2.** We will construct the ball B_r, where r will be determined later.
— **Step 3.** We will proof that $\mu(GX) \le q\mu(X)$ for all bounded subset X of B_r.

— **Step 4.** We will construct a nonempty closed convex weakly compact set M which we will need in the next steps.
— **Step 5.** $B(M)$ is relatively strongly compact in L_1.
— **Step 6.** We will check out the conditions needed in Schauder fixed point theorem [10] are fulfilled.

Step 1. First of all observe that by assumption (ii) and Theorem 1 we have that F is continuous mappings from L_1 into itself. By assumption (iv) and Theorem 3 we can deduce that the operators U and K map continuously the space L_1 into L_∞. Moreover, the operators T_i, $i = 1, 2, 3$ map continuously the space L_1 into itself (thanks for assumption (v)). From the Hölder inequality the operators A and B map L_1 into itself continuously. Finally, for a given $x \in L_1$ and by assumption (i) we infer that (Gx) belongs to L_1 and is continuous.

Step 2. In view of our assumptions we get:

$$\begin{aligned} \|Gx\|_{L_1} &= \int_0^\infty \bigg| g\bigg(t, (T_1x)(t), (T_2x)(t)\int_0^t k(t,s)f(s,x(s))ds, \\ &\qquad\qquad (T_3x)(t)\int_0^t u(t,s,x(s))ds\bigg)\bigg| dt \\ &\le \|a_4\|_{L_1} + b_4\int_0^\infty [a_1(t) + b_1|x(t)|]dt \\ &\quad + b_5\int_0^\infty [a_2(t) + b_2|x(t)|]\int_0^\infty k(t,s)[a(s) + b|x(s)|]dsdt \\ &\quad + b_6\int_0^\infty [a_3(t) + b_3|x(t)|]\int_0^\infty k_1(t,s)dsdt \\ &\le \big[\|a_4\|_{L_1} + b_4\|a_1\|_{L_1} + b_5\|a_2\|_{L_1}\|K\|_{L_\infty}\|a\|_{L_1} \\ &\quad + b_6\|K_1\|_{L_\infty}\|a_3\|_{L_1}\big] + r[b_1b_4 + b_3b_6\|K_1\|_{L_\infty} \\ &\quad + b_5\|K\|_{L_\infty}(b\|a_2\|_{L_1} + b_2\|a\|_{L_1})] + bb_2b_5\|K\|_{L_\infty}r^2 \le r, \end{aligned}$$

where $\|K\|_{L_\infty}$ and $\|K_1\|_{L_\infty}$ denote the norm of the Volterra integral operators K and K_1 respectively acting from L_1 to L_∞. From the above estimate, we have that $G(B_r) \subseteq B_r$ with

$$\begin{aligned} r &= \frac{W}{2bb_2b_5\|K\|_{L_\infty}} \\ &\quad - \frac{\sqrt{\begin{aligned}&W^2 - 4bb_2b_5\|K\|_{L_\infty}\big[\|a_4\|_{L_1} + b_4\|a_1\|_{L_1} \\ &\quad + b_5\|a_2\|_{L_1}\|K\|_{L_\infty}\|a\|_{L_1} + b_6\|K_1\|_{L_\infty}\|a_3\|_{L_1}\big]\end{aligned}}}{2bb_2b_5\|K\|_{L_\infty}} > 0. \end{aligned}$$

Assumption (vi) implies that W is positive and hence r is a positive constant.

Step 3. In what follows let us fix a nonempty subset X of the ball B_r. Take an arbitrary number $\varepsilon > 0$ and a set $D \subset \mathbb{R}^+$ such that $meas(D) \leq \varepsilon$. Then, fixing arbitrary $x \in X$, we have

$$\begin{aligned}\int_D |(Gx)(t)|dt \ \leq\ & \int_D a_4(t)dt + b_4\left[\int_D a_1(t)dt + b_1\int_D |x(t)|dt\right]\\ &+ b_5\left[\int_D a_2(t)dt + b_2\int_D |x(t)|dt\right]\|K\|_{L_\infty}[\|a\|_{L_1} + br]\\ &+ b_6\left[\int_D a_3(t)dt + b_3\int_D |x(t)|dt\right]\|K_1\|_{L_\infty},\end{aligned}$$

where $\|K\|_{L_\infty(D)}$ and $\|K_1\|_{L_\infty(D)}$ denote the norm of the Volterra integral operators K and K_1 respectively acting from $L_1(D)$ to $L_\infty(D)$.

Now, using the fact that

$$\lim_{\varepsilon\to\infty}\sup\left[\int_D a_i(t)dt : D \subset \mathbb{R}^+, meas(D) \leq \varepsilon\right] = 0, \quad \text{for } \ i = 1, 2, 3, 4.$$

From equation (2) it follows that

$$c(GX) \leq q = (b_1b_4 + b_2b_5\|K\|_{L_\infty}[\|a\|_{L_1} + br] + b_3b_6\|K_1\|_{L_\infty})c(X). \tag{6}$$

For fixed arbitrary number $\tau > 0$ and any $x \in X$, we have

$$\begin{aligned}\int_\tau^\infty |(Gx)(t)|dt \ \leq\ & \int_\tau^\infty a_4(t)dt + b_4\left[\int_\tau^\infty a_1(t)dt + b_1\int_\tau^\infty |x(t)|dt\right]\\ &+ b_5\left[\int_\tau^\infty a_2(t)dt + b_2\int_\tau^\infty |x(t)|dt\right][\|K\|_{L_\infty}[\|a\|_{L_1} + br]\\ &+ b_6\left[\int_\tau^\infty a_3(t)dt + b_3\int_\tau^\infty |x(t)|dt\right]\|K_1\|_{L_\infty}.\end{aligned}$$

Then as $\tau \to \infty$ and by equation (3) we get

$$d(GX) \leq q = (b_1b_4 + b_2b_5\|K\|_{L_\infty}[\|a\|_{L_1} + br] + b_3b_6\|K_1\|_{L_\infty})d(X). \tag{7}$$

By combining equation (6) and (7) and using equation (4), we have

$$\mu(GX) \leq q = (b_1b_4 + b_2b_5\|K\|_{L_\infty}[\|a\|_{L_1} + br] + b_3b_6\|K_1\|_{L_\infty})\mu(X).$$

Step 4. is similar as in [2].

Step 5. Let $\{x_n\} \subset M$ be an arbitrary sequence. Since $\mu(M) = 0$, $\exists\tau$, $\forall n$, the following inequality is satisfied:

$$\int_\tau^\infty |x_n(t)|dt \leq \frac{\epsilon}{4}. \tag{8}$$

Considering the functions $g(t,x,y,z)$ on $[0,\tau]\times\mathbb{R}\times\mathbb{R}\times\mathbb{R}$, $f(t,x)$ on $[0,\tau]\times\mathbb{R}$, $T_ix(t), i=1,2,3$ on $[0,\tau]\times\mathbb{R}$, $u(t,s,x)$ on $[0,\tau]\times\mathbb{R}^+\times\mathbb{R}$, $k(t,s)$ on $[0,\tau]\times[0,\tau]$ and $k_1(t,s)$ on $[0,\tau]\times[0,\tau]$ in view of Theorem 2 and Lemma 1 we can find a closed subset D_ϵ of the interval $[0,\tau]$, such that $meas(D_\epsilon^c)\leq\epsilon$, such that $g\mid_{D_\epsilon\times\mathbb{R}\times\mathbb{R}\times\mathbb{R}}$, $f\mid_{D_\epsilon\times\mathbb{R}}$, $T_i\mid_{D_\epsilon\times\mathbb{R}}$ $i=1,2,3$, $u\mid_{D_\epsilon\times\mathbb{R}^+\times\mathbb{R}}$, $k\mid_{D_\epsilon\times[0,\tau]}$ and $k_1\mid_{D_\epsilon\times[0,\tau]}$ are continuous. Especially $k\mid_{D_\epsilon\times[0,\tau]}$ and $k_1\mid_{D_\epsilon\times[0,\tau]}$ are uniformly continuous.

Let us take arbitrary $t_1,t_2\in D_\epsilon$ and assume $t_1<t_2$ without loss of generality. For an arbitrary fixed $n\in\mathbb{N}$ and denoting

$$T_{1n}(t)=(T_1x_n)(t),\qquad A_n(t)=(T_2x_n)(KFx_n)(t),$$
$$B_n(t)=(T_3x_n)(Ux_n)(t),$$

we obtain

$$\begin{aligned}
|A_n(t_2)-A_n(t_1)| &= |(T_2x_n)(t_2)\int_0^{t_2}k(t_2,s)f(s,x_n(s))ds\\
&\quad-(T_2x_n)(t_1)\int_0^{t_1}k(t_1,s)f(s,x_n(s))ds|\\
&\leq |(T_2x_n)(t_2)-(T_2x_n)(t_1)|\int_0^{t_2}|k(t_2,s)f(s,x_n(s))|ds\\
&\quad+|(T_2x_n)(t_1)\int_0^{t_2}k(t_2,s)f(s,x_n(s))ds\\
&\quad-(T_2x_n)(t_1)\int_0^{t_1}k(t_1,s)f(s,x_n(s))ds|\\
&\leq |(T_2x_n)(t_2)-(T_2x_n)(t_1)|\int_0^{t_2}k(t_2,s)[a(s)+b|x_n(s)|]ds\\
&\quad+[a_2(t_1)+b_2|x_n(t_1)|]\left[\int_{t_1}^{t_2}k(t_2,s)[a(s)+b|x_n(s)|]ds\right.\\
&\quad\left.+\int_0^{t_1}|k(t_2,s)-k(t_1,s)|[a(s)+b|x_n(s)|]ds\right].
\end{aligned}$$

Then we have

$$\begin{aligned}
|A_n(t_2)-A_n(t_1)| \;\leq\; &\omega^\tau(T_2,|t_2-t_1|)\tilde{k}\int_0^{t_2}[a(s)+b|x_n(s)|]ds\\
&+[a_2(t_1)+b_2|x_n(t_1)|]\left[\tilde{k}\int_{t_1}^{t_2}[a(s)+b|x_n(s)|]ds\right.\\
&\left.+\,\omega^\tau(k,|t_2-t_1|)\int_0^{t_1}[a(s)+b|x_n(s)]ds\right],
\end{aligned}$$

where $\omega^\tau(T_2,\cdot)$ and $\omega^\tau(k,\cdot)$ denotes the modulus continuity of the functions T_2 and k on the sets $D_\epsilon\times\mathbb{R}$ and $D_\epsilon\times[0,\tau]$, respectively and $\tilde{k}=$

$\max\{|k(t,s)| : (t,s) \in D_\epsilon \times [0,\tau]\}$. The last inequality (9) is obtained since $M \subset B_r$.

Taking into account the fact that $\mu(\{x_n\}) \leq \mu(M) = 0$, we infer that the terms of the sequence $\{\int_{t_1}^{t^2} |x_n(s)|ds\}$ are arbitrary small provided that the number $t_2 - t_1$ is small enough.

Since $\int_{t_1}^{t_2} a(s)ds$ is also arbitrary small provided the number $t_2 - t_1$ is small enough, the right hand side of (9) tends to zero independently on x_n as $t_2 - t_1$ tends to zero. We then have $\{A_n\}$ is equicontinuous in the space $C(D_\epsilon)$. Moreover,

$$\begin{aligned} |A_n(t)| &\leq |T_2(t)| \int_0^t k(t,s)|f(s,x_n(s))|ds \\ &\leq [|a_2(t)| + b_2|x_n(t)|] \int_0^t k(t,s)[a(s) + b|x_n(s)|]ds \\ &\leq \tilde{k}[d_1 + b_2 d_2][\|a\|_{L_1} + b \cdot r], \end{aligned}$$

where $|a_2(t)| \leq d_1$, $|x_n(t)| \leq d_2$ for $t \in D_\epsilon$. From the above inequality, we have that $\{A_n\}$ is equibounded in the space $C(D_\epsilon)$. In a similar way we can show that

$$\begin{aligned} |B_n(t_2) - B_n(t_1)| &\leq \omega^\tau(T_3, |t_2 - t_1|)\tau \tilde{k_1} \\ &\quad + [a_3(t_1) + b_3|x_n(t_1)|](t_2 - t_1)\tilde{k_1} \\ &\quad + [a_3(t_1) + b_3|x_n(t_1)|]\tau\omega^\tau(u, |t_2 - t_1|), \end{aligned} \tag{9}$$

where $\omega^\tau(T_3,\cdot)$ and $\omega^\tau(u,\cdot)$ denotes the modulus continuity of the functions T_3 and u on the sets $D_\epsilon \times \mathbb{R}$ and $D_\epsilon \times [0,\tau] \times \mathbb{R}$, respectively and $\tilde{k_1} = \max\{|k_1(t,s)| : (t,s) \in D_\epsilon \times [0,\tau]\}$. We have $\{B_n\}$ is equicontinuous in the space $C(D_\epsilon)$. Moreover,

$$|B_n(t)| \leq \tilde{k_1}\tau[d_3 + b_3 d_2],$$

where $|a_3(t)| \leq d_3$ for $t \in D_\epsilon$. From the above estimation, we have that $\{B_n\}$ is equibounded in the space $C(D_\epsilon)$. Furthermore, $\{T_{1n}\}$ is equicontinuous and equibounded in the space $C(D_\epsilon)$ (due to assumption (v)).

Put

$$Y_1 = \sup\{|T_{1n}(t)| : t \in D_\epsilon,\ n \in \mathbb{N}\}, \quad Y_2 = \sup\{|A_n(t)| : t \in D_\epsilon,\ n \in \mathbb{N}\}$$

$$\text{and} \quad Y_3 = \sup\{|B_n(t)| : t \in D_\epsilon,\ n \in \mathbb{N}\}.$$

Obviously Y_1, Y_2, Y_3 are finite in view of the choice of D_ϵ. Assumption (i) concludes that the function $g|_{D_\epsilon \times [-Y_1,Y_1] \times [-Y_2,Y_2] \times [-Y_3,Y_3]}$ is uniformly continuous. So the sequence $\{Gx_n\}$ is equibounded and equicontinuous in the space $C(D_\epsilon)$. Hence, by the Ascoli-Arzéla theorem [10], we obtain that the sequence $\{Gx_n\}$ forms a relatively compact set in the space $C(D_\epsilon)$.

Further observe that the above reasoning does not depend on the choice of ϵ. Thus we can construct a sequence D_l of closed subsets of the interval $[0,\tau]$ such that $meas(D_l^c) \to 0$ as $l \to \infty$ and such that the sequence $\{Gx_n\}$ is relatively compact in every space $C(D_l)$. Passing to subsequences if necessary we can assume that $\{Gx_n\}$ is a Cauchy sequence in each space $C(D_l)$, for $l = 1, 2, \cdots$.

In what follows, utilizing the fact that the set $G(M)$ is weakly compact, let us choose a number $\delta > 0$ such that for each closed subset D_δ of the interval $[0,\tau]$ such that $meas(D_\delta^c) \le \delta$, we have

$$\int_{D_\delta^c} |(Gx)(t)|dt \le \frac{\epsilon}{4} \tag{10}$$

for any $x \in M$.

Keeping in mind the fact that the sequence $\{Gx_n\}$ is a Cauchy sequence in each space $C(D_l)$ we can choose a natural number l_0 such that $meas(D_{l_0}^c) \le \delta$ and $meas(D_{l_0}) > 0$, and for arbitrary natural numbers $n, m \ge l_0$ the following inequality holds

$$|(Gx_n)(t) - (Gx_m)(t)| \le \frac{\epsilon}{4meas(D_{l_0})} \tag{11}$$

for any $t \in D_{l_0}$. Now use the above facts together with (8), (10) and (11) we obtain

$$\int_0^\infty |(Gx_n)(t) - (Gx_m)(t)|dt = \int_\tau^\infty |(Gx_n)(t) - (Gx_m)(t)|dt$$
$$+ \int_{D_{l_0}} |(Gx_n)(t) - (Gx_m)(t)|dt + \int_{D_{l_0}^c} |(Gx_n)(t) - (Gx_m)(t)|dt \le \epsilon,$$

which means that $\{G(x_n)\}$ is a Cauchy sequence in the space L_1. Hence we conclude that the set $G(M)$ is relatively strongly compact in the space L_1.

Step 6. Let us consider the set $M_0 = Conv(G(M))$. In view of the Mazur theorem we infer that the set M_0 is compact in the space L_1. Moreover, we have that the operator G transforms continuously the set M_0 into itself. Thus, we can apply the Schauder fixed point theorem and conclude that equation (1) has at least one integrable solution in $\mathbb{R}^+$. ■

Remark 2. It is easy to see that the operators $T_i x = 1$, $T_i x = x$, $T_i x = f(t,x)$ are examples of the operators T_i, $i = 1, 2, 3$, which satisfy assumption (v) of Theorem 4. Moreover, $T_i x$ can be also considered as the linear, nonlinear, Hammerstein or Urysohn integral operators.

Furthermore, assume that

$$f(t,x) = \frac{1}{1+t^2} + \frac{t}{2t+1}\sin(x), \quad k(t,s) = se^{-5(t^2+s^2)}$$

and

$$u(t,s,x) = \frac{t\cos(ts)}{1+x^2}.$$

One can easily prove that:

$$\int_0^t k(t,s)ds \le \int_0^\infty se^{-5(t^2+s^2)}\,ds \le \int_0^\infty se^{-5s^2}\,ds = \frac{1}{10}.$$

Since $\int_0^t u(t,s,x(s))ds \le \int_0^t t\cos(ts)\,ds = \sin t^2$, we get $|\int_0^t k_1(t,s)\,ds| \le 1$, which implies that $\|K_1\|_{L_\infty} \le 1$ and $\|K\|_{L_\infty} \le \frac{1}{10}$. Moreover, given arbitrary $h > 0$ and $|x_2 - x_1| \le \delta$ we have

$$\begin{aligned} |u(t,s,x_1) - u(t,s,x_2)| &\le |t\cos(ts)||\frac{x_2^2 - x_1^2}{(1+x_1^2)(1+x_2^2)}| \\ &\le \frac{2th\delta}{(1+x_1^2)(1+x_2^2)}. \end{aligned}$$

All assumption (ii)-(iv) of Theorem 4 are satisfied.

References

[1] APPELL J., ZABREJKO P.P., *Nonlinear Superposition Operators,* in: Cambridge Tracts in Mathematics, Cambridge University Press, 95(1990).

[2] BANAŚ J., CHLEBOWICZ A., On existence of integrable solutions of a functional integral equation under Carathéodory conditions, *Nonlinear Anal.,* 70(2009), 3172-3179.

[3] BANAŚ J., KNAP Z., Integrable solutions of a functional-integral equation, *Rev. Mat.Univ. Complut. Madrid,* 2(1989), 31-38.

[4] BANAŚ J., KNAP Z., Measures of weak noncompactness and nonlinear integral equations of convolution type, *J. Math. Anal. Appl.,* 146(1990), 353-362.

[5] BANAŚ J., RIVERO J., On measures of weak noncompactness, *Ann. Mat. Pura Appl.,* 151(1988), 213-224.

[6] CABALLERO J., MINGARELLI A.B., SADARANGANI K., Existence of solutions of an integral equation of Chandrasekhar type in the theory of radiative transfer, *Electr. Jour. Differ. Equat.,* 57(2006), 1-11.

[7] CHANDRASEKHAR S., *Radiative Transfer,* Dover Publications, New York, 1960.

[8] CICHOŃ M, METWALI M., Monotonic solutions for quadratic integral equations, *Discuss. Math. Diff. Incl.,* 31(2011), 157-171.

[9] DARWISH M., SADARANGANI K., Nonincreasing solutions of a functional integral equation with Carathéodory perturbed, *Mediterr. J. Math.,* 12(2015), 63-76.

[10] Deimling K., *Nonlinear Functional Analysis,* Springer-Verlag, Berlin, 1985.

[11] Gaidukevich O.I., Maslyuchenko V.K., New generalizations of the Scorza-Dragoni theorem, *Ukrainian Mathematical Journal,* 52(2000), 1010-1017.

[12] Kunze M., On a special class of nonlinear integral equations, *Jour. Integr. Equat. Appl.,* 7(1995), 329-350.

[13] Metwali M., Solvability of functional quadratic integral equations with perturbation, *Opuscula Math.,* 33(2013), 725-739.

[14] Scorza Dragoni G., Un teorema sulle funzioni continue rispetto ad une e misarubili rispetto ad un altra variable, *Rend. Sem. Mat. Univ. Padova,* 17(1948), 102-106.

[15] Zabrejko P.P., Koshlev A.I., Krasnoselskii M.A., Mikhlin S.G., Rakovshchik L.S., Stecenko V.J., Integral Equations, *Noordhoff,* Leyden, 1975.

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