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CONFORMABLE FRACTIONAL IYENGAR TYPE INEQUALITIES

Abstract. Here we present Conformable fractional Iyengar type inequalities with respect to L_p norms, with $1 < p \leq \infty$. The method is based on the right and left Conformable fractional Taylor's formulae.

Key words: Iyengar inequality, right and left conformable fractional Taylor formulae, conformable fractional derivative.

AMS Mathematics Subject Classification: 26A33, 26D10, 26D15.

1. Background

We are motivated by the following famous Iyengar inequality (1938), [3].

Theorem 1. *Let f be a differentiable function on $[a,b]$ and $|f'(x)| \leq M$. Then*

$$\left|\int_a^b f(x)\,dx - \frac{1}{2}(b-a)(f(a)+f(b))\right| \leq \frac{M(b-a)^2}{4} - \frac{(f(b)-f(a))^2}{4M}. \tag{1}$$

Here we follow [1] for the basics of generalized Conformable fractional calculus, see also [4].

We need

Definition 1 ([1])**.** *Let $a, b \in \mathbb{R}$. The left conformable fractional derivative starting from a of a function $f : [a,\infty) \to \mathbb{R}$ of order $0 < \alpha \leq 1$ is defined by*

$$(T_\alpha^a f)(t) = \lim_{\varepsilon \to 0} \frac{f\left(t + \varepsilon (t-a)^{1-\alpha}\right) - f(t)}{\varepsilon}. \tag{2}$$

If $(T_\alpha^a f)(t)$ exists on (a,b), then

$$(T_\alpha^a f)(a) = \lim_{t \to a+} (T_\alpha^a f)(t). \tag{3}$$

The right conformable fractional derivative of order $0 < \alpha \leq 1$ *terminating at* b *of* $f : (-\infty, b] \to \mathbb{R}$ *is defined by*

$$\left({}_{\alpha}^{b}Tf\right)(t) = -\lim_{\varepsilon \to 0} \frac{f\left(t + \varepsilon (b-t)^{1-\alpha}\right) - f(t)}{\varepsilon}. \tag{4}$$

If $\left({}_{\alpha}^{b}Tf\right)(t)$ *exists on* (a, b)*, then*

$$\left({}_{\alpha}^{b}Tf\right)(b) = \lim_{t \to b-} \left({}_{\alpha}^{b}Tf\right)(t). \tag{5}$$

Note that if f *is differentiable then*

$$\left(T_{\alpha}^{a}f\right)(t) = (t-a)^{1-\alpha} f'(t), \tag{6}$$

and

$$\left({}_{\alpha}^{b}Tf\right)(t) = -(b-t)^{1-\alpha} f'(t). \tag{7}$$

Denote by

$$\left(I_{\alpha}^{a}f\right)(t) = \int_{a}^{t} (x-a)^{\alpha-1} f(x)\, dx, \tag{8}$$

and

$$\left({}^{b}I_{\alpha}f\right)(t) = \int_{t}^{b} (b-x)^{\alpha-1} f(x)\, dx, \tag{9}$$

these are the left and right conformable fractional integrals of order $0 < \alpha \leq 1$.

In the higher order case we can generalize things as follows:

Definition 2 ([1])**.** *Let* $\alpha \in (n, n+1]$*, and set* $\beta = \alpha - n$*. Then, the left conformable fractional derivative starting from* a *of a function* $f : [a, \infty) \to \mathbb{R}$ *of order* α*, where* $f^{(n)}(t)$ *exists, is defined by*

$$\left(\mathbf{T}_{\alpha}^{a}f\right)(t) = \left(T_{\beta}^{a}f^{(n)}\right)(t), \tag{10}$$

The right conformable fractional derivative of order α *terminating at* b *of* $f : (-\infty, b] \to \mathbb{R}$*, where* $f^{(n)}(t)$ *exists, is defined by*

$$\left({}_{\alpha}^{b}\mathbf{T}f\right)(t) = (-1)^{n+1} \left({}_{\beta}^{b}Tf^{(n)}\right)(t). \tag{11}$$

If $\alpha = n+1$ *then* $\beta = 1$ *and* $\mathbf{T}_{n+1}^{a}f = f^{(n+1)}$.

If n *is odd, then* ${}_{n+1}^{b}\mathbf{T}f = -f^{(n+1)}$*, and if* n *is even, then* ${}_{n+1}^{b}\mathbf{T}f = f^{(n+1)}$.

When $n = 0$ *(or* $\alpha \in (0, 1]$*), then* $\beta = \alpha$*, and (10), (11) collapse to* $\{$*(2)-(5)*$\}$*, respectively.*

Lemma 1 ([1]). *Let $f : (a, b) \to \mathbb{R}$ be continuously differentiable and $0 < \alpha \leq 1$. Then, for all $t > a$ we have*

$$I_{\alpha}^{a} T_{\alpha}^{a} (f)(t) = f(t) - f(a). \tag{12}$$

We need

Definition 3 (see also [1]). *If $\alpha \in (n, n+1]$, then the left fractional integral of order α starting at a is defined by*

$$\left(\mathbf{I}_{\alpha}^{a} f\right)(t) = \frac{1}{n!} \int_{a}^{t} (t-x)^{n} (x-a)^{\beta-1} f(x)\, dx. \tag{13}$$

Similarly, (author's definition, see [2]) the right fractional integral of order α terminating at b is defined by

$$\left({}^{b}\mathbf{I}_{\alpha} f\right)(t) = \frac{1}{n!} \int_{t}^{b} (x-t)^{n} (b-x)^{\beta-1} f(x)\, dx. \tag{14}$$

We need

Proposition 1 ([1]). *Let $\alpha \in (n, n+1]$ and $f : [a, \infty) \to \mathbb{R}$ be $(n+1)$ times continuously differentiable for $t > a$. Then, for all $t > a$ we have*

$$\mathbf{I}_{\alpha}^{a} \mathbf{T}_{\alpha}^{a} (f)(t) = f(t) - \sum_{k=0}^{n} \frac{f^{(k)}(a)(t-a)^{k}}{k!}. \tag{15}$$

We also have

Proposition 2 ([2]). *Let $\alpha \in (n, n+1]$ and $f : (-\infty, b] \to \mathbb{R}$ be $(n+1)$ times continuously differentiable for $t < b$. Then, for all $t < b$ we have*

$$-{}^{b}\mathbf{I}_{\alpha} \, {}_{\alpha}^{b}\mathbf{T} (f)(t) = f(t) - \sum_{k=0}^{n} \frac{f^{(k)}(b)(t-b)^{k}}{k!}. \tag{16}$$

If $n = 0$ or $0 < \alpha \leq 1$, then (see also [1])

$${}^{b}I_{\alpha} \, {}_{\alpha}^{b}T (f)(t) = f(t) - f(b). \tag{17}$$

In conclusion we derive

Theorem 2 ([2]). *Let $\alpha \in (n, n+1]$ and $f \in C^{n+1}([a, b])$, $n \in \mathbb{N}$. Then*
1)

$$f(t) - \sum_{k=0}^{n} \frac{f^{(k)}(a)(t-a)^{k}}{k!} = \frac{1}{n!} \int_{a}^{t} (t-x)^{n} (x-a)^{\beta-1} \left(\mathbf{T}_{\alpha}^{a}(f)\right)(x)\, dx, \tag{18}$$

and

2)

$$f(t)-\sum_{k=0}^{n}\frac{f^{(k)}(b)(t-b)^k}{k!} = -\frac{1}{n!}\int_t^b (b-x)^{\beta-1}(x-t)^n \left({}_{\alpha}^{b}\mathbf{T}(f)\right)(x)\,dx, \quad \forall t\in[a,b]. \tag{19}$$

We need

Remark 1 ([2])**.** We notice the following: let $\alpha\in(n,n+1]$ and $f\in C^{n+1}([a,b])$, $n\in\mathbb{N}$. Then ($\beta:=\alpha-n$, $0<\beta\leq 1$)

$$\left(\mathbf{T}_{\alpha}^{a}(f)\right)(x)=\left(T_{\beta}^{\alpha}f^{(n)}\right)(x)=(x-a)^{1-\beta}f^{(n+1)}(x), \tag{20}$$

and

$$\begin{aligned}\left({}_{\alpha}^{b}\mathbf{T}(f)\right)(x) &= (-1)^{n+1}\left({}_{\beta}^{b}Tf^{(n)}\right)(x) \\ &= (-1)^{n+1}(-1)(b-x)^{1-\beta}f^{(n+1)}(x) \\ &= (-1)^{n}(b-x)^{1-\beta}f^{(n+1)}(x).\end{aligned} \tag{21}$$

Consequently we get that

$$\left(\mathbf{T}_{\alpha}^{a}(f)\right)(x),\ \left({}_{\alpha}^{b}\mathbf{T}(f)\right)(x)\in C([a,b]).$$

Furthermore it is obvious that

$$\left(\mathbf{T}_{\alpha}^{a}(f)\right)(a)=\left({}_{\alpha}^{b}\mathbf{T}(f)\right)(b)=0, \tag{22}$$

when $0<\beta<1$, i.e. when $\alpha\in(n,n+1)$.

If $f^{(k)}(a)=0$, $k=1,...,n$, then

$$f(t)-f(a)=\frac{1}{n!}\int_a^t (t-x)^n (x-a)^{\beta-1}\left(\mathbf{T}_{\alpha}^{a}(f)\right)(x)\,dx, \tag{23}$$

$\forall\ t\in[a,b]$.

If $f^{(k)}(b)=0$, $k=1,...,n$, then

$$f(t)-f(b)=-\frac{1}{n!}\int_t^b (b-x)^{\beta-1}(x-t)^n\left({}_{\alpha}^{b}\mathbf{T}(f)\right)(x)\,dx, \tag{24}$$

$\forall\ t\in[a,b]$.

2. Main Results

We present the following Conformable fractional Iyengar type inequalities:

Theorem 3. *Let* $\alpha \in (n, n+1]$ *and* $f \in C^{n+1}([a,b])$, $n \in \mathbb{N}$; $\beta = \alpha - n$. *Then*

i)

$$\left| \int_a^b f(t)\,dt - \sum_{k=0}^{n} \frac{1}{(k+1)!} \times \left[f^{(k)}(a)(z-a)^{k+1} + (-1)^k f^{(k)}(b)(b-z)^{k+1} \right] \right| \leq \frac{\Gamma(\beta)\max\left\{ \left\| \mathbf{T}_\alpha^a(f) \right\|_{\infty,[a,b]}, \left\| {}_\alpha^b\mathbf{T}(f) \right\|_{\infty,[a,b]} \right\}}{\Gamma(\alpha+2)} \times \left[(z-a)^{\alpha+1} + (b-z)^{\alpha+1} \right], \tag{25}$$

$\forall\ z \in [a,b]$,

ii) at $z = \frac{a+b}{2}$, *the right hand side of (25) is minimized, and we get:*

$$\left| \int_a^b f(t)\,dt - \sum_{k=0}^{n} \frac{1}{(k+1)!} \frac{(b-a)^{k+1}}{2^{k+1}} \left[f^{(k)}(a) + (-1)^k f^{(k)}(b) \right] \right| \leq \frac{\Gamma(\beta)\max\left\{ \left\| \mathbf{T}_\alpha^a(f) \right\|_{\infty,[a,b]}, \left\| {}_\alpha^b\mathbf{T}(f) \right\|_{\infty,[a,b]} \right\}}{\Gamma(\alpha+2)} \frac{(b-a)^{\alpha+1}}{2^\alpha}, \tag{26}$$

iii) assuming $f^{(k)}(a) = f^{(k)}(b) = 0$, *for* $k = 0, 1, ..., n$, *we obtain*

$$\left| \int_a^b f(t)\,dt \right| \leq \frac{\Gamma(\beta)\max\left\{ \left\| \mathbf{T}_\alpha^a(f) \right\|_{\infty,[a,b]}, \left\| {}_\alpha^b\mathbf{T}(f) \right\|_{\infty,[a,b]} \right\}}{\Gamma(\alpha+2)} \times \frac{(b-a)^{\alpha+1}}{2^\alpha}, \tag{27}$$

which is a sharp inequality,

iv) more generally, for $j = 0, 1, 2, ...,$ $N \in \mathbb{N}$, *it holds*

$$\left| \int_a^b f(t)\,dt - \sum_{k=0}^{n} \frac{1}{(k+1)!} \left(\frac{b-a}{N} \right)^{k+1} \times \left[f^{(k)}(a)\, j^{k+1} + (-1)^k f^{(k)}(b)(N-j)^{k+1} \right] \right| \tag{28}$$

$$\leq \frac{\Gamma(\beta)\max\left\{\left\|\mathbf{T}_{\alpha}^{a}(f)\right\|_{\infty,[a,b]}, \left\|{}_{\alpha}^{b}\mathbf{T}(f)\right\|_{\infty,[a,b]}\right\}}{\Gamma(\alpha+2)} \times \left(\frac{b-a}{N}\right)^{\alpha+1}\left[j^{\alpha+1}+(N-j)^{\alpha+1}\right],$$

v) if $f^{(k)}(a)=f^{(k)}(b)=0$, $k=1,...,n$, *from (28) we get:*

$$\left|\int_a^b f(t)\,dt-\left(\frac{b-a}{N}\right)[jf(a)+(N-j)f(b)]\right| \leq \frac{\Gamma(\beta)\max\left\{\left\|\mathbf{T}_{\alpha}^{a}(f)\right\|_{\infty,[a,b]}, \left\|{}_{\alpha}^{b}\mathbf{T}(f)\right\|_{\infty,[a,b]}\right\}}{\Gamma(\alpha+2)} \times \left(\frac{b-a}{N}\right)^{\alpha+1}\left[j^{\alpha+1}+(N-j)^{\alpha+1}\right], \quad j=0,1,2,...,N, \tag{29}$$

$j=0,1,2,...,N,$

vi) when $N=2$ *and* $j=1$, *(29) turns to*

$$\left|\int_a^b f(x)\,dx-\left(\frac{b-a}{2}\right)(f(a)+f(b))\right| \leq \frac{\Gamma(\beta)\max\left\{\left\|\mathbf{T}_{\alpha}^{a}(f)\right\|_{\infty,[a,b]}, \left\|{}_{\alpha}^{b}\mathbf{T}(f)\right\|_{\infty,[a,b]}\right\}}{\Gamma(\alpha+2)}\frac{(b-a)^{\alpha+1}}{2^{\alpha}}. \tag{30}$$

Proof. By Theorem 2 (18) we get

$$\left|f(t)-\sum_{k=0}^{n}\frac{f^{(k)}(a)}{k!}(t-a)^k\right| \leq \frac{1}{n!}\int_a^t (t-x)^n (x-a)^{\beta-1}\left|\left(\mathbf{T}_{\alpha}^{a}(f)\right)(x)\right|dx \leq \frac{\left\|\mathbf{T}_{\alpha}^{a}(f)\right\|_{\infty,[a,b]}}{n!}\int_a^t (t-x)^{(n+1)-1}(x-a)^{\beta-1}dx = \frac{\left\|\mathbf{T}_{\alpha}^{a}(f)\right\|_{\infty,[a,b]}}{n!}\frac{\Gamma(n+1)\Gamma(\beta)}{\Gamma(n+\beta+1)}(t-a)^{n+\beta}. \tag{31}$$

That is

$$\left|f(t)-\sum_{k=0}^{n}\frac{f^{(k)}(a)}{k!}(t-a)^k\right| \leq \left(\frac{\left\|\mathbf{T}_{\alpha}^{a}(f)\right\|_{\infty,[a,b]}\Gamma(\beta)}{\Gamma(n+\beta+1)}\right)(t-a)^{n+\beta}, \tag{32}$$

$\forall\ t\in[a,b]$.

By Theorem 2 (19) we get

$$\left|f(t)-\sum_{k=0}^{n}\frac{f^{(k)}(b)}{k!}(t-b)^{k}\right| \tag{33}$$
$$\leq \frac{1}{n!}\left|\int_{t}^{b}(b-x)^{\beta-1}(x-t)^{n}\left({}_{\alpha}^{b}\mathbf{T}(f)\right)(x)\right|dx$$
$$\leq \frac{1}{n!}\int_{t}^{b}(b-x)^{\beta-1}(x-t)^{n}\left|{}_{\alpha}^{b}\mathbf{T}(f)(x)\right|dx$$
$$\leq \frac{\left\|{}_{\alpha}^{b}\mathbf{T}(f)\right\|_{\infty,[a,b]}}{n!}\int_{t}^{b}(b-x)^{\beta-1}(x-t)^{(n+1)-1}dx$$
$$=\frac{\left\|{}_{\alpha}^{b}\mathbf{T}(f)\right\|_{\infty,[a,b]}}{n!}\frac{\Gamma(\beta)\Gamma(n+1)}{\Gamma(n+\beta+1)}(b-t)^{n+\beta}$$
$$=\left(\frac{\left\|{}_{\alpha}^{b}\mathbf{T}(f)\right\|_{\infty,[a,b]}\Gamma(\beta)}{\Gamma(n+\beta+1)}\right)(b-t)^{n+\beta}.$$

That is

$$\left|f(t)-\sum_{k=0}^{n}\frac{f^{(k)}(b)}{k!}(t-b)^{k}\right|\leq\left(\frac{\left\|{}_{\alpha}^{b}\mathbf{T}(f)\right\|_{\infty,[a,b]}\Gamma(\beta)}{\Gamma(n+\beta+1)}\right)(b-t)^{n+\beta}, \tag{34}$$

$\forall\ t\in[a,b]$.

Call

$$\gamma_{1}:=\frac{\left\|\mathbf{T}_{\alpha}^{a}(f)\right\|_{\infty,[a,b]}\Gamma(\beta)}{\Gamma(n+\beta+1)}, \tag{35}$$

and

$$\gamma_{2}:=\frac{\left\|{}_{\alpha}^{b}\mathbf{T}(f)\right\|_{\infty,[a,b]}\Gamma(\beta)}{\Gamma(n+\beta+1)}. \tag{36}$$

Set

$$\gamma:=\max(\gamma_{1},\gamma_{2}). \tag{37}$$

Therefore we have

$$\left|f(t)-\sum_{k=0}^{n}\frac{f^{(k)}(a)}{k!}(t-a)^{k}\right|\leq\gamma(t-a)^{n+\beta}, \tag{38}$$

and

$$\left|f(t)-\sum_{k=0}^{n}\frac{f^{(k)}(b)}{k!}(t-b)^{k}\right|\leq\gamma(b-t)^{n+\beta}, \tag{39}$$

$\forall\ t \in [a,b]$.

Hence it holds

$$\sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}(t-a)^k - \gamma (t-a)^{n+\beta} \leq f(t) \leq \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}(t-a)^k + \gamma (t-a)^{n+\beta} \tag{40}$$

and

$$\sum_{k=0}^{n} \frac{f^{(k)}(b)}{k!}(t-b)^k - \gamma (b-t)^{n+\beta} \leq f(t) \leq \sum_{k=0}^{n} \frac{f^{(k)}(b)}{k!}(t-b)^k + \gamma (b-t)^{n+\beta}, \tag{41}$$

$\forall\ t \in [a,b]$.

Let any $z \in [a,b]$, then by integration over $[a,z]$ and $[z,b]$, respectively, we obtain

$$\sum_{k=0}^{n} \frac{f^{(k)}(a)}{(k+1)!}(z-a)^{k+1} - \frac{\gamma}{(n+\beta+1)}(z-a)^{n+\beta+1} \leq \int_a^z f(t)\,dt \leq \sum_{k=0}^{n} \frac{f^{(k)}(a)}{(k+1)!}(z-a)^{k+1} + \frac{\gamma}{(n+\beta+1)}(z-a)^{n+\beta+1}, \tag{42}$$

and

$$-\sum_{k=0}^{n-1} \frac{f^{(k)}(b)}{(k+1)!}(z-b)^{k+1} - \frac{\gamma}{(n+\beta+1)}(b-z)^{n+\beta+1} \leq \int_z^b f(t)\,dt \leq -\sum_{k=0}^{n-1} \frac{f^{(k)}(b)}{(k+1)!}(z-b)^{k+1} + \frac{\gamma}{(n+\beta+1)}(b-z)^{n+\beta+1}. \tag{43}$$

Adding (42) and (43), we obtain

$$\left\{\sum_{k=0}^{n} \frac{1}{(k+1)!}\left[f^{(k)}(a)(z-a)^{k+1} - f^{(k)}(b)(z-b)^{k+1}\right]\right\} - \frac{\gamma}{(n+\beta+1)}\left[(z-a)^{n+\beta+1} + (b-z)^{n+\beta+1}\right] \leq \int_a^b f(t)\,dt$$
$$\leq \left\{\sum_{k=0}^{n} \frac{1}{(k+1)!}\left[f^{(k)}(a)(z-a)^{k+1} - f^{(k)}(b)(z-b)^{k+1}\right]\right\} + \frac{\gamma}{(n+\beta+1)}\left[(z-a)^{n+\beta+1} + (b-z)^{n+\beta+1}\right], \tag{44}$$

$\forall\ z \in [a, b]$.

Consequently we derive:

$$\left| \int_a^b f(t)\,dt - \sum_{k=0}^{n} \frac{1}{(k+1)!} \times \left[f^{(k)}(a)(z-a)^{k+1} + (-1)^k f^{(k)}(b)(b-z)^{k+1} \right] \right| \leq \frac{\gamma}{(n+\beta+1)} \left[(z-a)^{n+\beta+1} + (b-z)^{n+\beta+1} \right] = \frac{\gamma}{(\alpha+1)} \left[(z-a)^{\alpha+1} + (b-z)^{\alpha+1} \right], \tag{45}$$

$\forall\ z \in [a, b]$.

Let us consider

$$g(z) = (z-a)^{\alpha+1} + (b-z)^{\alpha+1},\ \forall\ z \in [a, b].$$

Hence

$$g'(z) = (\alpha+1)\left[(z-a)^{\alpha} - (b-z)^{\alpha}\right] = 0,$$

giving $(z-a)^{\alpha} = (b-z)^{\alpha}$ and $z - a = b - z$, that is $z = \frac{a+b}{2}$ the only critical number here.

We have $g(a) = g(b) = (b-a)^{\alpha+1}$, and $g\left(\frac{a+b}{2}\right) = \frac{(b-a)^{\alpha+1}}{2^{\alpha}}$, which is the minimum of g over $[a, b]$.

Consequently the right hand side of (45) is minimized when $z = \frac{a+b}{2}$, with value $\frac{\gamma}{(\alpha+1)} \frac{(b-a)^{\alpha+1}}{2^{\alpha}}$.

Assuming $f^{(k)}(a) = f^{(k)}(b) = 0$, for $k = 0, 1, ..., n$, then we obtain that

$$\left| \int_a^b f(t)\,dt \right| \leq \frac{\gamma}{(\alpha+1)} \frac{(b-a)^{\alpha+1}}{2^{\alpha}}, \tag{46}$$

which is a sharp inequality.

When $z = \frac{a+b}{2}$, then (45) becomes

$$\left| \int_a^b f(t)\,dt - \sum_{k=0}^{n} \frac{1}{(k+1)!} \frac{(b-a)^{k+1}}{2^{k+1}} \left[f^{(k)}(a) + (-1)^k f^{(k)}(b) \right] \right| \leq \frac{\gamma}{(\alpha+1)} \frac{(b-a)^{\alpha+1}}{2^{\alpha}}. \tag{47}$$

Next let $N \in \mathbb{N}$, $j = 0, 1, 2, ..., N$ and $z_j = a + j\left(\frac{b-a}{N}\right)$, that is $z_0 = a$, $z_1 = a + \frac{b-a}{N}, ..., z_N = b$.

Hence it holds

$$z_j - a = j\left(\frac{b-a}{N}\right),\quad (b - z_j) = (N - j)\left(\frac{b-a}{N}\right),\quad j = 0, 1, 2, ..., N. \tag{48}$$

We notice that

$$(z_j - a)^{\alpha+1} + (b - z_j)^{\alpha+1} = \left(\frac{b-a}{N}\right)^{\alpha+1} \left[j^{\alpha+1} + (N-j)^{\alpha+1}\right], \tag{49}$$

$j = 0, 1, 2, ..., N$, and (for $k = 0, 1, ..., n$)

$$\begin{aligned} &\left[f^{(k)}(a)(z_j - a)^{k+1} + (-1)^k f^{(k)}(b)(b - z_j)^{k+1}\right] \\ &= \left[f^{(k)}(a)\, j^{k+1} \left(\frac{b-a}{N}\right)^{k+1} \right. \\ &\quad \left. + (-1)^k f^{(k)}(b)(N-j)^{k+1} \left(\frac{b-a}{N}\right)^{k+1}\right] \\ &= \left(\frac{b-a}{N}\right)^{k+1} \left[f^{(k)}(a)\, j^{k+1} + (-1)^k f^{(k)}(b)(N-j)^{k+1}\right], \end{aligned} \tag{50}$$

$j = 0, 1, 2, ..., N$.

By (45) we get

$$\begin{aligned} &\left|\int_a^b f(t)\,dt - \sum_{k=0}^{n} \frac{1}{(k+1)!} \left(\frac{b-a}{N}\right)^{k+1} \left[f^{(k)}(a)\, j^{k+1} \right.\right. \\ &\quad \left.\left. + (-1)^k f^{(k)}(b)(N-j)^{k+1}\right]\right| \\ &\leq \frac{\gamma}{(\alpha+1)} \left(\frac{b-a}{N}\right)^{\alpha+1} \left[j^{\alpha+1} + (N-j)^{\alpha+1}\right], \end{aligned} \tag{51}$$

$j = 0, 1, 2, ..., N$.

If $f^{(k)}(a) = f^{(k)}(b) = 0$, $k = 1, ..., n$, then (51) becomes

$$\begin{aligned} &\left|\int_a^b f(t)\,dt - \left(\frac{b-a}{N}\right)[j f(a) + (N-j) f(b)]\right| \\ &\leq \frac{\gamma}{(\alpha+1)} \left(\frac{b-a}{N}\right)^{\alpha+1} \left[j^{\alpha+1} + (N-j)^{\alpha+1}\right], \end{aligned} \tag{52}$$

$j = 0, 1, 2, ..., N$.

When $N = 2$ and $j = 1$, then (52) becomes

$$\begin{aligned} &\left|\int_a^b f(x)\,dx - \left(\frac{b-a}{2}\right)(f(a) + f(b))\right| \\ &\leq \frac{\gamma}{(\alpha+1)} 2 \left(\frac{b-a}{2}\right)^{\alpha+1} = \frac{\gamma}{(\alpha+1)} \frac{(b-a)^{\alpha+1}}{2^{\alpha}}. \end{aligned} \tag{53}$$

The theorem is proved in all cases. ■

We give ($n = 0$ case)

Corollary 1. *Let* $0 < \alpha \leq 1$, $f \in C^1([a,b])$. *Then*
i)

$$\left| \int_a^b f(t)\,dt - [f(a)(z-a) + f(b)(b-z)] \right| \leq \frac{\max\left\{ \|T_\alpha^a(f)\|_{\infty,[a,b]}, \left\|{}_\alpha^b T(f)\right\|_{\infty,[a,b]} \right\}}{\alpha(\alpha+1)} \times \left[(z-a)^{\alpha+1} + (b-z)^{\alpha+1} \right], \quad \forall z \in [a,b], \tag{54}$$

ii) at $z = \frac{a+b}{2}$, *the right hand side of (54) is minimized, and we get:*

$$\left| \int_a^b f(t)\,dt - \left(\frac{b-a}{2}\right)[f(a) + f(b)] \right| \leq \frac{\max\left\{ \|T_\alpha^a(f)\|_{\infty,[a,b]}, \left\|{}_\alpha^b T(f)\right\|_{\infty,[a,b]} \right\}}{\alpha(\alpha+1)} \frac{(b-a)^{\alpha+1}}{2^\alpha}, \tag{55}$$

iii) assuming $f(a) = f(b) = 0$, *we obtain*

$$\left| \int_a^b f(t)\,dt \right| \leq \frac{\max\left\{ \|T_\alpha^a(f)\|_{\infty,[a,b]}, \left\|{}_\alpha^b T(f)\right\|_{\infty,[a,b]} \right\}}{\alpha(\alpha+1)} \frac{(b-a)^{\alpha+1}}{2^\alpha}, \tag{56}$$

which is a sharp inequality,
iv) more generally, for $j = 0, 1, 2, ..., N \in \mathbb{N}$, *it holds*

$$\left| \int_a^b f(t)\,dt - \left(\frac{b-a}{N}\right)[f(a)j + f(b)(N-j)] \right| \leq \frac{\max\left\{ \|T_\alpha^a(f)\|_{\infty,[a,b]}, \left\|{}_\alpha^b T(f)\right\|_{\infty,[a,b]} \right\}}{\alpha(\alpha+1)} \times \left(\frac{b-a}{N}\right)^{\alpha+1} \left[j^{\alpha+1} + (N-j)^{\alpha+1} \right], \tag{57}$$

v) when $N = 2$ *and* $j = 1$, *(57) turns to*

$$\left| \int_a^b f(x)\,dx - \left(\frac{b-a}{2}\right)(f(a) + f(b)) \right| \leq \frac{\max\left\{ \|T_\alpha^a(f)\|_{\infty,[a,b]}, \left\|{}_\alpha^b T(f)\right\|_{\infty,[a,b]} \right\}}{\alpha(\alpha+1)} \frac{(b-a)^{\alpha+1}}{2^\alpha}. \tag{58}$$

Proof. As in the proof of Theorem 3; case of $n = 0$, use of (12) and (17). ■

We continue with L_p conformable fractional Iyengar inequalities:

Theorem 4. *Let $\alpha \in (n, n+1]$ and $f \in C^{n+1}([a,b])$, $n \in \mathbb{N}$; $\beta = \alpha - n$. Let also $p_1, p_2, p_3 > 1 : \frac{1}{p_1} + \frac{1}{p_2} + \frac{1}{p_3} = 1$, with $\beta > \frac{1}{p_1} + \frac{1}{p_3}$. Then*

i)

$$\left| \int_a^b f(t)\,dt - \sum_{k=0}^{n} \frac{1}{(k+1)!} \left[f^{(k)}(a)(z-a)^{k+1} + (-1)^k f^{(k)}(b)(b-z)^{k+1} \right] \right| \leq \frac{\max\left\{ \left\| \mathbf{T}_{\alpha}^{a}(f) \right\|_{p_3,[a,b]}, \left\| {}_{\alpha}^{b}\mathbf{T}(f) \right\|_{p_3,[a,b]} \right\}}{n!\,(p_1 n+1)^{\frac{1}{p_1}} (p_2(\beta-1)+1)^{\frac{1}{p_2}} \left(\alpha + \frac{1}{p_1} + \frac{1}{p_2} \right)} \times \left[(z-a)^{\alpha+\frac{1}{p_1}+\frac{1}{p_2}} + (b-z)^{\alpha+\frac{1}{p_1}+\frac{1}{p_2}} \right], \tag{59}$$

$\forall\ z \in [a,b]$,

ii) at $z = \frac{a+b}{2}$, the right hand side of (59) is minimized, and we get:

$$\left| \int_a^b f(t)\,dt - \sum_{k=0}^{n} \frac{1}{(k+1)!} \frac{(b-a)^{k+1}}{2^{k+1}} \left[f^{(k)}(a) + (-1)^k f^{(k)}(b) \right] \right| \leq \frac{\max\left\{ \left\| \mathbf{T}_{\alpha}^{a}(f) \right\|_{p_3,[a,b]}, \left\| {}_{\alpha}^{b}\mathbf{T}(f) \right\|_{p_3,[a,b]} \right\}}{n!\,(p_1 n+1)^{\frac{1}{p_1}} (p_2(\beta-1)+1)^{\frac{1}{p_2}} \left(\alpha + \frac{1}{p_1} + \frac{1}{p_2} \right)} \times \frac{(b-a)^{\alpha+\frac{1}{p_1}+\frac{1}{p_2}}}{2^{\alpha-\frac{1}{p_3}}}, \tag{60}$$

iii) assuming $f^{(k)}(a) = f^{(k)}(b) = 0$, for $k = 0, 1, ..., n$, we obtain

$$\left| \int_a^b f(t)\,dt \right| \leq \frac{\max\left\{ \left\| \mathbf{T}_{\alpha}^{a}(f) \right\|_{p_3,[a,b]}, \left\| {}_{\alpha}^{b}\mathbf{T}(f) \right\|_{p_3,[a,b]} \right\}}{n!\,(p_1 n+1)^{\frac{1}{p_1}} (p_2(\beta-1)+1)^{\frac{1}{p_2}} \left(\alpha + \frac{1}{p_1} + \frac{1}{p_2} \right)} \times \frac{(b-a)^{\alpha+\frac{1}{p_1}+\frac{1}{p_2}}}{2^{\alpha-\frac{1}{p_3}}}, \tag{61}$$

which is a sharp inequality,

iv) more generally, for $j = 0, 1, 2, ..., N \in \mathbb{N}$, it holds

$$\left| \int_a^b f(t)\,dt - \sum_{k=0}^{n} \frac{1}{(k+1)!} \left(\frac{b-a}{N} \right)^{k+1} \times \left[f^{(k)}(a)\, j^{k+1} + (-1)^k f^{(k)}(b)(N-j)^{k+1} \right] \right| \tag{62}$$

$$\leq \frac{\max\left\{\left\|\mathbf{T}_{\alpha}^{a}(f)\right\|_{p_3,[a,b]}, \left\|{}_{\alpha}^{b}\mathbf{T}(f)\right\|_{p_3,[a,b]}\right\}}{n!\,(p_1 n+1)^{\frac{1}{p_1}}\,(p_2(\beta-1)+1)^{\frac{1}{p_2}}\left(\alpha+\frac{1}{p_1}+\frac{1}{p_2}\right)}$$
$$\times\left(\frac{b-a}{N}\right)^{\alpha+\frac{1}{p_1}+\frac{1}{p_2}}\left[j^{\alpha+\frac{1}{p_1}+\frac{1}{p_2}}+(N-j)^{\alpha+\frac{1}{p_1}+\frac{1}{p_2}}\right],$$

v) if $f^{(k)}(a)=f^{(k)}(b)=0$, $k=1,...,n$, from (62) we get:

$$\left|\int_a^b f(t)\,dt-\left(\frac{b-a}{N}\right)[jf(a)+(N-j)f(b)]\right| \tag{63}$$
$$\leq \frac{\max\left\{\left\|\mathbf{T}_{\alpha}^{a}(f)\right\|_{p_3,[a,b]}, \left\|{}_{\alpha}^{b}\mathbf{T}(f)\right\|_{p_3,[a,b]}\right\}}{n!\,(p_1 n+1)^{\frac{1}{p_1}}\,(p_2(\beta-1)+1)^{\frac{1}{p_2}}\left(\alpha+\frac{1}{p_1}+\frac{1}{p_2}\right)}$$
$$\times\left(\frac{b-a}{N}\right)^{\alpha+\frac{1}{p_1}+\frac{1}{p_2}}\left[j^{\alpha+\frac{1}{p_1}+\frac{1}{p_2}}+(N-j)^{\alpha+\frac{1}{p_1}+\frac{1}{p_2}}\right],$$

for $j=0,1,2,...,N$,
vi) when $N=2$ and $j=1$, (63) turns to

$$\left|\int_a^b f(x)\,dx-\left(\frac{b-a}{2}\right)(f(a)+f(b))\right| \tag{64}$$
$$\leq \frac{\max\left\{\left\|\mathbf{T}_{\alpha}^{a}(f)\right\|_{p_3,[a,b]}, \left\|{}_{\alpha}^{b}\mathbf{T}(f)\right\|_{p_3,[a,b]}\right\}}{n!\,(p_1 n+1)^{\frac{1}{p_1}}\,(p_2(\beta-1)+1)^{\frac{1}{p_2}}\left(\alpha+\frac{1}{p_1}+\frac{1}{p_2}\right)}$$
$$\times\frac{(b-a)^{\alpha+\frac{1}{p_1}+\frac{1}{p_2}}}{2^{\alpha-\frac{1}{p_3}}}.$$

Proof. By Theorem 2 (18) we get

$$\left|f(t)-\sum_{k=0}^{n}\frac{f^{(k)}(a)}{k!}(t-a)^k\right| \tag{65}$$
$$\leq \frac{1}{n!}\int_a^t (t-x)^n (x-a)^{\beta-1}\left|\left(\mathbf{T}_{\alpha}^{a}(f)\right)(x)\right|dx$$
$$\leq \frac{1}{n!}\left(\int_a^t (t-x)^{p_1 n}\,dx\right)^{\frac{1}{p_1}}\left(\int_a^t (x-a)^{p_2(\beta-1)}\,dx\right)^{\frac{1}{p_2}}$$
$$\times\left(\int_a^t \left|\left(\mathbf{T}_{\alpha}^{a}(f)\right)(x)\right|^{p_3}dx\right)^{\frac{1}{p_3}}$$

$$\leq \frac{\|\mathbf{T}_{\alpha}^{a}(f)\|_{p_3,[a,b]}}{n!} \frac{(t-a)^{n+\frac{1}{p_1}}}{(p_1 n+1)^{\frac{1}{p_1}}} \frac{(t-a)^{\beta-1+\frac{1}{p_2}}}{(p_2(\beta-1)+1)^{\frac{1}{p_2}}}$$

$$= \frac{\|\mathbf{T}_{\alpha}^{a}(f)\|_{p_3,[a,b]}}{n!} \frac{(t-a)^{n+\beta-\frac{1}{p_3}}}{(p_1 n+1)^{\frac{1}{p_1}}(p_2(\beta-1)+1)^{\frac{1}{p_2}}}.$$

Notice that $p_2(\beta-1)+1>0$, iff $\beta>\frac{1}{p_1}+\frac{1}{p_3}$.

We have proved that

$$\left| f(t) - \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}(t-a)^k \right| \leq \left(\frac{\|\mathbf{T}_{\alpha}^{a}(f)\|_{p_3,[a,b]}}{n!(p_1 n+1)^{\frac{1}{p_1}}(p_2(\beta-1)+1)^{\frac{1}{p_2}}} \right)(t-a)^{n+\beta-\frac{1}{p_3}}, \tag{66}$$

$\forall\ t\in[a,b]$.

Similarly, from Theorem 2 (19) we obtain

$$\left| f(t) - \sum_{k=0}^{n} \frac{f^{(k)}(b)}{k!}(t-b)^k \right| \tag{67}$$

$$\leq \frac{1}{n!}\int_t^b (b-x)^{\beta-1}(x-t)^n \left| \left({}_{\alpha}^{b}\mathbf{T}(f) \right)(x) \right| dx$$

$$\leq \frac{1}{n!}\left(\int_t^b (x-t)^{p_1 n}\,dx\right)^{\frac{1}{p_1}}$$

$$\times \left(\int_t^b (b-x)^{p_2(\beta-1)}\,dx\right)^{\frac{1}{p_2}} \left\| {}_{\alpha}^{b}\mathbf{T}(f) \right\|_{p_3,[a,b]}$$

$$= \frac{\left\| {}_{\alpha}^{b}\mathbf{T}(f) \right\|_{p_3,[a,b]}}{n!(p_1 n+1)^{\frac{1}{p_1}}(p_2(\beta-1)+1)^{\frac{1}{p_2}}}(b-t)^{n+\frac{1}{p_1}}(b-t)^{\beta-1+\frac{1}{p_2}}$$

$$= \frac{\left\| {}_{\alpha}^{b}\mathbf{T}(f) \right\|_{p_3,[a,b]}}{n!(p_1 n+1)^{\frac{1}{p_1}}(p_2(\beta-1)+1)^{\frac{1}{p_2}}}(b-t)^{n+\beta-\frac{1}{p_3}}.$$

We have proved that

$$\left| f(t) - \sum_{k=0}^{n} \frac{f^{(k)}(b)}{k!}(t-b)^k \right| \leq \left(\frac{\left\| {}_{\alpha}^{b}\mathbf{T}(f) \right\|_{p_3,[a,b]}}{n!(p_1 n+1)^{\frac{1}{p_1}}(p_2(\beta-1)+1)^{\frac{1}{p_2}}} \right)(b-t)^{n+\beta-\frac{1}{p_3}}, \tag{68}$$

$\forall\ t\in[a,b]$.

Since $\beta > \frac{1}{p_3}$, then $\beta - \frac{1}{p_3} > 0$, and $m := \alpha - \frac{1}{p_3} = n + \beta - \frac{1}{p_3} > n \in \mathbb{N}$. Call

$$\rho_1 := \frac{\left\| \mathbf{T}_{\alpha}^{a}\left(f\right)\right\|_{p_3,[a,b]}}{n!\left(p_1 n+1\right)^{\frac{1}{p_1}}\left(p_2\left(\beta-1\right)+1\right)^{\frac{1}{p_2}}}, \tag{69}$$

and

$$\rho_2 := \frac{\left\| {}_{\alpha}^{b}\mathbf{T}\left(f\right)\right\|_{p_3,[a,b]}}{n!\left(p_1 n+1\right)^{\frac{1}{p_1}}\left(p_2\left(\beta-1\right)+1\right)^{\frac{1}{p_2}}}. \tag{70}$$

Set

$$\rho := \max\left(\rho_1, \rho_2\right). \tag{71}$$

We have

$$\left| f\left(t\right) - \sum_{k=0}^{n} \frac{f^{(k)}\left(a\right)}{k!}\left(t-a\right)^{k}\right| \leq \rho\left(t-a\right)^{m}, \tag{72}$$

and

$$\left| f\left(t\right) - \sum_{k=0}^{n} \frac{f^{(k)}\left(b\right)}{k!}\left(t-b\right)^{k}\right| \leq \rho\left(b-t\right)^{m}, \tag{73}$$

$\forall\ t \in [a,b]$.

As in the proof of Theorem 3 (45) we derive

$$\begin{aligned} &\left| \int_a^b f\left(t\right)dt - \sum_{k=0}^{n} \frac{1}{(k+1)!} \right. \\ &\qquad \left. \times \left[f^{(k)}\left(a\right)\left(z-a\right)^{k+1} + \left(-1\right)^{k} f^{(k)}\left(b\right)\left(b-z\right)^{k+1}\right] \right| \\ &\leq \frac{\rho}{(m+1)}\left[\left(z-a\right)^{m+1} + \left(b-z\right)^{m+1}\right] \\ &= \frac{\rho}{\left(\alpha + \frac{1}{p_1} + \frac{1}{p_2}\right)}\left[\left(z-a\right)^{\alpha+\frac{1}{p_1}+\frac{1}{p_2}} + \left(b-z\right)^{\alpha+\frac{1}{p_1}+\frac{1}{p_2}}\right], \end{aligned} \tag{74}$$

$\forall\ z \in [a,b]$.

The rest of the proof is similar to the proof of Theorem 3. ■

We give ($n = 0$ case)

Corollary 2. *Let $0 < \alpha \leq 1$ and $f \in C^1([a,b])$. Let $p, q > 1 : \frac{1}{p} + \frac{1}{q} = 1$, with $\alpha > \frac{1}{q}$. Then*

i)

$$\left| \int_a^b f(t)\,dt - [f(a)(z-a) + f(b)(b-z)] \right| \leq \frac{\max\left\{ \|T_\alpha^a(f)\|_{q,[a,b]}, \left\| {}_\alpha^b T(f) \right\|_{q,[a,b]} \right\}}{(p(\alpha-1)+1)^{\frac{1}{p}} \left(\alpha + \frac{1}{p}\right)} \left[(z-a)^{\alpha+\frac{1}{p}} + (b-z)^{\alpha+\frac{1}{p}} \right], \tag{75}$$

$\forall\, z \in [a,b]$,

ii) at $z = \frac{a+b}{2}$, the right hand side of (75) is minimized, and we get:

$$\left| \int_a^b f(t)\,dt - \left(\frac{b-a}{2}\right)[f(a) + f(b)] \right| \leq \frac{\max\left\{ \|T_\alpha^a(f)\|_{q,[a,b]}, \left\| {}_\alpha^b T(f) \right\|_{q,[a,b]} \right\}}{(p(\alpha-1)+1)^{\frac{1}{p}} \left(\alpha + \frac{1}{p}\right)} \frac{(b-a)^{\alpha+\frac{1}{p}}}{2^{\alpha-\frac{1}{q}}}, \tag{76}$$

iii) assuming $f(a) = f(b) = 0$, we obtain

$$\left| \int_a^b f(t)\,dt \right| \leq \frac{\max\left\{ \|T_\alpha^a(f)\|_{q,[a,b]}, \left\| {}_\alpha^b T(f) \right\|_{q,[a,b]} \right\}}{(p(\alpha-1)+1)^{\frac{1}{p}} \left(\alpha + \frac{1}{p}\right)} \frac{(b-a)^{\alpha+\frac{1}{p}}}{2^{\alpha-\frac{1}{q}}}, \tag{77}$$

which is a sharp inequality,

iv) more generally, for $j = 0, 1, 2, \ldots, N \in \mathbb{N}$, it holds

$$\left| \int_a^b f(t)\,dt - \left(\frac{b-a}{N}\right)[jf(a) + (N-j)f(b)] \right| \leq \frac{\max\left\{ \|T_\alpha^a(f)\|_{q,[a,b]}, \left\| {}_\alpha^b T(f) \right\|_{q,[a,b]} \right\}}{(p(\alpha-1)+1)^{\frac{1}{p}} \left(\alpha + \frac{1}{p}\right)} \times \left(\frac{b-a}{N}\right)^{\alpha+\frac{1}{p}} \left[j^{\alpha+\frac{1}{p}} + (N-j)^{\alpha+\frac{1}{p}} \right], \tag{78}$$

v) when $N = 2$ and $j = 1$, (78) turns to

$$\left| \int_a^b f(x)\,dx - \left(\frac{b-a}{2}\right)(f(a) + f(b)) \right| \leq \frac{\max\left\{ \|T_\alpha^a(f)\|_{q,[a,b]}, \left\| {}_\alpha^b T(f) \right\|_{q,[a,b]} \right\}}{(p(\alpha-1)+1)^{\frac{1}{p}} \left(\alpha + \frac{1}{p}\right)} \frac{(b-a)^{\alpha+\frac{1}{p}}}{2^{\alpha-\frac{1}{q}}}. \tag{79}$$

Proof. As in the proof of Theorem 4, case of $n = 0$, use of (12) and (17). ■

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