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OSCILLATION CRITERIA FOR HIGHER ORDER DIFFERENTIAL EQUATIONS WITH POSITIVE AND NEGATIVE COEFFICIENTS

Abstract. A new sufficient conditions for the oscillation of all solutions of higher order neutral delay differential equations with positive and negative coefficients are given. Because, we did not find a paper which gave conditions to guarantee the existence of oscillatory solutions for those equations with positive and negative coefficients. The main distinguishing feature of results is oscillation theorems for all solutions of those homogeneous or non-homogeneous neutral equations are derived. These oscillation criteria extend and improve the results given in the recent papers.

Key words: oscillation criteria, higher order, delay differential equations, positive and negative coefficients.

AMS Mathematics Subject Classification: 34C10, 34K11, 34K40.

1. Introduction

In this paper we consider the oscillation of the higher order neutral delay differential equations

$$(\mathrm{E}_\pm) \qquad \left[x(t) \pm \sum_{i=1}^{l} h_i(t)x(\alpha_i(t))\right]^{(N)} + \sum_{i=1}^{m} p_i(t)G_1(x(\beta_i(t))) - \sum_{i=1}^{n} q_i(t)G_2(x(\gamma_i(t))) = 0,\ t > 0,$$

$$(\tilde{\mathrm{E}}_\pm) \qquad \left[x(t) \pm \sum_{i=1}^{l} h_i(t)x(\alpha_i(t))\right]^{(N)} + \sum_{i=1}^{m} p_i(t)G_1(x(\beta_i(t))) - \sum_{i=1}^{n} q_i(t)G_2(x(\gamma_i(t))) = f(t),\ t > 0,$$

where $x^{(N)}(t) \equiv d^N x/dt^N$, and N is an integer $N \geq 2$. Throughout, we assume that the following hypotheses are satisfied:

(H1) $h_i(t)(i=1,2,\dots,l) \in C^N([0,\infty);[0,\infty))$,
$p_i(t)(i=1,2,\dots,m)$, $q_i(t)(i=1,2,\dots,n) \in C([0,\infty);[0,\infty))$,
$f(t) \in C([0,\infty);\mathbb{R})$, $\mathbb{R}$ is real line;

(H2) $\alpha_i(t) \in C([0,\infty);\mathbb{R})$, $\lim_{t\to\infty} \alpha_i(t) = \infty$, $\alpha_i(t) \le t$ $(i=1,2,\dots,l)$,
$\beta_i(t) \in C([0,\infty);\mathbb{R})$, $\lim_{t\to\infty} \beta_i(t) = \infty$, $\beta_i(t) \le t$ $(i=1,2,\dots,m)$,
$\gamma_i(t) \in C([0,\infty);\mathbb{R})$, $\lim_{t\to\infty} \gamma_i(t) = \infty$, $\gamma_i(t) \le t$ $(i=1,2,\dots,n)$;

(H3) $h_i(t) \le h_i$ $(i=1,2,\dots,l)$, where h_i are nonnegative constants;

(H4) $G_i(\xi) \in C(\mathbb{R};\mathbb{R})$, $uG_i(u) > 0$ $(i=1,2)$ for $u \ne 0$, $G_1(\xi)$ is nondecreasing and there exists positive constants M such that

$$\liminf_{|u|\to\infty} \frac{G_2(u)}{u} \le M;$$

(H5) there exist a bounded function $F(t) \in C^N([0,\infty);\mathbb{R})$ such that $\lim_{t\to\infty} F^{(i)}(t) = 0$ $(i=0,1,\dots,N)$, where

$$F(t) = \int_t^\infty \int_{s_1}^\infty \cdots \int_{s_N}^\infty f(\xi)d\xi ds_{N-1}\cdots ds_1.$$

Definition 1. *By a solution of* $(\mathrm{E}_\pm)$ *(or* $(\tilde{\mathrm{E}}_\pm)$*) we mean a continuous function* $x(t)$ *which is defined for* $t \ge T$*, and satisfies* $(\mathrm{E}_\pm)$ *(or* $(\tilde{\mathrm{E}}_\pm)$*), where* $T = \min\{\alpha,\beta,\gamma\}$ *and*

$$\alpha = \inf_{t>0}\left\{\min_{1\le i\le l} \alpha_i(t)\right\},\ \beta = \inf_{t>0}\left\{\min_{1\le i\le m} \beta_i(t)\right\},\ \gamma = \inf_{t>0}\left\{\min_{1\le i\le n} \gamma_i(t)\right\}.$$

Definition 2. *A solution of* $(\mathrm{E}_\pm)$ *(or* $(\tilde{\mathrm{E}}_\pm)$*) is called oscillatory if it has arbitrary large zeros, otherwise, it is called nonoscillatory.*

Lemma 1 ([8], p.193)**.** *Let* $u(t) \in C^N([0,\infty);\mathbb{R})$ *be of constant sign and not identically zero on any interval* $[T,\infty)$*,* $T \ge 0$*, and* $u^{(N)}(t)u(t) \le 0$*. Then there exists a number* $T_0 \ge 0$ *such that the function* $u^{(j)}(t)$ $(j = 1,2,\dots,N-1)$ *are of the constant sign on* $[T_0,\infty)$ *and there exists a number* $j_0 \in \{1,3,\dots,N-1\}$ *when* N *is even or* $j_0 \in \{0,2,4,\dots,N-1\}$ *when* N *is odd such that*

$$\begin{aligned} u(t)u^{(j)}(t) > 0 \quad &for \quad j = 0,1,2,\dots,j_0, \\ (-1)^{N+j-1}u(t)u^{(j)}(t) > 0 \quad &for \quad j = j_0+1,\dots,N-1. \end{aligned}$$

Lemma 2 ([1], p.169)**.** *If* $u(t)$ *is an* N*-times differentiable function on* $[T,\infty)$ *with* $u^{(N)}(t)$ *of constant sign on* $[T,\infty)$ *, then for any* $i = 0,1,\dots,N-2$ *with* $\lim_{t\to\infty} u^{(i)}(t) = c$*,* $c \in \mathbb{R}$*, it follows that* $\lim_{t\to\infty} u^{(i+1)}(t) = 0$*.*

The oscillation and asymptotic behavior of homogeneous or non-homogeneous differential equations with positive and negative coefficients has been widely studied by numerous authors (see, [2]–[8], [10]–[17]).

In 2008, Kurpuz, Padhy and Rath [8] studied higher order neutral differential equations with positive and negative coefficients ($\tilde{\mathrm{E}}_{\pm}$), and they obtained various sufficient conditions for the oscillation of solutions of ($\tilde{\mathrm{E}}_{\pm}$). However, they were not established the oscillatory conditions for all solutions of higher order neutral differential equations with positive and negative coefficients ($\tilde{\mathrm{E}}_{\pm}$).

Our aim in this paper, we derive the sufficient conditions for the oscillation of all solutions of higher order neutral delay differential equations with positive and negative coefficients ($\mathrm{E}_{\pm}$) and ($\tilde{\mathrm{E}}_{\pm}$), and improve the results of [8]. As a consequence, we success to erase restrictive conditions of oscillatory solution of ($\tilde{\mathrm{E}}_{\pm}$), and establish the new oscillation criteria.

2. Oscillatory solutions of the equations ($\mathrm{E}_{\pm}$)

Theorem 1. *If for some $j \in \{1, 2, \dots, m\}$*

$$\int_0^\infty p_j(s)ds = \infty \tag{1}$$

and

$$\sum_{i=1}^{n} \int_0^\infty \int_{s_1}^\infty \cdots \int_{s_{N-1}}^\infty q_i(\xi)d\xi ds_{N-1}\cdots ds_1 < \frac{1}{M}, \tag{2}$$

then every solution of (E_+) *oscillates.*

Proof. Suppose that $x(t)$ is a nonoscillatory solution of (E_+). Without any loss of generality, we assume that $x(t) > 0$, $t \ge t_0$ for some $t_0 > 0$. We set

$$\begin{aligned}(3)\quad z(t) &= x(t) + \sum_{i=1}^{l} h_i(t)x(\alpha_i(t)) \\ &\quad + (-1)^N \sum_{i=1}^{n} \int_{t_0}^t \int_{s_1}^\infty \cdots \int_{s_{N-1}}^\infty q_i(\xi)G_2(x(\gamma_i(\xi)))d\xi ds_{N-1}\cdots ds_1 \\ &= X(t) + (-1)^N \sum_{i=1}^{n} \int_{t_0}^t \int_{s_1}^\infty \cdots \int_{s_{N-1}}^\infty q_i(\xi)G_2(x(\gamma_i(\xi)))d\xi ds_{N-1}\cdots ds_1\end{aligned}$$

for $t \ge t_0$. Differentiating the above equation N-times and noting (E_+) yields

$$z^{(N)}(t) = X^{(N)}(t) - \sum_{i=1}^{n} q_i(t)G_2(x(\gamma_i(t))) = -\sum_{i=1}^{m} p_i(t)G_1(x(\beta_i(t))), \tag{4}$$

which can be rewritten as

$$z^{(N)}(t) \leq -p_j(t)G_1(x(\beta_j(t))) \leq 0,\ t \geq t_0 \tag{5}$$

for some $j \in \{1, 2, \ldots, m\}$. Hence, $z^{(N)}(t)$ is nonincreasing. By applying Lemma 1, we see that $z(t), z'(t), \ldots, z^{(N-1)}(t)$ are monotonic and single sign for $t \geq t_0$. ∎

If N is odd, then

$$z(t) = X(t) - \sum_{i=1}^{n} \int_{t_0}^{t} \int_{s_1}^{\infty} \cdots \int_{s_{N-1}}^{\infty} q_i(\xi) G_2(x(\gamma_i(\xi))) d\xi ds_{N-1} \cdots ds_1$$

and

$$\lim_{t\to\infty} z(t) = \mu \in [-\infty, \infty]$$

exists, because of the monotonic property of $z(t)$.

Case 1. $\mu \in [-\infty, 0)$. If $x(t)$ is not bounded from above, there exists a number $T \geq t_0$ such that

$$\max_{t_0 \leq t \leq T} x(t) = x(T).$$

Thus we see that

$$z(T) \geq \left(\sum_{i=1}^{l} h_i(T) - M \sum_{i=1}^{n} \int_{t_0}^{T} \int_{s_1}^{\infty} \cdots \int_{s_{N-1}}^{\infty} q_i(\xi) d\xi ds_{N-1} \cdots ds_1 \right) x(T) \geq 0, \tag{6}$$

which is a contradiction. Hence, $x(t)$ is bounded from above. There exists a positive constant L such that

$$x(t) \leq L \quad \text{and} \quad L = \limsup_{t\to\infty} x(t), \tag{7}$$

and so,

$$z(t) \geq x(t) - ML \sum_{i=1}^{n} \int_{0}^{\infty} \int_{s_1}^{\infty} \cdots \int_{s_{N-1}}^{\infty} q_i(\xi) d\xi ds_{N-1} \cdots ds_1.$$

Taking the superior limit as $t \to \infty$ yields

$$\lim_{t\to\infty} z(t) \geq \left(1 - M \sum_{i=1}^{n} \int_{0}^{\infty} \int_{s_1}^{\infty} \cdots \int_{s_{N-1}}^{\infty} q_i(\xi) d\xi ds_{N-1} \cdots ds_1 \right) L \geq 0. \tag{8}$$

This is a contradiction.

Case 2. $\mu = 0$. If $z(t) \geq 0$ and $z'(t) \geq 0$, there exists a constant $k_0 > 0$ such that $z(t) \geq k_0$, which contradicts $\mu = 0$. Therefore $z'(t) < 0$. If $z'(t) < 0$ and $z''(t) < 0$, then $z'(t) \leq -k_0$. Integrating $z'(t) \leq -k_0$, we see that $\lim_{t\to\infty} z(t) = -\infty$. This is a contradiction, and so, $z''(t) \geq 0$. Proceeding as the above, we obtain

$$(-1)^i z(t) z^{(i)}(t) > 0 \ (i = 1, 2, \ldots, N-1).$$

Using this fact and Lemma 2, we obtain

$$\lim_{t\to\infty} z^{(i)} = 0 \ (i = 0, 1, \ldots, N-1).$$

Furthermore, (6) and (8) implies that

$$\lim_{t\to\infty} x(t) = 0, \tag{9}$$

which lead to $\lim_{t\to\infty} X(t) = 0$. From (9) we see that

$$0 < x(t) < \varepsilon \tag{10}$$

for some sufficiently small $\varepsilon > 0$. On the other hand, it follows from (3) that

$$z'(t) \leq X'(t) \leq z'(t) + \varepsilon M \sum_{i=1}^{n} \int_t^\infty \int_{s_2}^\infty \cdots \int_{s_{N-1}}^\infty q_i(\xi) d\xi ds_{N-1} \cdots ds_2,$$

and

$$z''(t) - \varepsilon M \sum_{i=1}^{n} \int_t^\infty \int_{s_3}^\infty \cdots \int_{s_{N-1}}^\infty q_i(\xi) d\xi ds_{N-1} \cdots ds_3 \leq X''(t) \leq z''(t).$$

Repeating the same method as in the above proof, we can show that

$$\lim_{t\to\infty} X^{(i)}(t) = 0 \ (i = 0, 1, \ldots, N-1).$$

Integrating (E_+) and (4), N times from t to ∞, we obtain (cf. [7])

$$\begin{aligned} X(t) &- \sum_{i=1}^{m} \int_t^\infty \int_{s_1}^\infty \cdots \int_{s_{N-1}}^\infty p_i(\xi) G_1(x(\beta_i(\xi))) d\xi ds_{N-1} \cdots ds_1 \\ &+ \sum_{i=1}^{n} \int_t^\infty \int_{s_1}^\infty \cdots \int_{s_{N-1}}^\infty q_i(\xi) G_2(x(\gamma_i(\xi))) d\xi ds_{N-1} \cdots ds_1 = 0 \end{aligned}$$

and

$$z(t) + \sum_{i=1}^{m} \int_t^\infty \int_{s_1}^\infty \cdots \int_{s_{N-1}}^\infty p_i(\xi) G_1(x(\beta_i(\xi))) d\xi ds_{N-1} \cdots ds_1 = 0.$$

Since

$$z(t) \le X(t)$$

holds, it is easy to see that

$$\sum_{i=1}^{n} \int_t^\infty \int_{s_1}^\infty \cdots \int_{s_{N-1}}^\infty q_i(\xi) G_2(x(\gamma_i(\xi))) d\xi ds_{N-1} \cdots ds_1 \le 0, \tag{11}$$

which is a contradiction.

Case 3. $\mu \in (0, \infty]$. Then it is easy to see from Lemma 1 that

$$z^{(N-1)}(t) > 0, \ t \ge t_1$$

for some $t_1 \ge t_0$. There exists a constant $k_1 > 0$ such that

$$z(t) \ge k_1, \ t \ge t_2$$

for some $t_2 \ge t_1$. Then we see from $z(t) \le X(t)$ that

$$k_1 \le z(t) \le x(t) + \sum_{i=1}^{l} h_i x(\alpha_i(t)).$$

Taking inferior limit as $t \to \infty$ yields

$$k_1 \le \left(1 + \sum_{i=1}^{l} h_i\right) \liminf_{t \to \infty} x(t),$$

that is,

$$x(\beta_j(t)) \ge \frac{k_1}{2}, \ t \ge t_3 \tag{12}$$

for some $t_3 \ge t_2$. Integrating (5) over $[t_3, t]$ yields

$$G_1\left(\frac{k_1}{2}\right) \int_{t_3}^t p_j(s) ds \le -z^{(N-1)}(t) + z^{(N-1)}(t_3) < \infty. \tag{13}$$

This contradicts the condition (1).

Proof. If N is even, then

$$z(t) = X(t) + \sum_{i=1}^{n} \int_{t_0}^t \int_{s_1}^\infty \cdots \int_{s_{N-1}}^\infty q_i(\xi) G_2(x(\gamma_i(\xi))) d\xi ds_{N-1} \cdots ds_1. \tag{14}$$

and $z(t) > 0$. It follows from Lemma 1 that

$$z'(t) > 0 \quad \text{and} \quad z^{(N-1)}(t) > 0,\ t \geq t_1.$$

Since $z(t) > 0$ and $z'(t) > 0$, there exists a constant $k_0 > 0$ such that

$$z(t) \geq k_0,\ t \geq t_2.$$

Substituting $z(t) \geq x(t)$ into (14) and noting $z'(t) > 0$, we obtain

$$\begin{aligned} z(t) &\leq X(t) + M \sum_{i=1}^{n} \int_{t_0}^{t} \int_{s_1}^{\infty} \cdots \int_{s_{N-1}}^{\infty} q_i(\xi) x(\gamma_i(\xi)) d\xi ds_{N-1} \cdots ds_1 \\ &\leq X(t) + M z(t) \sum_{i=1}^{n} \int_{t_0}^{t} \int_{s_1}^{\infty} \cdots \int_{s_{N-1}}^{\infty} q_i(\xi) d\xi ds_{N-1} \cdots ds_1. \end{aligned}$$

Obviously we see that

$$K_0 \equiv k_0 \left(1 - M \sum_{i=1}^{n} \int_{t_0}^{t} \int_{s_1}^{\infty} \cdots \int_{s_{N-1}}^{\infty} q_i(\xi) d\xi ds_{N-1} \cdots ds_1\right) \leq X(t).$$

Taking inferior limit, we show that

$$K_0 \leq \left(1 + \sum_{i=1}^{l} h_i\right) \liminf_{t\to\infty} x(t).$$

This means that

$$\liminf_{t\to\infty} x(t) \geq \frac{K_0}{\left(1 + \sum_{i=1}^{l} h_i\right)} \equiv K_1,$$

that is, (12) holds. Integrating (5) over $[t_3, t]$ yields (13), which contradicts the condition (1). We complete the proof of the theorem. ■

3. Oscillatory solutions of equation (E_-)

Theorem 2. *If* (1) *for some* $j \in \{1, 2, \ldots, m\}$ *and*

$$\begin{cases} \textit{if } N \textit{ is odd:} \\ \qquad \displaystyle\sum_{i=1}^{l} h_i + M \sum_{i=1}^{n} \int_{0}^{\infty} \int_{s_1}^{\infty} \cdots \int_{s_{N-1}}^{\infty} q_i(\xi) d\xi ds_{N-1} \cdots ds_1 \leq 1, \\ \textit{if } N \textit{ is even:} \\ \qquad \displaystyle M \sum_{i=1}^{n} \int_{0}^{\infty} \int_{s_1}^{\infty} \cdots \int_{s_{N-1}}^{\infty} q_i(\xi) d\xi ds_{N-1} \cdots ds_1 \leq \sum_{i=1}^{l} h_i \leq 1, \end{cases}$$

then every solution of (E_-) *oscillates.*

Proof. Let $x(t)$ be a nonoscillatory solution of (E_-). Without loss of generality, we assume that $x(t) > 0$, $t \geq t_0$ for some $t_0 > 0$. Putting

$$
\begin{aligned}
(15)\ w(t) &= x(t) - \sum_{i=1}^{l} h_i(t)x(\alpha_i(t)) \\
&\quad + (-1)^N \sum_{i=1}^{n} \int_{t_0}^{t} \int_{s_1}^{\infty} \cdots \int_{s_{N-1}}^{\infty} q_i(\xi)G_2(x(\gamma_i(\xi)))d\xi ds_{N-1}\cdots ds_1 \\
&= Y(t) + (-1)^N \sum_{i=1}^{n} \int_{t_0}^{t} \int_{s_1}^{\infty} \cdots \int_{s_{N-1}}^{\infty} q_i(\xi)G_2(x(\gamma_i(\xi)))d\xi ds_{N-1}\cdots ds_1.
\end{aligned}
$$

Differentiating (15) N-times and combining (E_-), we obtain

$$
(16)\quad w^{(N)}(t) = Y^{(N)}(t) - \sum_{i=1}^{n} q_i(t)G_2(x(\gamma_i(t))) = -\sum_{i=1}^{m} p_i(t)G_1(x(\beta_i(t))).
$$

This can be rewritten

$$
(17)\qquad w^{(N)}(t) \leq -p_j(t)G_1(x(\beta_j(t))) \leq 0,\ t \geq t_0
$$

for some $j \in \{1, 2, \ldots, m\}$. Then we conclude that $w^{(N)}(t)$ is nonincreasing. Clearly, $w(t), w'(t), \ldots, w^{(N-1)}(t)$ are monotonic and single sign for $t \geq t_0$. ■

If N is odd, then

$$
w(t) = Y(t) - \sum_{i=1}^{n} \int_{t_0}^{t} \int_{s_1}^{\infty} \cdots \int_{s_{N-1}}^{\infty} q_i(\xi)G_2(x(\gamma_i(\xi)))d\xi ds_{N-1}\cdots ds_1
$$

and $\lim_{t\to\infty} w(t) = \mu \in [-\infty, \infty]$ exists.

Case 1. $\mu \in [-\infty, 0)$. If $x(t)$ is not bounded from above, there exists a sequence $\{t_{\bar{n}}\}_{\bar{n}=1}^{\infty}$ such that

$$
(18)\qquad \lim_{\bar{n}\to\infty} t_{\bar{n}} = \infty \quad \text{and} \quad \max_{t_1 \leq t \leq t_{\bar{n}}} x(t) = x(t_{\bar{n}}).
$$

Then we have

$$
w(t_{\bar{n}}) \geq \left(1 - \sum_{i=1}^{l} h_i - M\sum_{i=1}^{n} \int_{0}^{\infty} \int_{s_1}^{\infty} \cdots \int_{s_{N-1}}^{\infty} q_i(\xi)d\xi ds_{N-1}\cdots ds_1\right) x(t_{\bar{n}}).
$$

Taking the limit as $\bar{n} \to \infty$ yields

$$
\begin{aligned}
\lim_{\bar{n}\to\infty} w(t_{\bar{n}}) \geq \Bigg(1 &- \sum_{i=1}^{l} h_i \\
&- M\sum_{i=1}^{n} \int_{0}^{\infty} \int_{s_1}^{\infty} \cdots \int_{s_{N-1}}^{\infty} q_i(\xi)d\xi ds_{N-1}\cdots ds_1\Bigg) \lim_{\bar{n}\to\infty} x(t_{\bar{n}}) \geq 0,
\end{aligned}
$$

which contradicts the assumption. Hence $x(t)$ is bounded from above. There exists a constant $L > 0$ such that (7) holds. Then we have

$$w(t) \geq x(t) - L\sum_{i=1}^{l} h_i - ML\sum_{i=1}^{n}\int_0^\infty\int_{s_1}^\infty\cdots\int_{s_{N-1}}^\infty q_i(\xi)d\xi ds_{N-1}\cdots ds_1.$$

Taking superior limit as $t\to\infty$ yields

$$\begin{aligned}&\lim_{t\to\infty} w(t)\\ &\geq \left(1-\sum_{i=1}^{l} h_i - M\sum_{i=1}^{n}\int_0^\infty\int_{s_1}^\infty\cdots\int_{s_{N-1}}^\infty q_i(\xi)d\xi ds_{N-1}\cdots ds_1\right)L \geq 0,\end{aligned}$$

which is a contradiction.

Case 2. $\mu = 0$. By the same proof of Theorem 1, we observe that

$$(-1)^i w(t)w^{(i)}(t) > 0\ (i = 1, 2, \ldots, N-1),\ t \geq t_1$$

for some $t_1 \geq t_0$, and

$$\lim_{t\to\infty} x(t) = \lim_{t\to\infty} w^{(i)}(t) = 0\ (i = 0, 1, \ldots, N-1).$$

From the definition of $Y(t)$ we obtain

$$x(t) - \sum_{i=1}^{l} h_i x(\alpha_i(t)) \leq Y(t) \leq x(t),$$

which implies that $\lim_{t\to\infty} Y(t) = 0$. Now, there exists a small number $\varepsilon > 0$ such that (10). Then we show that

$$w'(t) \leq Y'(t) \leq w'(t) + \varepsilon M\sum_{i=1}^{n}\int_t^\infty\int_{s_2}^\infty\cdots\int_{s_{N-1}}^\infty q_i(\xi)d\xi ds_{N-1}\cdots ds_2,$$

and

$$w''(t) - \varepsilon M\sum_{i=1}^{n}\int_t^\infty\int_{s_3}^\infty\cdots\int_{s_{N-1}}^\infty q_i(\xi)d\xi ds_{N-1}\cdots ds_3 \leq Y''(t) \leq w''(t).$$

Repeating the same method as in the above, we have

$$\lim_{t\to\infty} Y^{(i)}(t) = 0\ (i = 0, 1, \ldots, N-1).$$

Integrating (16) and (E_+) N-times, we have

$$Y(t)+\sum_{i=1}^{m}\int_t^\infty\int_{s_1}^\infty\cdots\int_{s_{N-1}}^\infty p_i(\xi)G_1(x(\beta_i(\xi)))d\xi ds_{N-1}\cdots ds_1 \\ -\sum_{i=1}^{n}\int_t^\infty\int_{s_1}^\infty\cdots\int_{s_{N-1}}^\infty q_i(\xi)G_2(x(\gamma_i(\xi)))d\xi ds_{N-1}\cdots ds_1=0 \tag{19}$$

and

$$w(t)-\sum_{i=1}^{m}\int_t^\infty\int_{s_1}^\infty\cdots\int_{s_{N-1}}^\infty p_i(\xi)G_1(x(\beta_i(\xi)))d\xi ds_{N-1}\cdots ds_1=0. \tag{20}$$

Combining (19), (20) and $w(t)\le Y(t)$, we obtain the contradiction (11).

Case 3. $\mu\in(0,\infty]$. It follows from Lemma 1 that

$$w^{(N-1)}(t)>0,\ t\ge t_2$$

for some $t_2\ge t_0$. There exists a constant $k_0>0$ and a number $t_3\ge t_2$ such that

$$x(t)\ge w(t)\ge k_0,\ t\ge t_3,$$

which implies that (12) holds. By integrating (17) we obtain the contradiction

$$G_1\left(\frac{k_1}{2}\right)\int_{t_3}^t p_j(s)ds\le -w^{(N-1)}(t)+w^{(N-1)}(t_3)<\infty. \tag{21}$$

Proof. If N is even, then

$$w(t)=Y(t)+\sum_{i=1}^{n}\int_{t_0}^t\int_{s_1}^\infty\cdots\int_{s_{N-1}}^\infty q_i(\xi)G_2(x(\gamma_i(\xi)))d\xi ds_{N-1}\cdots ds_1 \tag{22}$$

and $\lim_{t\to\infty}w(t)=\mu\in[-\infty,\infty]$ exists.

Case 1. $\mu\in[-\infty,0)$. If $x(t)$ is not bounded from above, then there exists a sequence $\{t_{\bar n}\}_{\bar n=1}^\infty$ such that (18) holds. Hence we have

$$w(t_{\bar n})\ge\left(1-\sum_{i=1}^{l}h_i\right)x(t_{\bar n}),$$

that is,

$$\lim_{\bar n\to\infty}w(t_{\bar n})\ge\left(1-\sum_{i=1}^{l}h_i\right)\lim_{\bar n\to\infty}x(t_{\bar n})\ge 0$$

as $\bar{n} \to \infty$. This is a contradiction. Therefore, $x(t)$ is bounded from above. There exists a positive constant L satisfying (7). Then

$$w(t) \geq x(t) - L \sum_{i=1}^{l} h_i.$$

By taking superior limit as $t \to \infty$, we obtain

$$\lim_{t\to\infty} w(t) \geq \left(1 - \sum_{i=1}^{l} h_i\right) L \geq 0,$$

which is a contradiction.

Case 2. $\mu = 0$. Applying the same proof of the case when N is odd, we can lead to a contradiction.

Case 3. $\mu \in (0, \infty]$. It follows from Lemma that $w^{(N-1)} > 0$. There exists a constant $k_0 > 0$ such that $w(t) \geq k_0$. If $x(t)$ is not bounded from above, there exists a sequence $\{t_n\}_{n=1}^{\infty}$ satisfying (18). Then

$$\begin{aligned} k_0 &\leq \left(1 + M \sum_{i=1}^{n} \int_0^{\infty} \int_{s_1}^{\infty} \cdots \int_{s_{N-1}}^{\infty} q_i(\xi) d\xi ds_{N-1} \cdots ds_1\right) x(t_n) \\ &\leq 2x(t_n), \end{aligned}$$

which means that (12) holds. Thus we can lead to the contradiction (21). Therefore, $x(t)$ is bounded from above. There exists a constant $L > 0$ such that (7) holds. Then

$$\begin{aligned} k_0 &\leq x(t) - \sum_{i=1}^{l} h_i x(\alpha_i(t)) \\ &\quad + ML \sum_{i=1}^{n} \int_0^{\infty} \int_{s_1}^{\infty} \cdots \int_{s_{N-1}}^{\infty} q_i(\xi) d\xi ds_{N-1} \cdots ds_1. \end{aligned}$$

Taking inferior limit as $t \to \infty$, we observe that

$$\begin{aligned} k_0 &\leq \liminf_{t\to\infty} x(t) \\ &\quad + \left(M \sum_{i=1}^{n} \int_0^{\infty} \int_{s_1}^{\infty} \cdots \int_{s_{N-1}}^{\infty} q_i(\xi) d\xi ds_{N-1} \cdots ds_1 - \sum_{i=1}^{l} h_i\right) L \\ &\leq \liminf_{t\to\infty} x(t), \end{aligned}$$

which implies that (12) is satisfied, moreover, (21) holds. This is a contradiction. Therefore, we complete the proof. ∎

4. Oscillatory solutions of equations $(\tilde{\mathrm{E}}_+)$

Theorem 3. *If* (1) *for some* $j \in \{1, 2, \ldots, m\}$, *and* (2) *holds, then every solution of* $(\tilde{\mathrm{E}}_+)$ *oscillates.*

Proof. Suppose that $x(t)$ is a nonoscillatory solution of $(\tilde{\mathrm{E}}_+)$. We may assume that $x(t) > 0$, $t \geq t_0$ for some $t_0 > 0$. In view of (H5) there exists a $\varepsilon_F > 0$ such that $F(t) \leq \varepsilon_F$. If we now define

$$Z(t) = z(t) + \tilde{F}, \;\; t \geq t_1, \tag{23}$$

where

$$\tilde{F} = \begin{cases} F(t), & N \text{ is odd}, \\ -F(t) + \varepsilon_F, & N \text{ is even} \end{cases} \tag{24}$$

for sufficiently large $t_1 > t_0$. Differentiating (23) N-times and substituting $(\tilde{\mathrm{E}}_+)$, we obtain

$$\begin{aligned} Z^{(N)}(t) \;&=\; X^{(N)}(t) - \sum_{i=1}^{n} q_i(t) G_2(x(\gamma_i(t))) - f(t) \\ &= -\sum_{i=1}^{m} p_i(t) G_1(x(\beta_i(t))), \;\; t \geq t_1, \end{aligned} \tag{25}$$

which can be rewritten as follows

$$Z^{(N)}(t) = -p_j(t) G_1(x(\beta_j(t))) \leq 0, \;\; t \geq t_1 \tag{26}$$

for some $j \in \{1, 2, \ldots, m\}$. Then we conclude that $Z^{(N)}(t)$ is nonincreasing, and $Z(t), Z'(t), \ldots, Z^{(N-1)}(t)$ are monotonic and single sign for $t \geq t_1$. ■

If N is odd, then

$$Z(t) = X(t) - \sum_{i=1}^{n} \int_{t_0}^{t} \int_{s_1}^{\infty} \cdots \int_{s_{N-1}}^{\infty} q_i(\xi) G_2(x(\gamma_i(\xi))) d\xi ds_{N-1} \cdots ds_1 + F(t)$$

and $\lim_{t \to \infty} Z(t) = \mu \in [-\infty, \infty]$ exists.

Case 1. $\mu \in [-\infty, 0)$. If $x(t)$ is not bounded from above, there exists a sequence $\{t_{\bar{n}}\}_{\bar{n}=1}^{\infty}$ such that (18) holds. Hence we see that

$$Z(t_{\bar{n}}) \geq \left(1 - M \sum_{i=1}^{n} \int_{0}^{\infty} \int_{s_1}^{\infty} \cdots \int_{s_{N-1}}^{\infty} q_i(\xi) d\xi ds_{N-1} \cdots ds_1 \right) x(t_{\bar{n}}) + F(t_{\bar{n}}).$$

Taking limit as $\bar{n} \to \infty$, we obtain

$$\begin{aligned}
&\lim_{n\to\infty} Z(t_{\bar{n}})\\
&\geq \left(1 - M\sum_{i=1}^{n}\int_0^{\infty}\int_{s_1}^{\infty}!\cdots\int_{s_{N-1}}^{\infty} q_i(\xi)d\xi ds_{N-1}\cdots ds_1\right)\lim_{\bar{n}\to\infty} x(t_{\bar{n}}) \geq 0,
\end{aligned}$$

which is a contradiction. Consequently, $x(t)$ is bounded from above. There exists a positive constant L such that (7) holds. Then we have

$$Z(t) \geq x(t) - ML\sum_{i=1}^{n}\int_0^{\infty}\int_{s_1}^{\infty}\cdots\int_{s_{N-1}}^{\infty} q_i(\xi)d\xi ds_{N-1}\cdots ds_1 + F(t).$$

Taking superior limit as $t \to \infty$ yields

$$\lim_{t\to\infty} Z(t) \geq \left(1 - M\sum_{i=1}^{n}\int_0^{\infty}\int_{s_1}^{\infty}\cdots\int_{s_{N-1}}^{\infty} q_i(\xi)d\xi ds_{N-1}\cdots ds_1\right) L \geq 0.$$

This is a contradiction.

Case 2. $\mu = 0$. From the same proof of Theorem 1, it follows that

$$\lim_{t\to\infty} x(t) = \lim_{t\to\infty} X(t) = \lim_{t\to\infty} Z^{(i)} = 0 \ (i = 0, 1, \ldots, N-1).$$

Thus there exists a small number $\varepsilon > 0$ such that (10). Hence we see that

$$\begin{aligned}
Z'(t) - F'(t) &\leq X'(t)\\
&\leq Z'(t) + \varepsilon M\sum_{i=1}^{n}\int_t^{\infty}\int_{s_2}^{\infty}\cdots\int_{s_{N-1}}^{\infty} q_i(\xi)d\xi ds_{N-1}\cdots ds_2,
\end{aligned}$$

and

$$\begin{aligned}
&Z''(t) - \varepsilon M\sum_{i=1}^{n}\int_t^{\infty}\int_{s_3}^{\infty}\cdots\int_{s_{N-1}}^{\infty} q_i(\xi)d\xi ds_{N-1}\cdots ds_3 - F''(t)\\
&\leq X''(t) \leq Z''(t) - F''(t).
\end{aligned}$$

By the similar proof of Theorem 1, we state that

$$\lim_{t\to\infty} X^{(i)}(t) = 0 \ (i = 0, 1, \ldots, N-1).$$

Integrating (25) and $(\tilde{\mathrm{E}}_{+})$ N-times yields

$$\begin{aligned}
X(t) &- \sum_{i=1}^{m}\int_t^{\infty}\int_{s_1}^{\infty}\cdots\int_{s_{N-1}}^{\infty} p_i(\xi)G_1(x(\beta_i(\xi)))d\xi ds_{N-1}\cdots ds_1\\
&+ \sum_{i=1}^{n}\int_t^{\infty}\int_{s_1}^{\infty}\cdots\int_{s_{N-1}}^{\infty} q_i(\xi)G_2(x(\gamma_i(\xi)))d\xi ds_{N-1}\cdots ds_1 = -F(t)
\end{aligned}$$

and

$$Z(t) - \sum_{i=1}^{m} \int_t^\infty \int_{s_1}^\infty \cdots \int_{s_{N-1}}^\infty p_i(\xi) G_1(x(\beta_i(\xi))) d\xi ds_{N-1} \cdots ds_1 = 0.$$

Substituting the above equations into

$$Z(t) \le X(t) + F(t),$$

we can lead to the contradiction (11).

Case 3. $\mu \in (0, \infty]$. From Lemma we see that

$$Z^{(N-1)}(t) > 0, \ t \ge t_1$$

for some $t_1 \ge t_0$. There exists a constant $k_0 > 0$ such that

$$Z(t) \ge k_0, \ t \ge t_2$$

for some $t_2 \ge t_1$. Then we see that

$$\begin{aligned} k_0 &\le X(t) + F(t) \\ &\le x(t) + \sum_{i=1}^{l} h_i x(\alpha_i(t)) + F(t). \end{aligned}$$

Taking inferior limit as $t \to \infty$ yields (12) holds. By integrating (26) over $[t_2, t]$, we have

$$G_1\left(\frac{k_1}{2}\right) \int_{t_2}^t p_j(s) ds \le -Z^{(N-1)}(t) + Z^{(N-1)}(t_2) < \infty,$$

which is a contradiction.

Proof. If N is even, then

$$\begin{aligned} Z(t) &= X(t) + \sum_{i=1}^{n} \int_{t_0}^t \int_{s_1}^\infty \cdots \int_{s_{N-1}}^\infty q_i(\xi) G_2(x(\gamma_i(\xi))) d\xi ds_{N-1} \cdots ds_1 \\ &\quad - F(t) + \varepsilon_F, \end{aligned}$$

which means that $Z(t) \ge x(t) > 0$. Using the similar proof of Theorem 1, we can prove the rest part of this proof, and hence we omit its proof. We complete the proof of Theorem. ■

Example 1. Consider the equation

$$\left[x(t)+\frac{3}{4}x(t-\pi)\right]^{(4)}+\left(\frac{1}{4}+e^{-t}\right)x(t-3\pi) \\ -\frac{1}{2}e^{-t}x(t-\pi)=-\frac{1}{2}e^{-t}\cos t,\ t>0. \tag{27}$$

It is easy to see that all conditions of Theorem 3 holds. Therefore, every solutions of (27) oscillates. In fact, $x(t)=\cos t$ is such a solution.

5. Oscillatory solutions of equations $(\tilde{\mathrm{E}}_{-})$

Theorem 4. *If all the conditions of Theorem 2 hold, then every solution of* $(\tilde{\mathrm{E}}_{-})$ *oscillates.*

Proof. Suppose that $x(t)$ is a nonoscillatory solution of $(\tilde{\mathrm{E}}_{-})$. We may assume that $x(t)>0$, $t\ge t_0$ for some $t_0>0$. The function $W(t)$ defined with

$$W(t)=w(t)+\tilde{F},\ t\ge t_1, \tag{28}$$

where $\tilde{F}$ is defined by (24). Differentiating (28) N-times and using $(\tilde{\mathrm{E}}_{-})$, we obtain

$$\begin{aligned} W^{(N)}(t) &= Y^{(N)}(t)-\sum_{i=1}^{n}q_i(t)G_2(x(\gamma_i(t)))-f(t) \\ &= -\sum_{i=1}^{m}p_i(t)G_1(x(\beta_i(t))). \end{aligned} \tag{29}$$

Rewrite (29) in the form

$$W^{(N)}(t)\le -p_j(t)G_1(x(\beta_j(t)))\le 0,\ t\ge t_0. \tag{30}$$

for some $j\in\{1,2,\dots,m\}$. Therefore, $W(t),W'(t),\dots,W^{(N-1)}(t)$ are monotonic and single sign for $t\ge t_1$. ■

If N is odd, then

$$W(t)=Y(t)-\sum_{i=1}^{n}\int_{t_0}^{t}\int_{s_1}^{\infty}\cdots\int_{s_{N-1}}^{\infty}q_i(\xi)G_2(x(\gamma_i(\xi)))d\xi ds_{N-1}\cdots ds_1+F(t)$$

and $\lim\limits_{t\to\infty}W(t)=\mu\in[-\infty,\infty]$ exists.

Case 1. $\mu \in [-\infty, 0)$. If $x(t)$ is not bounded from above, there exists a sequence $\{t_{\bar{n}}\}_{\bar{n}=1}^{\infty}$ satisfying (18). Then we obtain

$$\begin{aligned} W(t_{\bar{n}}) \ \geq \ & \Bigg(1 - \sum_{i=1}^{l} h_i \\ & - M \sum_{i=1}^{n} \int_0^\infty \int_{s_1}^\infty \cdots \int_{s_{N-1}}^\infty q_i(\xi) d\xi ds_{N-1} \cdots ds_1 \Bigg) x(t_{\bar{n}}) + F(t_{\bar{n}}). \end{aligned}$$

Taking limit as $\bar{n} \to \infty$ yields

$$\begin{aligned} \lim_{\bar{n}\to\infty} W(t_{\bar{n}}) \geq & \Bigg(1 - \sum_{i=1}^{l} h_i \\ & - M \sum_{i=1}^{n} \int_0^\infty \int_{s_1}^\infty \cdots \int_{s_{N-1}}^\infty q_i(\xi) d\xi ds_{N-1} \cdots ds_1 \Bigg) \lim_{\bar{n}\to\infty} x(t_{\bar{n}}) \geq 0, \end{aligned}$$

which is a contradiction. Next, we assume that $x(t)$ is bounded from above. There exists a positive constant L such that (7) holds. Then it is clear that

$$\begin{aligned} W(t) \ \geq \ & x(t) - L \sum_{i=1}^{l} h_i \\ & - ML \sum_{i=1}^{n} \int_0^\infty \int_{s_1}^\infty \cdots \int_{s_{N-1}}^\infty q_i(\xi) d\xi ds_{N-1} \cdots ds_1 + F(t), \end{aligned}$$

and taking superior limit as $t \to \infty$ yields

$$\begin{aligned} \varlimsup_{t\to\infty} W(t) \ \geq \ & \Bigg(1 - \sum_{i=1}^{l} h_i \\ & - M \sum_{i=1}^{n} \int_0^\infty \int_{s_1}^\infty \cdots \int_{s_{N-1}}^\infty q_i(\xi) d\xi ds_{N-1} \cdots ds_1 \Bigg) L \geq 0. \end{aligned}$$

This is a contradiction.

Case 2. $\mu = 0$. From the same proof of Theorem 2, we see that (10) and

$$\lim_{t\to\infty} x(t) = \lim_{t\to\infty} Y(t) = \lim_{t\to\infty} W^{(i)}(t) = 0 \ (i = 0, 1, \ldots, N-1).$$

for sufficiently small $\varepsilon > 0$. Hence, we obtain

$$\begin{aligned} & W'(t) - F'(t) \leq Y'(t) \\ & \quad \leq W'(t) + \varepsilon M \sum_{i=1}^{n} \int_t^\infty \int_{s_2}^\infty \cdots \int_{s_{N-1}}^\infty q_i(\xi) d\xi ds_{N-1} \cdots ds_2 - F'(t) \end{aligned}$$

and

$$W''(t) - \varepsilon M \sum_{i=1}^{n} \int_t^\infty \int_{s_3}^\infty \cdots \int_{s_{N-1}}^\infty q_i(\xi) d\xi ds_{N-1} \cdots ds_3 - F''(t)$$
$$\le Y''(t) \le W''(t) - F''(t).$$

From the same proof of Theorem 2 we have

$$\lim_{t\to\infty} Y^{(i)}(t) = 0 \ (i = 0, 1, \ldots, N-1).$$

Integrating (29) and ($\tilde{\mathrm{E}}_-$) N-times and noting $W(t) \le Y(t) + F(t)$, we obtain (11), which is a contradiction.

Case 3. $\mu \in (0, \infty]$. It follows from Lemma 1 that

$$W^{(N-1)}(t) > 0, \ t \ge t_2$$

for some $t_2 \ge t_1$. There exists a constant $k_0 > 0$ such that

$$x(t) + F(t) \ge W(t) \ge k_0, \ t \ge t_2.$$

By taking inferior limit as $t \to \infty$, we show that (12) holds for some $t_3 \ge t_2$. Therefore we obtain the contradiction

$$G_1\left(\frac{k_1}{2}\right) \int_{t_3}^t p_j(s) ds \le -W^{(N-1)}(t) + W^{(N-1)}(t_3) < \infty \tag{31}$$

by integrating (30) over $[t_3, t]$.

If N is even, then

$$W(t) = Y(t) + \sum_{i=1}^{n} \int_{t_0}^t \int_{s_1}^\infty \cdots \int_{s_{N-1}}^\infty q_i(\xi) G_2(x(\gamma_i(\xi))) d\xi ds_{N-1} \cdots ds_1$$
$$- F(t) + \varepsilon_F,$$

and $\lim_{t\to\infty} W(t) = \mu \in [-\infty, \infty]$ exists.

Case 1. $\mu \in [-\infty, 0)$. If $x(t)$ is not bounded from above, there exists a sequence $\{t_{\bar{n}}\}_{\bar{n}=1}^\infty$ such that (18) holds. Hence we obtain

$$W(t_{\bar{n}}) \ge \left(1 - \sum_{i=1}^{l} h_i\right) x(t_{\bar{n}}),$$

that is,

$$\lim_{\bar{n}\to\infty} W(t_{\bar{n}}) \ge \left(1 - \sum_{i=1}^{l} h_i\right) \lim_{\bar{n}\to\infty} x(t_{\bar{n}}) \ge 0.$$

This is a contradiction. Therefore $x(t)$ is bounded from above. There exists a constant $L > 0$ satisfies (7). It is obvious that

$$W(t) \geq x(t) - L \sum_{i=1}^{l} h_i.$$

Taking superior limit as $t \to \infty$ yields

$$\varlimsup_{t\to\infty} W(t) \geq \left(1 - \sum_{i=1}^{l} h_i\right) L \geq 0,$$

which is a contradiction.

Case 2. $\mu = 0$. From the same proof of the case when N is odd, we obtain

$$\lim_{t\to\infty} Y^{(i)}(t) = \lim_{t\to\infty} W^{(i)}(t) = 0 \ (i = 0, 1, \ldots, N-1).$$

Integrating (29) and ($\tilde{\text{E}}_-$) N times and noting $W(t) \geq Y(t) - F(t) + \varepsilon_F$, we can lead to the contradiction

$$-\varepsilon_F \geq \sum_{i=1}^{n} \int_t^{\infty} \int_{s_1}^{\infty} \cdots \int_{s_{N-1}}^{\infty} q_i(\xi) G_2(x(\gamma_i(\xi))) d\xi ds_{N-1} \cdots ds_1.$$

Case 3. $\mu \in (0, \infty]$. There exists a constant $k_1 > 0$ such that

$$W(t) \geq k_1, \ t \geq t_4$$

for some $t_4 \geq t_0$. If $x(t)$ is not bounded from above, there exists a sequence $\{t_{\bar{n}}\}_{\bar{n}=1}^{\infty}$ such that (18) is satisfied. Then we obtain

$$\begin{aligned} k_1 \ &\leq \ \left(1 + M \sum_{i=1}^{n} \int_0^{\infty} \int_{s_1}^{\infty} \cdots \int_{s_{N-1}}^{\infty} q_i(\xi) d\xi ds_{N-1} \cdots ds_1\right) x(t_{\bar{n}}) \\ &\quad - F(t_{\bar{n}}) + \varepsilon_F \\ &\leq \ 2x(t_{\bar{n}}) - F(t_{\bar{n}}) + \varepsilon_F. \end{aligned}$$

Proof. Taking limit as $\bar{n} \to \infty$ yields

$$\varliminf_{\bar{n}\to\infty} x(t_{\bar{n}}) \geq \frac{(k_1 - \varepsilon_F)}{2}$$

for $k_1 > \varepsilon_F > 0$. If we choose $\varepsilon_F = k_1/2$, then we see that (12) holds. Taking the account into $W^{(N-1)}(t) > 0$, we obtain the contradiction (31). Hence, $x(t)$ is bounded from above. Applying the same proof of Theorem 2, we can lead to a contradiction. Therefore we complete the proof. ■

Example 2. Consider the equation

$$
\begin{aligned}
(32)\qquad \left[x(t)-\frac{1}{2}x(t-2\pi)\right]''' &+ \frac{1}{2}\left(1-e^{-t}\right)x(t-\frac{3}{2}\pi) \\
&- \frac{1}{4}e^{-t}x(t-\frac{\pi}{2}) = -\frac{1}{4}e^{-t}\cos t,\ t>0.
\end{aligned}
$$

It is easy to see that all conditions of Theorem 4 holds. Therefore, every solutions of (32) oscillates. In fact, $x(t)=\sin t$ is such a solution.

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