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NORMAL FAMILIES AND SHARED FUNCTION II

Abstract. Let $k, n \in \mathbb{N}, l \in \mathbb{N} \backslash \{1\}, m \in \mathbb{N} \cup \{0\}$, and $a(z)(\not\equiv 0)$ be a holomorphic function, all of whose zeros have multiplicities at most m. Let $\mathcal{F}$ be a family of meromorphic functions in D such that multiplicities of zeros of each $f \in \mathcal{F}$ are at least $k+m$. If for $f, g \in \mathcal{F}$ satisfy $f^l(f^{(k)})^n$ and $g^l(g^{(k)})^n$ share $a(z)$, then $\mathcal{F}$ is normal in D. The examples are provided to show that the result is sharp. The result extends the related theorems [9,10,12]. we also omit the conditions "m is divisible by $n+l$" and "all poles of f have multiplicities at least $m+1$" in the result due to Meng, Liu and Xu [12] [Journal of Computational Analysis and Applications 27(3)(2019), 511-526].

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1. Introduction and main results

Let D be a domain in $\mathbb{C}$ and $\mathcal{F}$ be a family of meromorphic functions in D. A family $\mathcal{F}$ is said to be normal in D, if for each sequence f_n in $\mathcal{F}$ there exists a subsequence f_{n_j} converges spherically locally uniformly to a meromorphic function or ∞ in D.

Let $f(z)$ and $g(z)$ be two meromorphic functions in D. Given a function $\varphi(z)$, if $f(z)-\varphi(z)$ and $g(z)-\varphi(z)$ have the same zeros without multiplicity in D, we said that $f(z)$ and $g(z)$ share a function $\varphi(z)$ IM.

In 1967, Hayman proposed the following normal conjecture.

Theorem A. [1]. *Let $n \in \mathbb{N}$, and $a \in \mathbb{C} \backslash \{0\}$. let $\mathcal{F}$ be a family of meromorphic function in D. If $f^n f' \neq a$, for each $f \in \mathcal{F}$, then $\mathcal{F}$ is normal in D.*

This normal conjecture was showed by L. Yang and G. Zhang [2] (for $n \geq 5$), Y. X. Gu [3] (for $n = 4, 3$), X. C. Pang [4] (for $n \geq 2$) and Chen and Fang [5] (for $n = 1$).

In 1999, Pang and Zalcman proved the following result.

Theorem B. [6]. *Let $k, n \in \mathbb{N}$, and $a \in \mathbb{C}\setminus\{0\}$. Let $\mathcal{F}$ be a family of holomorphic functions in a unit disc Δ such that each $f \in \mathcal{F}$ has only zeros of multiplicity at least k. If $f^n f^{(k)} \neq a$ for each $f \in \mathcal{F}$ in Δ, then $\mathcal{F}$ is normal in Δ.*

In 2008, using the shared values, Zhang proved.

Theorem C. *Let $n \in \mathbb{N}\setminus\{1\}$, $a \in \mathbb{C}\setminus\{0\}$. Let $\mathcal{F}$ be a family of meromorphic functions in D. If for $f, g \in \mathcal{F}$, $f^n f'$ and $g^n g'$ share a, then $\mathcal{F}$ is normal in D.*

In 2009, Meng and Hu [8] extended Theorem B-C, later Deng, Lei and Fang [9] improved Meng's result and obtained.

Theorem D. *Let $k \in \mathbb{N}, n \in \mathbb{N}\setminus\{1\}, m \in \mathbb{N}\cup\{0\}$, and let $a(z)(\not\equiv 0)$ be a holomorphic function, all zeros of $a(z)$ have multiplicities at most m. Let $\mathcal{F}$ be a family of meromorphic functions in D. If for each $f \in \mathcal{F}$, the zeros of f have multiplicity at least $k+m$, and for $f, g \in \mathcal{F}$, $f^n f^{(k)}$ and $g^n g^{(k)}$ share $a(z)$, then $\mathcal{F}$ is normal in D.*

In 2011, Jiang and Gao [10] considered the case of $f(f^{(k)})^n$ and proved.

Theorem E. *Let $k \in \mathbb{N}\setminus\{1\}, m \in \mathbb{N}\cup\{0\}, n(\geq 2m+2) \in \mathbb{N}$, and let $a(z)(\not\equiv 0)$ be a holomorphic function, all zeros of $a(z)$ have multiplicities at most m, which is divisible by $n+1$. Let $\mathcal{F}$ be a family of meromorphic functions in D. If for each $f \in \mathcal{F}$, the zeros of f have multiplicity at least $max\{k+m, 2m+2\}$, and for $f, g \in \mathcal{F}$, $f(f^{(k)})^n$ and $g(g^{(k)})^n$ share $a(z)$, then $\mathcal{F}$ is normal in D.*

In 2013, Ding, Ding and Yuan [11] studied the general case of $f^l(f^{(k)})^n$ and obtained.

Theorem F. *Let $k, l \in \mathbb{N}, n \in \mathbb{N}\setminus\{1\}, a \in \mathbb{C}\setminus\{0\}$. Let $\mathcal{F}$ be a family of meromorphic functions in D. If for each $f \in \mathcal{F}$, the zeros of f have multiplicity at least $max\{k, 2\}$, and for $f, g \in \mathcal{F}$, $f^l(f^{(k)})^n$ and $g^l(g^{(k)})^n$ share a, then $\mathcal{F}$ is normal in D.*

Recently, Meng, Liu and Xu [12] considered the case of sharing a holomorphic function and promoted Ding's result.

Theorem G. *Let $k, l \in \mathbb{N}, n \in \mathbb{N}\setminus\{1\}, m \in \mathbb{N}\cup\{0\}$, and let $a(z)(\not\equiv 0)$ be a holomorphic function, all zeros of $a(z)$ have multiplicities at most m, which is divisible by $n+l$. Let $\mathcal{F}$ be a family of meromorphic functions in D. If for each $f \in \mathcal{F}$, the zeros of f have multiplicities at least $k+m+1$ and all poles of f are of multiplicity at least $m+1$, and for $f, g \in \mathcal{F}$, $f^l(f^{(k)})^n$ and $g^l(g^{(k)})^n$ share $a(z)$, then $\mathcal{F}$ is normal in D.*

According to the above results, naturally, we ask the following questions.

Question 1. Can we omit the conditions “all zeros of $a(z)$ have multiplicity divisible by $n+l$” and “all poles of f have multiplicity at least $m+1$” in Theorem G?

Question 2. Can we reduce the condition “the multiplicity of the zeros from $k+m+1$ to $k+m$” in Theorem G?

In this paper, our main goal is to solve the above questions and obtain the following results.

Theorem 1. *Let $k,n \in \mathbb{N}, l \in \mathbb{N}\backslash\{1\}, m \in \mathbb{N}\cup\{0\}$, and let $a(z)(\not\equiv 0)$ be a holomorphic function, all of whose zeros have multiplicity at most m. Let $\mathcal{F}$ be a family of meromorphic functions in D. If for every $f\in\mathcal{F}$, the zeros of f have multiplicity at least $k+m$, and for $f,g\in\mathcal{F}$, $f^l(f^{(k)})^n$ and $g^l(g^{(k)})^n$ share $a(z)$, then $\mathcal{F}$ is normal in D.*

Theorem 2. *Let $k,n \in \mathbb{N}, l \in \mathbb{N}\backslash\{1\}, m \in \mathbb{N}\cup\{0\}$, and let $a(z)(\not\equiv 0)$ be a holomorphic function, all zeros of $a(z)$ have multiplicities at most m. Let $\mathcal{F}$ be a family of meromorphic functions in D. If for each $f\in\mathcal{F}$, the zeros of f have multiplicity at least $k+m$, and for $f\in\mathcal{F}$, $f^l(f^{(k)})^n - a(z)$ has at most one zero in D, then $\mathcal{F}$ is normal in D.*

Example 1. Let $D=\{z:|z|<1\}$ and $a(z)\equiv 0$. Let $\mathcal{F}=\{f_j(z)\}$, where

$$f_j(z)=e^{jz}, z\in D, j=1,2\cdots.$$

Then $f_j^l(z)\left(f_j^{(k)}\right)^n(z)-a(z)$ does not have zero in D, however $\mathcal{F}$ is not normal at $z=0$. This shows that $a(z)\not\equiv 0$ is necessary in Theorem 1 and 2.

Example 2. Let $D=\{z:|z|<1\}$ and $a(z)=\frac{1}{z^{l+k+n}}$. Let $\mathcal{F}=\{f_j(z)\}$, where

$$f_j(z)=\frac{1}{jz}, z\in D, j=1,2\cdots, j^{l+n}\neq(-1)^k k!.$$

Then $f_j^l(z)\left(f_j^{(k)}\right)^n(z)-a(z)$ does not have zero in D, however $\mathcal{F}$ is not normal at $z=0$. This shows that Theorem 1 and 2 are not valid if $a(z)$ is a meromorphic function in D.

Example 3. Let $D=\{z:|z|<1\}$, $a(z)=a$. Let $\mathcal{F}=\{f_j(z)\}$, where

$$f_j(z)=jz^{k-1}, z\in D, j=1,2\cdots.$$

Then $f_j^l(z)\left(f_j^{(k)}\right)^n(z)-a$, which has no zero in D , however $\mathcal{F}$ is not normal at $z=0$. This shows that the condition “ all zeros of f have multiplicity at least $k+m$ ” in Theorem 1 and 2 is sharp.

Example 4. Let $D = \{z : |z| < 1\}$, $a(z) = a$. Let $\mathcal{F} = \{f_j(z)\}$, where

$$f_j(z) = jz^k, \quad z \in D, j = 1, 2 \cdots .$$

Then $f_j^l(z)\left(f_j^{(k)}\right)^n(z) - a = j^{l+n}(k!)^n z^{lk} - a$, which has at least $l \geq 2$ distinct zeros in D , however $\mathcal{F}$ is not normal at $z = 0$. This shows that the condition "$f^l(f^{(k)})^n - a(z)$ has at most one zero" in Theorem 2 is necessary.

2. Some lemmas

Lemma 1 ([13]). *Let $\mathcal{F}$ be a family of functions meromorphic in the unit disc Δ, all of whose zeros have multiplicity at least k. Then if $\mathcal{F}$ is not normal in any neighbourhood of $z_0 \in \Delta$, there exist, for each α, $0 \leq \alpha < k$,*
(i) points z_n, $z_n \to z_0$, $z_0 \in \Delta$;
(ii) functions $f_n \in \mathcal{F}$; and
(iii) positive numbers $\rho_n \to 0^+$, such that $g_n(\xi) = \rho_n^{-\alpha} f_n(z_n + \rho_n \xi) \to g(\xi)$ spherically uniformly on compact subsets of $\mathbb{C}$, where g is a non-constant meromorphic function, all of whose zeros have multiplicity at least k.

Lemma 2 ([14]). *Let $k, n \in \mathbb{N}$, $l \in \mathbb{N}\setminus\{1\}$, $a \in \mathbb{C} \setminus \{0\}$, and let $f(z)$ be a non-constant meromorphic with all zeros that have multiplicity at least k. Then $f^l(z)(f^{(k)})^n(z) - a$ has at least two distinct zeros.*

Using the idea of Chang[15], we get the following lemma.

Lemma 3. *Let $k, l, n, m \in \mathbb{N}$, let $q(z)$ be a polynomial of degree m, and let $f(z)$ be a non-constant rational function with $f(z) \neq 0$. Then $f^l(z)(f^{(k)})^n(z) - q(z)$ has at least $l + kn + n$ distinct zeros.*

The proof of Lemma 3 is almost exactly the same with Lemma 11 in Deng etc. [16], here, we omit the detail.

Lemma 4 ([17]). *Let $f_j (j = 1, 2)$ be nonconstant meromorphic function, then*

$$N(r, f_1 f_2) - N(r, \frac{1}{f_1 f_2}) = N(r, f_1) + N(r, f_2) - N(r, \frac{1}{f_1}) - N(r, \frac{1}{f_2}).$$

Lemma 5. *Let $k, m, n \in \mathbb{N}$, $l \in \mathbb{N}\setminus\{1\}$, let $q(z)$ be a polynomial of degree m, and let $f(z)$ be a non-constant meromorphic function in $\mathbb{C}$, the zeros of $f(z)$ have multiplicities at least $k + m$.Then $(f(z))^l(f^{(k)})^n(z) - q(z)$ has at least two distinct zeros.*

Proof. Since

$$\begin{aligned}\frac{1}{f^{l+n}} &= \left(\frac{f^{(k)}}{f}\right)^n \frac{1}{q} - \frac{f^l\left(f^{(k)}\right)^n - q}{qf^{l+n}} \\ &= \frac{f^l\left(f^{(k)}\right)^n}{qf^{l+n}} - \frac{\left[f^l\left(f^{(k)}\right)^n\right]' q - q'\left[f^l\left(f^{(k)}\right)^n\right]}{qf^{l+n}} \\ &\quad \times \frac{f^l\left(f^{(k)}\right)^n - q}{\left[f^l\left(f^{(k)}\right)^n\right]' q - q'\left[f^l\left(f^{(k)}\right)^n\right]}.\end{aligned}$$

Noticing that $m(r, \frac{f^{(k)}}{f}) = S(r,f)$, $m(r, \frac{1}{q}) = O(1)$, and $m(r,q) = m\ logr + O(1)$. Applying the First Fundamental Theorem, we get

$$\begin{aligned}m\left(r, \frac{1}{f^{l+n}}\right) &= (l+n)\, m\left(r, \frac{1}{f}\right) \\ &\leq m\left(r, \frac{\left[f^l\left(f^{(k)}\right)^n\right]' q - q'\left[f^l\left(f^{(k)}\right)^n\right]}{qf^{l+n}}\right) \\ &\quad + m\left(r, \frac{f^l\left(f^{(k)}\right)^n - q}{\left[f^l\left(f^{(k)}\right)^n\right]' q - q'\left[f^l\left(f^{(k)}\right)^n\right]}\right) + S(r,f) \\ &\leq T\left(r, \frac{f^l\left(f^{(k)}\right)^n - q}{\left[f^l\left(f^{(k)}\right)^n\right]' q - q'\left[f^l\left(f^{(k)}\right)^n\right]}\right) \\ &\quad - N\left(r, \frac{f^l\left(f^{(k)}\right)^n - q}{\left[f^l\left(f^{(k)}\right)^n\right]' q - q'\left[f^l\left(f^{(k)}\right)^n\right]}\right) + S(r,f) \\ &\leq T\left(r, \frac{\left[f^l\left(f^{(k)}\right)^n\right]' q - q'\left[f^l\left(f^{(k)}\right)^n\right]}{f^l\left(f^{(k)}\right)^n - q}\right) \\ &\quad - N\left(r, \frac{f^l\left(f^{(k)}\right)^n - q}{\left[f^l\left(f^{(k)}\right)^n\right]' q - q'\left[f^l\left(f^{(k)}\right)^n\right]}\right) + S(r,f)\end{aligned}$$

By Lemma 4, we can have

$$(l+n)\, m\left(r, \frac{1}{f}\right) \leq m\left(r, \frac{\left[\frac{f^l\left(f^{(k)}\right)^n - 1}{q}\right]'}{\frac{f^l\left(f^{(k)}\right)^n - 1}{q}}\right) + N\left(r, \frac{1}{f^l\left(f^{(k)}\right)^n - q}\right)$$

$$- N\left(r, f^l\left(f^{(k)}\right)^n - q\right) + N\left(r, \left[f^l\left(f^{(k)}\right)^n\right]' q - q'\left[f^l\left(f^{(k)}\right)^n\right]\right)$$
$$-N\left(r, \frac{1}{\left[f^l\left(f^{(k)}\right)^n\right]' q - q'\left[f^l\left(f^{(k)}\right)^n\right]}\right) + m\log r + S(r,f).$$

This is

$$(l+n)\,m\left(r, \frac{1}{f}\right) \le \overline{N}(r,f) + N\left(r, \frac{1}{f^l\left(f^{(k)}\right)^n - q}\right)$$
$$- N\left(r, \frac{1}{\left[f^l\left(f^{(k)}\right)^n\right]' q - q'\left[f^l\left(f^{(k)}\right)^n\right]}\right) + m\log r + S(r,f).$$

We add $(l+n)N(r,\frac{1}{f})$ to both sides, then

$$(l+n)\,T\left(r, \frac{1}{f}\right) \le (l+n)N(r,\frac{1}{f}) + \overline{N}(r,f) + N\left(r, \frac{1}{f^l\left(f^{(k)}\right)^n - q}\right)$$
$$- N\left(r, \frac{1}{\left[f^l\left(f^{(k)}\right)^n\right]' q - q'\left[f^l\left(f^{(k)}\right)^n\right]}\right) + m\log r + S(r,f).$$

Noticing that

$$\left[f^l\left(f^{(k)}\right)^n\right]' q - q'\left[f^l\left(f^{(k)}\right)^n\right] = \left[f^l\left(f^{(k)}\right)^n - q\right]' q - q'\left[f^l\left(f^{(k)}\right)^n - q\right],$$

which implies

$$N\left(r, \frac{1}{\left[f^l\left(f^{(k)}\right)^n\right]' q - q'\left[f^l\left(f^{(k)}\right)^n\right]}\right)$$
$$\ge N\left(r, \frac{1}{f^l\left(f^{(k)}\right)^n - q}\right) - \overline{N}\left(r, \frac{1}{f^l\left(f^{(k)}\right)^n - q}\right).$$

Therefore, we get

$$(l+n)\,T(r,f) \;\le\; (kn+1)\overline{N}(r,\frac{1}{f}) + \overline{N}(r,f)$$
$$+ \overline{N}\left(r, \frac{1}{f^l\left(f^{(k)}\right)^n - q}\right) + m\log r + S(r,f).$$

i.e.,

$$\left(l+n-1-\frac{kn+1}{k+m}\right) T(r,f) \le \overline{N}\left(r, \frac{1}{f^l\left(f^{(k)}\right)^n - q}\right) + m\log r + S(r,f),$$

where we denote

$$M = l + n - 1 - \frac{kn+1}{k+m} = l - 1 + \frac{mn-1}{k+m}.$$

Suppose that $f^l(z)(f^{(k)})^n(z) - q(z)$ has at most one zero.

Next we complete our proof in two steps.

Step 1: $n \geq 2$. By the assumptions,

$$M \geq 1 + \frac{1}{k+m}.$$

Then

$$T(r,f) < MT(r,f) \leq (m+1)\log r + S(r,f).$$

It follows that $f(z)$ is a rational function of degree $< m+1$. Since the zeros of $f(z)$ have multiplicities at least $k+m \geq m+1$, then we get $f(z) \neq 0$. Thus, by Lemma 3, we obtain that $f^l(z)(f^{(k)})^n(z) - q(z)$ has at least $l+kn+n \geq 6$ distinct zeros, which is a contradiction.

Step 2: $n = 1$. Then $M = l - \frac{k+1}{k+m}$.

Sub-step 2.1: $m \geq 2$. By the assumptions, $M > 1$ and

$$T(r,f) < (m+1)\log r + S(r,f).$$

It follows that $f(z)$ is a rational function of degree $< m+1$. Since the zeros of $f(z)$ have multiplicities at least $k+m \geq m+1$, then we get $f(z) \neq 0$. Thus, by Lemma 3, we obtain that $f^l(z)(f^{(k)})^n(z) - q(z)$ has at least $l+k+1 \geq 4$ distinct zeros, which is a contradiction.

Sub-step 2.2: $m = 1$. Then

$$(l-1)T(r,f) \leq \overline{N}\left(r, \frac{1}{f^l f^{(k)} - q}\right) + \log r + S(r,f),$$

Sub-step 2.2.1: $f^l(z)f^{(k)}(z) - q(z) \neq 0$. By the assumptions, we get

$$T(r,f) \leq (l-1)T(r,f) \leq \log r + S(r,f).$$

It follows that $f(z)$ is a rational function of degree ≤ 1. Since the zeros of $f(z)$ have multiplicities at least $k+1 \geq 2$, then we get $f(z) \neq 0$. Thus, by Lemma 3, we obtain that $f^l(z)(f^{(k)})^n(z) - q(z)$ has at least $l+k+1 \geq 4$ distinct zeros, which is a contradiction.

Sub-step 2.2.2: $f^l(z)f^{(k)}(z) - q(z) = 0$. By the assumptions, we get $f^l(z)f^{(k)}(z) - q(z)$ has only one zero. Then we obtain

$$(l-1)T(r,f) \leq 2\log r + S(r,f).$$

Sub-step 2.2.2.1: $l \geq 3$, then

$$T(r,f) \leq \log r + S(r,f).$$

It follows that $f(z)$ is a rational function of degree ≤ 1. Since the zeros of $f(z)$ have multiplicities at least $k+1 \geq 2$, then we get $f(z) \neq 0$. Thus, by Lemma 3, we obtain that $f^l(z)(f^{(k)})^n(z) - q(z)$ has at least $l+k+1 \geq 5$ distinct zeros, which is a contradiction.

Sub-step 2.2.2.2: $l = 2$, then

$$T(r,f) \leq 2\log r + S(r,f).$$

It follows that $f(z)$ is a rational function of degree ≤ 2.

Sub-step 2.2.2.2.1: $k \geq 2$. Since the zeros of $f(z)$ have multiplicities at least $k+1 \geq 3$, then we get $f(z) \neq 0$. Thus, by Lemma 3, we obtain that $f^l(z)(f^{(k)})^n(z) - q(z)$ has at least $l+k+1 \geq 5$ distinct zeros, which is a contradiction.

Sub-step 2.2.2.2.2: $k = 1$. Then we get $f(z) \neq 0$ or $f(z)$ has only one zero with multiplicity 2.

The former case can be ruled out from Lemma 3. Hence $f(z)$ has the following forms:

(i) $f(z) = A(z-z_0)^2$;

(ii) $f(z) = \frac{A(z-z_0)^2}{(z-z_1)}$;

(iii) $f(z) = \frac{A(z-z_0)^2}{(z-z_1)^2}$;

(iv) $f(z) = \frac{A(z-z_0)^2}{(z-z_1)(z-z_2)}$,

where A, z_0 are nonzero constants, and z_1, z_2 are distinct constants. Clearly, $z_0 \neq z_1, z_0 \neq z_2$, and $T(r,f) = 2\log r + O(1)$.

We now show (i). Obviously, $\overline{N}(r,\frac{1}{f}) \leq \frac{1}{2}T(r,f) + O(1)$. Noticing that

$$3T(r,f) \leq 2\overline{N}(r,\frac{1}{f}) + \overline{N}(r,f) + 2\log r + S(r,f).$$

Then

$$T(r,f) \leq \log r + S(r,f),$$

a contradiction.

We now show (ii) or (iii). Obviously, $\overline{N}(r,\frac{1}{f}) \leq \frac{1}{2}T(r,f) + O(1), \overline{N}(r,f) = \log r$ or $\overline{N}(r,f) \leq \frac{1}{2}T(r,f) + O(1)$. Noticing that

$$3T(r,f) \leq 2\overline{N}(r,\frac{1}{f}) + \overline{N}(r,f) + 2\log r + S(r,f).$$

Then

$$T(r,f) \leq \frac{4}{3}\log r + S(r,f),$$

we also get a contradiction.

We now show (iv). Then

$$f^2(z)\,f'(z) = \frac{A^3\,(z-z_0)^5\,[(2z_0-(z_1+z_2))\,z+2z_1z_2-z_0\,(z_1+z_2)]}{(z-z_1)^4\,(z-z_2)^4}.$$

Since $q(z)=Bz+C$, where $B\neq 0, C$ are constants, and $f^l(z)f^{(k)}(z)-q(z)$ has only one zero. Then we have

$$f^2(z)f'(z) = Bz+C+\frac{d(z-Z_0)^t}{(z-z_1)^4\,(z-z_2)^4}.$$

Obviously, By calculation, we get $d=-B, t=9$, and $Z_0\neq z_0$.

Differentiating the above two equations separately, we obtain

$$[f^2(z)f'(z)]'' = \frac{(z-z_0)^3 g(z)}{(z-z_1)^6\,(z-z_2)^6},$$

where $g(z)$ is a polynomial of degree ≤ 5, and

$$[f^2(z)f'(z)]'' = \frac{(z-Z_0)^7 h(z)}{(z-z_1)^6\,(z-z_2)^6},$$

where $h(z)$ is a polynomial of degree ≤ 4.

Since $z_0\neq Z_0$, then $(z-Z_0)^7$ is a factor of $g(z)$. Thus $g(z)$ is a polynomial of degree ≥ 7, which is impossible. ∎

Lemma 6. *Let $k,n\in\mathbb{N}, l\in\mathbb{N}\backslash\{1\}$, and let $\mathcal{F}=\{f_m\}$ be a sequence of meromorphic functions, $g_m(z)$ be a sequence of holomorphic functions in D such that $g_m(z)\longrightarrow g(z)$, where $g(z)(\neq 0)$ be a holomorphic function. If all zeros of function $f_m(z)$ have multiplicity at least k, and $f_m^l(z)(f_m^{(k)}(z))^n - g_n(z)$ has at most one zero, then $\mathcal{F}$ is normal in D.*

Proof. Suppose that $\mathcal{F}$ is not normal at $z_0\in D$. By Lemma 1, there exists $z_m\to z_0$, $\rho_m\to 0^+$, and $f_m\in\mathcal{F}$ such that

$$h_m(\xi) = \frac{f_m(z_m+\rho_m\xi)}{\rho_m^{\frac{kn}{l+n}}} \longrightarrow h(\xi)$$

locally uniformly on compact subsets of $\mathbb{C}$, where $h(\xi)$ is a non-constant meromorphic function in $\mathbb{C}$. By Hurwitz's theorem, all zeros of $h(\xi)$ have multiplicity at least k.

For each $\xi\in\mathbb{C}/\{h^{-1}(\infty)\}$, we have

$$\begin{aligned} h_m^l(\xi)(h_m^{(k)}(\xi))^n - g_m(z_m+\rho_m\xi) &= f_m^l(z_m+\rho_m\xi)(f_m^{(k)})^n(z_m+\rho_m\xi)\\ &\quad - g_m(z_m+\rho_m\xi) \longrightarrow h^l(\xi)(h^{(k)})^n(\xi)-g(z_0). \end{aligned}$$

Claim 1: $h^l(\xi)(h^{(k)})^n(\xi) - g(z_0) \not\equiv 0$.

Suppose that $h^l(\xi)(h^{(k)})^n(\xi) - g(z_0) \equiv 0$, then $h(\xi) \neq 0$ since $g(z_0) \neq 0$. It follows that

$$\frac{1}{h^{l+n}(\xi)} \equiv \frac{1}{g(z_0)}[\frac{h^{(k)}(\xi)}{h(\xi)}]^n.$$

Thus

$$(l+n)m(r, \frac{1}{h}) = m(r, \frac{1}{g(z_0)}[\frac{h^{(k)}(\xi)}{h(\xi)}]^n) = S(r, h).$$

Then $T(r, h) = S(r, h)$ since $h \neq 0$. we can deduce that $h(\xi)$ is a constant, a contradiction. The claim is proved.

Claim 2: $h^l(\xi)(h^{(k)})^n(\xi) - g(z_0)$ has at most one zero.

Otherwise, suppose that ξ_1, ξ_2 are two distinct zeros of $h^l(\xi)(h^{(k)})^n(\xi) - g(z_0)$. We choose a positive number δ small enough such that $D_1 \cap D_2 = \emptyset$ and $h^l(\xi)(h^{(k)})^n(\xi) - g(z_0)$ has no other zeros in $D_1 \cup D_2$ except for ξ_1 and ξ_2,where $D_1 = \{\xi : |\xi - \xi_1| < \delta\}$ and $D_2 = \{\xi : |\xi - \xi_2| < \delta\}$.

By Hurwitz's theorem, for sufficiently large m, there exist points $\xi_{1,m} \to \xi_1$ and $\xi_{2,m} \to \xi_2$ such that

$$f_m^l(z_m + \rho_m\xi_{1,m})(f_m^{(k)})^n(z_m + \rho_m\xi_{1,m}) - g_m(z_m + \rho_m\xi_{1,m}) = 0,$$

and

$$f_m^l(z_m + \rho_m\xi_{2,m})(f_m^{(k)})^n(z_m + \rho_m\xi_{2,m}) - g_m(z_m + \rho_m\xi_{2,m}) = 0.$$

Since $f_m^l(z)(f_m^{(k)}(z))^n - g_m(z)$ has at most one zero in D, then

$$z_m + \rho_m\xi_{1,m} = z_m + \rho_m\xi_{2,m},$$

this is

$$\xi_{1,m} = \xi_{2,m} = \frac{z_0 - z_m}{\rho_m},$$

which contradicts the fact $D_1 \cap D_2 = \emptyset$. The claim is proved.

From Lemma 2, we get $h^l(z)(h^{(k)})^n(z) - g(z_0)$ has at least two distinct zeros, a contradiction. Therefore $\mathcal{F}$ is normal in D. ■

3. Proof of Theorem 2

Proof. Suppose that $\mathcal{F}$ is not normal at z_0. From Lemma 6, we obtain $a(z_0) = 0$. Without loss of generality, we assume that $z_0 = 0$ and $a(z) =$

$z^t b(z)$, where $1 \le t \le m$, $b(0) = 1$. Then by Lemma 1, there exists $z_j \longrightarrow 0$, $f_j \in \mathcal{F}$ and $\rho_j \longrightarrow 0^+$ such that

$$g_j(\xi) = \frac{f_j(z_j + \rho_j \xi)}{\rho_j^{\frac{kn+t}{l+n}}} \longrightarrow g(\xi)$$

locally uniformly on compact subsets of $\mathbb{C}$, where $g(\xi)$ is a non-constant meromorphic functions in $\mathbb{C}$. By Hurwitz's theorem, all zeros of $g(\xi)$ have multiplicity at least $k + m$.

We now consider the following two steps.

Step I. Let $\frac{z_n}{\rho_n} \to \alpha, \alpha \in \mathbb{C}$.

For each $\xi \in \mathbb{C}/\{g^{-1}(\infty)\}$, we can be easily calculated that

$$\begin{aligned} & g_j^l(\xi)\,(g_j^{(k)}(\xi))^n - \left(\xi + \frac{z_j}{\rho_j}\right)^t b\,(z_j + \rho_j \xi) \\ &= \frac{f_j^l(z_j + \rho_j \xi)\,(f_j^{(k)}(z_j + \rho_j \xi))^n - a\,(z_j + \rho_j \xi)}{\rho_j^t} \\ &\longrightarrow g^l(\xi)\,(g^{(k)}(\xi))^n - (\xi + \alpha)^t . \end{aligned}$$

Since for sufficiently large j, $f_j^l(z_j + \rho_j \xi)\,(f_j^{(k)}(z_j + \rho_j \xi))^n - a\,(z_j + \rho_j \xi)$ has one zero, from the proof Lemma 6, we can deduce that $g^l(\xi)\,(g^{(k)}(\xi))^n - (\xi + \alpha)^t$ has at most one distinct zero.

By Lemma 5, $g^l(\xi)\,(g^{(k)}(\xi))^n - (\xi + \alpha)^t$ have at least two distinct zeros. Thus $g(\xi)$ is a constant, we can get a contradiction.

Step II. Let $\frac{z_n}{\rho_n} \to \infty$.

Set

$$F_j(\xi) = \frac{f_j(z_j + \rho_j \xi)}{\rho_j^{\frac{kn+t}{l+n}}}.$$

It follows that

$$\begin{aligned} & F_j^l(\xi)(F_j^{(k)}(\xi))^n - (1 + \xi)^t b(z_j + z_j \xi) \\ &= \frac{f_j^l(z_j + z_j \xi)(f_j^{(k)}(z_j + z_j \xi))^n - a(z_j + z_j \xi)}{z_j^t}. \end{aligned}$$

As the same argument as in Lemma 6, we can deduce that $F_j^l(\xi)(F_j^{(k)}(\xi))^n - (1 + \xi)^t b(z_j + z_j \xi)$ has at most one zero in $\Delta = \{\xi : |\xi| < 1\}$.

Since all zeros of F_j have multiplicity at least $k + m$, and $(1 + \xi)^t b(z_j + z_j \xi) \to (1 + \xi)^t \neq 0$ for $\xi \in \Delta$. Then by Lemma 6, $\{F_n\}$ is normal in Δ.

Therefore, there exists a subsequence of $\{F_n(z)\}$(we still express it as $\{F_n(z)\}$) such that $\{F_n(z)\}$ converges spherically locally uniformly to a meromorphic function $F(z)$ or ∞.

If $F(0) \neq \infty$, then, for each $\xi \in \mathbb{C}/\{g^{-1}(\infty)\}$, we have

$$g^{(k+m-1)}(\xi) = \lim_{j\to\infty} g_j^{(k+m-1)}(\xi) = \lim_{j\to\infty} \frac{f_j^{(k+m-1)}(z_j+\rho_j\xi)}{\rho_j^{\frac{kn+t}{l+n}-(k+m-1)}}$$
$$= \lim_{j\to\infty} \left(\frac{\rho_j}{z_j}\right)^{k+m-1-\frac{kn+t}{l+n}} F_j^{(k+m-1)}\left(\frac{\rho_j}{z_j}\xi\right) = 0.$$

Hence $g^{(k+m-1)} \equiv 0$. It follows that g is a polynomial of degree $\leq k+m-1$. Note that all zeros of g have multiplicity at least $k+m$, then we get that g is a constant, which is a contradiction.

If $F(0) = \infty$, then, for each $\xi \in \mathbb{C}/\{g^{-1}(0)\}$, we get

$$\frac{1}{F_j\left(\frac{\rho_j}{z_j}\xi\right)} = \frac{z_j^{\frac{kn+t}{l+n}}}{f_j(z_j+\rho_j\xi)} \to \frac{1}{F(0)} = 0,$$

It follows that we have

$$\frac{1}{g(\xi)} = \lim_{j\to\infty} \frac{\rho_j^{\frac{kn+t}{l+n}}}{f_j(z_j+\rho_j\xi)} = \lim_{j\to\infty}\left(\frac{\rho_j}{z_j}\right)^{\frac{kn+t}{l+n}} \frac{z_j^{\frac{kn+t}{l+n}}}{f_j(z_j+\rho_j\xi)} = 0.$$

Thus $g(\xi) = \infty$, which contradicts that $g(\xi)$ is a non-constant meromorphic function.

Therefore $\mathcal{F}$ is normal at $z_0 = 0$. Hence $\mathcal{F}$ is normal in D. ■

4. Proof of Theorem 1

Proof. Let $z_0 \in D$, $f \in \mathcal{F}$, we show that $\mathcal{F}$ is normal at z_0.

Step I. If $f^l(z_0)(f^{(k)}(z_0))^n \neq a(z_0)$.

Then there exists $D_\delta(z_0) = \{z : |z - z_0| < \delta\}$ such that

$$f^l(z)(f^{(k)}(z))^n \neq a(z)$$

in $D_\delta(z_0)$.

Since $f, g \in \mathcal{F}$, $f^l(z)(f^{(k)}(z))^n$ and $g^l(z)(g^{(k)}(z))^n$ share $a(z)$ in D. So, for each $g \in \mathcal{F}$, $g^l(z)(g^{(k)}(z))^n \neq a(z)$ in $D_\delta(z_0)$. By Theorem 2, $\mathcal{F}$ is normal in $D_\delta(z_0)$. Hence $\mathcal{F}$ is normal at z_0.

Step II. If $f^l(z_0)(f^{(k)}(z_0))^n = a(z_0)$.

Then there exists $D_\delta(z_0) = \{z : |z - z_0| < \delta\}$ such that

$$f^l(z)(f^{(k)}(z))^n \neq a(z)$$

in $D^0_\delta(z_0) = \{z : 0 < |z - z_0| < \delta\}$.

Since $f, g \in \mathcal{F}$, $f^l(z)(f^{(k)}(z))^n$ and $g^l(z)(g^{(k)}(z))^n$ share $a(z)$ in D. Thus, for each $g \in \mathcal{F}$, $g^l(z)(g^{(k)}(z))^n \neq a(z)$ in $D^0_\delta(z_0)$ and $g^l(z_0)(g^{(k)}(z_0))^n = a(z_0)$. Therefore, $g^l(z)(g^{(k)}(z))^n - a(z)$ have only one zero in $D_\delta(z_0)$. By Theorem 2, $\mathcal{F}$ is normal in $D_\delta(z_0)$. Thus $\mathcal{F}$ is normal at z_0. Hence $\mathcal{F}$ is normal in D. ■

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