F A S C I C U L I   M A T H E M A T I C I

Nr 62 2019

DOI: 10.21008/j.0044-4413.2019.0012

S. Touati, S. Lemita, M. Ghiat and M.-Z. Aissaoui

SOLVING A NONLINEAR VOLTERRA-FREDHOLM INTEGRO-DIFFERENTIAL EQUATION WITH WEAKLY SINGULAR KERNELS

Abstract. The aim of the current work is to investigate the numerical study of an integro-differential nonlinear Volterra -Fredholm equation with a weakly singular kernels. Our approximation technique is based on the product integration method in conjunction with an iterative scheme. The existence and uniqueness of the solution have been proved. We conclude the paper with numerical examples to illustrate the effectiveness of our method.

Key words: Volterra-Fredholm, fixed point, product integration method, weakly singular kernels.

AMS Mathematics Subject Classification: 45B05, 45D05, 47G20, 65R20.

1. Introduction

Integral and integro-differential equations are important in many applications of applied mathematics. Since the analytical solution of these equations is difficult to find, hence it is important to develop a numerical methods to approach their solutions. Several numerical methods are used for that, for example: meshless method [4], radial basis functions (RBFs) [5, 15], via Bell polynomials [9], modification of hat functions (MHFs) [10], operational matrix method [11], collocation method [12].

An important kind of integral and integro-differential equations is the Volterra-Fredholm type arising in many physical and biological problems, for instance: epidemic development in a population living in a habitat, also, we can find this type of equations in the parabolic boundary value problems [3, 8]. The mathematical theory of these problems is still under construction, where we can mention the work of [2] and [19].

So far, we propose a general equations type, by considering the following integro-differential nonlinear Volterra-Fredholm equation with a weakly singular kernels

$$\begin{aligned} u(t) &= f(t) + \int_a^t p_1(t-s) K_1(t,s,u(s),u'(s))\,ds \\ &\quad + \int_a^b p_2(\mid t-s \mid) K_2(t,s,u(s),u'(s))\,ds, \quad t \in [a,b], \end{aligned} \tag{1}$$

where, the unknown is $u \in C^1([a,b])$ and f is a given function in the same space.

Recently, in[7] the authors studied a Volterra type equation similar to equation (1) with $K_2 = 0$, and in [17] the authors studied a Fredholm type equation similar to equation (1) for $K_1 = 0$. Notice that the derivative of the unknown u' appear inside the integral so, we need to join the equation (1) with another equation contains more information about the solution u, so if we derive the both sides of equation (1) we get the following equation:

$$\begin{aligned} u'(t) &= f'(t) + \int_a^t p_1'(t-s) K_1(t,s,u(s),u'(s))\,ds \\ &\quad + \int_a^t p_1(t-s) \frac{\partial K_1}{\partial t}(t,s,u(s),u'(s))\,ds \\ &\quad + \int_a^b \gamma\, p_2'(\mid t-s \mid) K_2(t,s,u(s),u'(s))\,ds \\ &\quad + \int_a^b p_2(\mid t-s \mid) \frac{\partial K_2}{\partial t}(t,s,u(s),u'(s))\,ds. \end{aligned} \tag{2}$$

Where γ represents the sign of $(t-s)$ as follow:

$$\gamma = sign(t-s) = \begin{cases} 1 & t > s, \\ -1 & t < s, \\ 0 & t = s. \end{cases}$$

We take into consideration from [7], that the singularity came from the derivative of functions $p_1(s)$ and $p_2(s)$ as the following form:

$$(H1) \quad \left\| \begin{array}{ll} (1) & p_m \in W^{1,1}(0,b-a), \quad m = 1,2, \\ (2) & p_1(0) = 0, \\ (3) & \lim_{s \to t} p_1'(t-s) = +\infty, \\ (4) & \lim_{s \to 0^+} p_2'(s) = +\infty, \end{array} \right.$$

where the Banach space:

$$W^{1,1}(0,b-a) = \{p \in L^1(0,b-a), p' \in L^1(0,b-a)\}$$

is equipped with the norm:

$$\|p\|_{W^{1,1}(0,b-a)} = \|p\|_{L^1(0,b-a)} + \|p'\|_{L^1(0,b-a)}.$$

In this paper, firstly we investigate the existence and uniqueness of equation (1), then we develop an iterative scheme to find its approximate solution.

2. Analytical study

We suppose that $K_m, m = 1, 2$, satisfy the following hypotheses:

$$(H2)\quad \left\| \begin{array}{ll} (1) & f \in C^1([a,b],\mathbb{R}),\ \dfrac{\partial K_m}{\partial t} \in C^0([a,b]^2 \times \mathbb{R}^2),\ m = 1,2. \\ & \exists A,\ B,\ \bar{A},\ \bar{B},\ C,\ D,\ \bar{C},\ \bar{D} > 0, \\ & \qquad \forall t, s \in [a,b],\ \forall x,\ y,\ \bar{x},\ \bar{y} \in \mathbb{R}, \text{ such that :} \\ & |K_1(t,s,x,y) - K_1(t,s,\bar{x},\bar{y})| \le A|x-y| + B|\bar{x}-\bar{y}|, \\ (2) & |K_2(t,s,x,y) - K_2(t,s,\bar{x},\bar{y})| \le C|x-y| + D|\bar{x}-\bar{y}|, \\ & \left|\dfrac{\partial K_1}{\partial t}(t,s,x,y) - \dfrac{\partial K_1}{\partial t}(t,s,\bar{x},\bar{y})\right| \le \bar{A}|x-y| + \bar{B}|\bar{x}-\bar{y}|, \\ & \left|\dfrac{\partial K_2}{\partial t}(t,s,x,y) - \dfrac{\partial K_2}{\partial t}(t,s,\bar{x},\bar{y})\right| \le \bar{C}|x-y| + \bar{D}|\bar{x}-\bar{y}|. \end{array} \right.$$

Now, to prove the existence and uniqueness of the solution of (1), we define for all $f \in C^1([a,b])$ the functional T_f by:

$$\begin{aligned} \forall t \in [a,b],\quad & T_f : C^1([a,b]) \longrightarrow C^1([a,b]) \\ & \xi \longmapsto T_f(\xi)(t) = f(t) + \int_a^t p_1(|\,t-s\,|)K_1(t,s,u(s),u'(s))\,ds \\ & \qquad\qquad + \int_a^b p_2(|\,t-s\,|)K_2(t,s,u(s),u'(s))\,ds. \end{aligned}$$

For all $\xi \in C^1([a,b])$ equipped with norm $\|\xi\|_{C^1([a,b])} = \|\xi\|_\infty + \|\xi'\|_\infty$ such functional T_f is continuous from $C^1([a,b])$ into itself, because it is the sum of three well defined operators (see [7]).

Theorem 1. *According to hypotheses $(H1)$, $(H2)$ and if we suppose that there exist $\varrho > 0$ such that, for all $t \in [a,b]$*

$$\max\{A,B,\bar{A},\bar{B}\}\int_a^b |p_1(t-s)|\,ds + \max\{C,D,\bar{C},\bar{D}\}\int_a^b |p_2(|\,t-s\,|)|\,ds \le \varrho < \frac{1}{3},$$

$$\max\{A,B\}\int_a^b \left|p_1'(t-s)\right|\,ds + \max\{C,D\}\int_a^b \left|p_2'(|\,t-s\,|)\right|\,ds \le \varrho < \frac{1}{3},$$

then the solution u of (1) exists and unique in $C^1([a,b])$.

Proof. We apply the Banach fixed point theorem on the functional T_f in the space $C^1([a,b])$ equipped with the norm defined above.

For all $t \in [a,b]$, we have:

$$
\begin{aligned}
&|T_f(u_1(t)) - T_f(u_2(t))| \\
&\leq \int_a^t |p_1(t-s)||K_1(t,s,u_1(s),u_1'(s)) - K_1(t,s,u_2(s),u_2'(s))|\,ds \\
&\quad + \int_a^b |p_2(\mid t-s \mid)||K_2(t,s,u_1(s),u_1'(s)) - K_2(t,s,u_2(s),u_2'(s))|\,ds \\
&\leq A\int_a^t |p_1(t-s)||u_1(s)-u_2(s)|\,ds + B\int_a^t |p_1(t-s)||u_1'(s)-u_2'(s)|\,ds \\
&\quad + C\int_a^b |p_2(\mid t-s \mid)||u_1(s)-u_2(s)|\,ds \\
&\quad + D\int_a^b |p_2(\mid t-s \mid)||u_1'(s)-u_2'(s)|\,ds \\
&\leq A\int_a^t |p_1(t-s)|\,ds\|u_1-u_2\|_\infty + B\int_a^t |p_1(t-s)|\,ds\|u_1'-u_2'\|_\infty \\
&\quad + C\int_a^b |p_2(\mid t-s \mid)|\,ds\|u_1-u_2\|_\infty + D\int_a^b |p_2(|t-s|)|\,ds\|u_1'-u_2'\|_\infty \\
&\leq \Bigg(\max\{A,B,\bar{A},\bar{B}\}\int_a^b |p_1(t-s)|\,ds \\
&\quad + \max\{C,D,\bar{C},\bar{D}\}\int_a^b |p_2(|t-s|)|\,ds\Bigg)\|u_1-u_2\|_{C^1([a,b])},
\end{aligned}
$$

then

$$\|T_f(u_1) - T_f(u_2)\|_\infty \leq \varrho\,\|u_1-u_2\|_{C^1([a,b])}. \tag{3}$$

In the same way, we get

$$
\begin{aligned}
&||T_f'(u_1(t)) - T_f'(u_2(t))|| \\
&\leq \Bigg(\max\{A,B\}\int_a^b \left|p_1'(t-s)\right|\,ds + \max\{A,B,\bar{A},\bar{B}\}\int_a^b |p_1(t-s)|\,ds \\
&\quad + \max\{C,D\}\int_a^b \left|p_2'(\mid t-s \mid)\right|\,ds \\
&\quad + \max\{C,D,\bar{C},\bar{D}\}\int_a^b |p_2(\mid t-s \mid)|\,ds\Bigg)\|u_1-u_2\|_{C^1([a,b])},
\end{aligned}
$$

then

$$\|T_f'(u_1) - T_f'(u_2)\|_\infty \leq 2\varrho\|u_1-u_2\|_{C^1([a,b])}. \tag{4}$$

Therefore, from (3) and (4) we deduce that

$$\|T_f(u_1) - T_f(u_2)\|_{C^1([a,b])} \leq 3\varrho\|u_1 - u_2\|_{C^1([a,b])}.$$

Hence the functional T_f has a one fixed point, which conclude the proof of the theorem. ■

3. Numerical results

In general, to solve the equations (1) and (2), there are many numerical methods which based on the construction of a nonlinear algebraic system according to the exact equations. In our paper, we interest on the method described in [18] to study the following nonlinear Volterra-Fredholm integral equation defined by:

$$x(t) = f(t) + \lambda_1 \int_a^t K_1(t, s, x(s))\, ds + \lambda_2 \int_a^b K_2(t, s, x(s))\, ds, \quad a \leq t \leq b,$$

with $\lambda_1, \lambda_1 \in \mathbb{R}$, $K_1, K_2 \in C^1([a, b])$. The method based on the transformation of the equation into a discretized form using Newton-Cotes integration formula, then, with some conditions authors proved that the solution of the discretized form in conjunction with an iterative scheme converges to the exact solution.

Throughout this section, following the idea mentioned above, we give an iterative scheme to approach the solution of our equations (1) and (2), wish is more difficult due to the derivative and weakly singularity of some terms.

First; let $N \in \mathbb{N}^*$, consider the equidistance partition Δ_N by:

$$\Delta_N = \left\{ t_i = a + ih,\, h = \frac{b-a}{N}, \quad i = 0 \cdots N \right\}.$$

For any partition Δ_N, if we denote $u(t_i) = u_i$, $u'(t_i) = u'_i$, $f(t_i) = f_i$, $f'(t_i) = f'_i$ and for all $i = 0 \cdots N$, then according to equations (1) and (2), we can write:

$$\begin{aligned} u_i &= f_i + \int_a^{t_i} p_1(t_i - s) K_1(t_i, s, u(s), u'(s))\, ds \\ &\quad + \int_a^b p(|t_i - s|) K_2(t_i, s, u(s), u'(s))\, ds, \end{aligned} \tag{5}$$

$$\begin{aligned} u'_i &= f'_i + \int_a^{t_i} p'_1(t_i - s) K_1(t_i, s, u(s), u'(s))\, ds \\ &\quad + \int_a^{t_i} p_1(t_i - s) \frac{\partial K_1}{\partial t}(t_i, s, u(s), u'(s))\, ds \end{aligned} \tag{6}$$

$$+ \int_a^b \gamma_i p_2'(\mid t_i - s \mid) K_2(t_i, s, u(s), u'(s))\, ds$$
$$+ \int_a^b p_2(\mid t_i - s \mid) \frac{\partial K_2}{\partial t}(t_i, s, u(s), u'(s))\, ds.$$

Where $\gamma_i = sign(t_i - s)$. In our case, as the functions $p_1(s), p_2(s)$ satisfy the hypothesis $(H1)$, so to approach the integral terms of (5) and (6), we use the product integration method (see [1, 16]), the concept of this method is only approximate the well-behaved part K_m and $\frac{\partial K_m}{\partial t}$ for $m = 1, 2$ on Δ_N using the piecewise linear functions in every subinterval $[t_j, t_{j+1}], j = 0 \cdots N$, then we obtain:

$$\begin{aligned} K_m(t_i, s, u(s), u'(s)) \; &\simeq \; \left(\frac{s - t_j}{h}\right) K_m(t_i, t_{j+1}, u_{j+1}, u'_{j+1}) \\ &+ \left(\frac{t_{j+1} - s}{h}\right) K_m(t_i, t_j, u_j, u'_j), \quad s \in [t_j, t_{j+1}], \end{aligned}$$

$$\begin{aligned} \frac{\partial K_m}{\partial t}(t_i, s, u(s), u'(s)) \; &\simeq \; \left(\frac{s - t_j}{h}\right) \frac{\partial K_m}{\partial t}(t_i, t_{j+1}, u_{j+1}, u'_{j+1}) \\ &+ \left(\frac{t_{j+1} - s}{h}\right) \frac{\partial K_m}{\partial t}(t_i, t_j, u_j, u'_j), \quad s \in [t_j, t_{j+1}]. \end{aligned}$$

Then, for all $i = 0 \cdots N$, equations (5) and (6) can be written as follow:

$$u_i = f_i + \sum_{j=0}^{i} \alpha_j K_1(t_i, t_j, u_j, u'_j) + \sum_{j=0}^{N} \overline{\alpha}_j K_2(t_i, t_j, u_j, u'_j) \; + O_1(h), \tag{7}$$

$$\begin{aligned} u'_i \; &= \; f'_i + \sum_{j=0}^{i} \beta_j K_1(t_i, t_j, u_j, u'_j) + \alpha_j \frac{\partial K_1}{\partial t}(t_i, t_j, u_j, u'_j) \\ &+ \sum_{j=0}^{N} \overline{\beta}_j K_2(t_i, t_j, u_j, u'_j) + \overline{\alpha}_j \frac{\partial K_2}{\partial t}(t_i, t_j, u_j, u'_j) \; + O_2(h), \end{aligned} \tag{8}$$

where, $O_1(h)$, $O_2(h)$ are the convergence orders of the product integration rule and α_j, β_j, $\overline{\alpha_j}$, $\overline{\beta_j}$ are given by:

$$\begin{aligned} \alpha_0 \; &= \; \frac{1}{h} \int_a^{t_1} (t_1 - s) p_1(t_i - s)\, ds, \\ \alpha_j \; &= \; \frac{1}{h} \left(\int_{t_{j-1}}^{t_j} (s - t_{j-1}) p_1(t_i - s)\, ds + \int_{t_j}^{t_{j+1}} (t_{j+1} - s) p_1(t_i - s)\, ds \right), \\ & \quad j = 1 \ldots i - 1, \end{aligned}$$

$$\begin{aligned}
\alpha_i &= \frac{1}{h}\int_{t_{i-1}}^{t_i}(s-t_{i-1})p_1(t_i-s)\,ds,\\
\beta_0 &= \frac{1}{h}\int_a^{t_1}(t_1-s)\gamma_i\,p_1'(t_i-s)\,ds,\\
\beta_j &= \frac{1}{h}\left(\int_{t_{j-1}}^{t_j}(s-t_{j-1})\gamma_i\,p_1'(t_i-s)\,ds+\int_{t_j}^{t_{j+1}}(t_{j+1}-s)\gamma_i\,p_1'(t_i-s)\,ds\right),\\
&\quad j=1\dots i-1,\\
\beta_i &= \frac{1}{h}\int_{t_{i-1}}^{t_i}(s-t_{i-1})\gamma_i\,p_1'(t_i-s)\,ds.\\
\overline{\alpha}_0 &= \frac{1}{h}\int_a^{t_1}(t_1-s)p_2(|t_i-s|)\,ds,\\
\overline{\alpha}_j &= \frac{1}{h}\left(\int_{t_{j-1}}^{t_j}(s-t_{j-1})p_2(|t_i-s|)\,ds+\int_{t_j}^{t_{j+1}}(t_{j+1}-s)p_2(|t_i-s|)\,ds\right),\\
&\quad j=1\dots N-1,\\
\overline{\alpha}_N &= \frac{1}{h}\int_{t_{N-1}}^{b}(s-t_{N-1})p_2(|t_i-s|)\,ds,\\
\overline{\beta}_0 &= \frac{1}{h}\int_a^{t_1}(t_1-s)\gamma_i\,p_2'(|t_i-s|)\,ds,\\
\overline{\beta}_j &= \frac{1}{h}\left(\int_{t_{j-1}}^{t_j}(s-t_{j-1})\gamma_i\,p_2'(|t_i-s|)\,ds+\int_{t_j}^{t_{j+1}}(t_{j+1}-s)\gamma_i\,p_2'(|t_i-s|)\,ds\right),\\
&\quad j=1\dots N-1,\\
\overline{\beta}_N &= \frac{1}{h}\int_{t_{N-1}}^{b}(s-t_{N-1})\gamma_i\,p_2'(|t_i-s|)\,ds.
\end{aligned}$$

If we neglect the error terms $O_1(h)$ and $O_2(h)$ in (7) and (8), we obtain the following nonlinear approached equations:

$$v_i=f_i+\sum_{j=0}^{i}\alpha_jK_1(t_i,t_j,v_j,w_j)+\sum_{j=0}^{N}\overline{\alpha}_jK_2(t_i,t_j,v_j,w_j)\,, \tag{9}$$

$$\begin{aligned}
w_i \;=\; & f_i'+\sum_{j=0}^{i}\beta_jK_1(t_i,t_j,v_j,w_j)+\alpha_j\frac{\partial K_1}{\partial t}(t_i,t_j,v_j,w_j)\\
&+\sum_{j=0}^{N}\overline{\beta}_jK_2(t_i,t_j,v_j,w_j)+\overline{\alpha}_j\frac{\partial K_2}{\partial t}(t_i,t_j,v_j,w_j)\,,
\end{aligned} \tag{10}$$

where, the couple $(v,w)=(v_0,\dots,v_N,w_0,\dots,w_N)$ is the solution of (9) and (10) respectively.

Before studying our system (9) and (10), we mention that our numerical process is clear, simple and easy to structure. In addition, in spite of we already injected it with another iterative scheme presented below, the process remains rapid in the execution. On the other hand, our scheme remains very effective, because we don't need to add of another conditions to confirm its convergence, but we have taken the same analytical conditions proposed in the first.

3.1. System study

Theorem 2. *The system (9)-(10) has a unique solution, for* $0 < \varrho < \frac{1}{3}$.

Proof. In order to facilitate the formulas, we use a functional denoted by $\Psi(V)$ to equip the system (9)-(10) as $V = \Psi(V)$, where, $V = (v_0, \ldots, v_N, w_0, \ldots, w_N)$ is the unknown vector in $\mathbb{R}^{2N+2}$, withe the following norm:

$$\|V\|_{\mathbb{R}^{2N+2}} = \max_{0\leq i\leq N}\{|v_i|\} + \max_{0\leq i\leq N}\{|w_i|\}.$$

Directly, it is clear that:

$$\begin{aligned}
|\Psi(V) - \Psi(\bar{V})| \leq\ & \max\{A,B\}\int_a^{t_i} |p_1(t_i-s)|\,ds\|V-\bar{V}\|_{\mathbb{R}^{2N+2}} \\
&+ \max\{C,D\}\int_a^{b} |p_2(|t_i-s|)|\,ds\|V-\bar{V}\|_{\mathbb{R}^{2N+2}}. \\
&+ \left(\max\{A,B,\overline{A},\overline{B}\}\int_a^{t_i} |p_1(t_i-s)|\right. \\
&+ \left.\max\{A,B\}\int_a^{b} |p_1'(t_i-s)|\,ds\right)\|V-\bar{V}\|_{\mathbb{R}^{2N+2}}. \\
&+ \left(\max\{C,D,\overline{C},\overline{D}\}\int_a^{t_i} |p_2(|t_i-s|)|\right. \\
&+ \left.\max\{C,D\}\int_a^{t_i} |p_2'(|t_i-s|)|\,ds\right)\|V-\bar{V}\|_{\mathbb{R}^{2N+2}}.
\end{aligned}$$

Finally

$$\|\Psi(V) - \Psi(\bar{V})\|_{\mathbb{R}^{2N+2}} \leq 3\varrho\|V-\bar{V}\|_{\mathbb{R}^{2N+2}},$$

With Banach fixed point theorem, we obtain the result. ■

3.2. Convergence analysis

Now, we are going to show that (v, w) converges to (u, u') when h is small enough.

Proposition 1. *Under the hypotheses* $(H1)$, $(H2)$ *and for* $0 < \varrho < \frac{1}{3}$, *we have:*

$$\|u - v\|_\infty + \|u' - w\|_\infty \leq \frac{O_1(h) + O_2(h)}{1 - 3\varrho}.$$

Proof. First, we can see that there exists $q, r \in \{0 \dots N\}$ such that:

$$\|u - v\|_\infty + \|u' - w\|_\infty = |u_q - v_q| + |u'_r - w_r|,$$

thus we have:

$$\begin{aligned}
&|u_q - v_q| + |u'_r - w_r| \\
&\qquad \leq \sum_{j=0}^{i} |\alpha_j||K_1(t_q, t_j, u_j, u'_j) - K_1(t_q, t_j, v_j, w_j)| \\
&\qquad\quad + |\beta_j||K_1(t_r, t_j, u_j, u'_j) - K_1(t_r, t_j, v_j, w_j)| \\
&\qquad\quad + |\alpha_j| \left| \frac{\partial K_1}{\partial t}(t_r, t_j, u_j, u'_j) - \frac{\partial K_1}{\partial t}(t_r, t_j, v_j, w_j) \right| + O_1(h) \\
&\qquad\quad + \sum_{j=0}^{N} |\overline{\alpha}_j||K_2(t_q, t_j, u_j, u'_j) - K_2(t_q, t_j, v_j, w_j)| \\
&\qquad\quad + |\overline{\beta}_j||K_2(t_r, t_j, u_j, u'_j) - K_2(t_r, t_j, v_j, w_j)| \\
&\qquad\quad + |\overline{\alpha}_j| \left| \frac{\partial K_2}{\partial t}(t_r, t_j, u_j, u'_j) - \frac{\partial K_2}{\partial t}(t_r, t_j, v_j, w_j) \right| + O_2(h).
\end{aligned}$$

According hypothesis $(H2)$ we obtain

$$\begin{aligned}
&|u_q - v_q| + |u'_r - w_r| \\
&\qquad \leq \sum_{j=0}^{i} (|\alpha_j| + |\beta_j|)(A|u_j - v_j| + B|u'_j - w_j|) \\
&\qquad\quad + |\alpha_j|(\bar{A}|u_j - v_j| + \bar{B}|u'_j - w_j|) \\
&\qquad\quad + \sum_{j=0}^{N} (|\overline{\alpha}_j| + |\overline{\beta}_j|)(C|u_j - v_j| + D|u'_j - w_j|) \\
&\qquad\quad + |\overline{\alpha}_j|(\bar{C}|u_j - v_j| + \bar{D}|u'_j - w_j|) + O_1(h) + O_2(h).
\end{aligned}$$

Therefore

$$\begin{aligned}
&|u_q - v_q| + |u'_r - w_r| \\
&\qquad \leq \left\{ \max\{A, B\} \left(\int_a^{t_q} |p_1(t_q - s)|\, ds + \int_a^{t_q} \left| p'_1(t_q - s) \right| \, ds \right) \right. \\
&\qquad\quad \left. + \max\{\bar{A}, \bar{B}\} \int_a^{t_r} |p_1(t_r - s)|\, ds \right\} (|u_q - v_q| + |u'_r - w_r|)
\end{aligned}$$

$$
\begin{aligned}
&+ \left\{ \max\{C, D\} \left(\int_a^b |p_2(|t_q - s|)| \, ds + \int_a^b \left| p_2'(|t_q - s|) \right| \, ds \right) \right. \\
&+ \left. \max\{\bar{C}, \bar{D}\} \int_a^b |p(|t_r - s|)| \, ds \right\} (|u_q - v_q| + |u_r' - w_r|) \\
&+ O_1(h) + O_2(h), \\
\leq\ & 3\varrho(|u_q - v_q| + |u_r' - w_r|) + O_1(h) + O_2(h).
\end{aligned}
$$

Which implies that

$$\|u - v\|_\infty + \|u' - w\|_\infty \leq \frac{O_1(h) + O_2(h)}{1 - 3\varrho}.$$

Finally, if h converges to 0, then (v, w) converges to (u, u'). ■

In practice, it is clear that, the solution (v, w), can not be found exactly. For that we need to approach them using the iterative scheme of the successive approximation method defined by the formula: for all $i = 0 \dots N$,

$$v_i^{k+1} = f_i + \sum_{j=0}^{i} \alpha_j K_1(t_i, t_j, v_j^k, w_j^k) + \sum_{j=0}^{N} \overline{\alpha}_j K_2(t_i, t_j, v_j^k, w_j^k), \tag{11}$$

$$
\begin{aligned}
w_i^{k+1} \ =\ & f_i' + \sum_{j=0}^{i} \beta_j K_1(t_i, t_j, v_j^k, w_j^k) + \alpha_j \frac{\partial K_1}{\partial t}(t_i, t_j, v_j^k, w_j^k) \\
& + \sum_{j=0}^{N} \overline{\beta}_j K_2(t_i, t_j, v_j^k, w_j^k) + \overline{\alpha}_j \frac{\partial K_2}{\partial t}(t_i, t_j, v_j^k, w_j^k).
\end{aligned} \tag{12}
$$

In the next theorem we will prove that the iterative solution

$$(v^{k+1}, w^{k+1}) = (v_0^{k+1}, \dots, v_N^{k+1}, w_0^{k+1}, \dots, w_N^{k+1})$$

of (11) and (12) successively converges to (v, w) when $k \to \infty$.

Theorem 3. *Under the assumptions $(H1)$, $(H2)$ and for any arbitrary initial vector (v^0, w^0), we have:*

$$\|v^{k+1} - v\|_\infty + \|w^{k+1} - w\|_\infty \leq (3\varrho)^{k+1}(\|v^0 - v\|_\infty + \|w^0 - w\|_\infty).$$

Proof. By using (9), (10), (11) and (12) we have for all $i = 0 \cdots N$:

$$v_i^{k+1} - v_i \ =\ \sum_{j=0}^{i} \alpha_j (K_1(t_i, t_j, v_j^k, w_j^k) - K_1(t_i, t_j, v_j, w_j))$$

$$
\begin{aligned}
&+\sum_{j=0}^{N}\overline{\alpha}_j(K_2(t_i,t_j,v_j^k,w_j^k)-K_2(t_i,t_j,v_j,w_j)),\\
w_i^{k+1}-w_i \;=\; &\sum_{j=0}^{i}\beta_j(K_1(t_i,t_j,v_j^k,w_j^k)-K_1(t_i,t_j,v_j,w_j))\\
&+\alpha_j(\frac{\partial K_1}{\partial t}(t_r,t_j,v_j^k,w_j^k)-\frac{\partial K_1}{\partial t}(t_r,t_j,v_j,w_j))\\
&+\sum_{j=0}^{N}\overline{\beta}_j(K_2(t_i,t_j,v_j^k,w_j^k)-K_2(t_i,t_j,v_j,w_j))\\
&+\overline{\alpha}_j(\frac{\partial K_2}{\partial t}(t_r,t_j,v_j^k,w_j^k)-\frac{\partial K_2}{\partial t}(t_r,t_j,v_j,w_j)).
\end{aligned}
$$

In the same manner that's applied in the previous proposition, we obtain directly

$$\|v^{k+1}-v\|_\infty+\|w^{k+1}-w\|_\infty \le 3\varrho(\|v^k-v\|_\infty+\|w^k-w\|_\infty).$$

By induction, we get

$$\|v^{k+1}-v\|_\infty+\|w^{k+1}-w\|_\infty \le (3\varrho)^{k+1}(\|v^0-v\|_\infty+\|w^0-w\|_\infty).$$

Since $0<\varrho<\frac{1}{3}$, we have $\|v^{k+1}-v\|_\infty+\|w^{k+1}-w\|_\infty$ vanishes when $k\to\infty$. ■

4. Numerical examples

In the following examples we apply our numerical processes to calculate v^{k+1} and w^{k+1} according schemes (11) and (12) successively, where the arbitrary initial vector $v^0 = w^0 = 0$ and the stopping criterion on the parameter k is taken as:

$$|v^{k+1}-v^k|+|w^{k+1}-w^k|\le 10^{-7}.$$

We denote that the error function E_N computed using our numerical method is given by:

$$E_N=\max_{0\le i\le N}\{|u(t_i)-v_i^{k+1}|+|u'(t_i)-w_i^{k+1}|\}.$$

We mention that numerical results are obtained by using Matlab computation software, with a computer of type Intel Core i5 Duo processor 2.6 GHz and 4 GB RAM.

Example 1. Consider the following integro-differential equation

$$
\begin{aligned}
u(t) &= \frac{1}{4}\int_0^1 \sqrt{|t-s|}\ \cos\left(e^{2s} + \arccos\left(\frac{s+t}{3}\right) + u(s) - u'(s)\right) ds \\
&+ \frac{1}{2}\int_0^t \sqrt{(t-s)}\ \sin\left(e^{2s} + \arcsin\left(\frac{t-s}{5}\right) + u(s) - u'(s)\right) ds \\
&+ e^{2t} - \frac{(7t+3)(1-t)^{3/2}}{90} - \frac{53t^{5/2}}{450}, \quad t \in [0,1].
\end{aligned}
$$

The exact solution is $u(t) = e^{2t}$.

Table 1. Numerical Results of (1).

N	E_N
10	3.03E-3
50	3.00E-4
100	1.08E-4
200	3.92E-5
500	1.01E-5
1000	3.60E-6

Example 2. Consider the second equation

$$
\begin{aligned}
u(t) &= \frac{1}{20}\int_0^1 \sqrt{|t-s|}\ \cos\left(1 - s + \arccos\left(\frac{s+t}{3}\right) + u(s) - u'(s)\right) ds \\
&+ \int_0^t \sqrt{(t-s)}\frac{ts^2(1+e^{-s}+e^{-1})}{1+e^{-u(s)}+e^{-u'(s)}} ds \\
&+ t - \frac{(7t+3)(1-t)^{3/2} + 7t^{5/2}}{450} - \frac{16t^{9/2}}{105}, \quad t \in [0,1].
\end{aligned}
$$

The exact solution is $u(t) = t$.

Table 2. Numerical Results of (2).

N	E_N
10	1.30E-2
50	1.28E-3
100	4.61E-4
200	1.66E-4
500	4.28E-5
1000	1.52E-5

Table 1 and 2 shows us that the error function E_N converges to zero when N increase, that confirms the convergence of our method.

5. Conclusion

We have introduced a coherent numerical method for approximating a nonlinear weakly singular Volterra-Fredholm integro-differential equation, preserving the same condition that occurs in analytical study. As future work, we will apply other iterative schemes with suitable conditions to improve the convergence of the approached solutions.

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Sami Touati
University of 8 Mai 1945
B.P.401, 24000, Guelma, Algeria
e-mail: touati.sami@univ-guelma.dz
touati1sami@gmail.com

Samir Lemita
Ecole normale supérieure
30000 Ouargla, Algeria
e-mail: lem.samir@gmail.com

Mourad Ghiat
University of 8 Mai 1945
B.P. 401, 24000, Guelma, Algeria
e-mail: mourad.ghi24@gmail.com

Mohamed-Zine Aissaoui
University of 8 Mai 1945
B.P.401, 24000, Guelma, Algeria
e-mail: aissaouizine@gmail.com

Received on 06.04.2019 and, in revised form, on 19.07.2019.