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IMPLICIT FUNCTIONS UNDER FIXED POINT CONSIDERATION IN PROBABILISTIC MENGER SPACES

Abstract. In the present paper, we have introduced a pair of weakly-biased maps in the Probabilistic Menger Spaces under the implicit relation. Our results proved herein is the partial extension and mild improvement of the results due to Imdad, Tanveer and Hasan [10]. We have discussed an example in support of our main theorem.

Key words: fixed point, non expansive mappings, weakly commuting maps, compatible maps, compatible maps of type (A), ibased maps, common property(E.A).

AMS Mathematics Subject Classification: 47H09, 47H10.

1. Introduction

The concept of Probabilistic Metric Space (or Statistical Metric Spaces), which is a generalization of the metric space was introduced initially by Menger [8] in 1942. Especially, the theory of Probabilistic Metric Spaces has fundamental importance in probabilistic functional analysis. In 1958, Schweizer and Sklar [1] has developed this theory extensively. In 1972, Sehgal and Reid [18] generalized the Banach contraction condition in metric space into Menger space. Since then a spat of results in the area of fixed point theory and applications established.
Let (X, d) be a metric space and T be a mapping from X into itself such that for all $x, y \in X$.

$$d(Tx, Ty) \leq \alpha d(x, y) \tag{1}$$

where $\alpha \in (0, 1)$ condition (1) is the putting of PM-space was as under

$$F_{Tx,Ty}(\alpha t) \geq F_{x,y}(t) \tag{2}$$

$t \in [0, 1]$ and $\alpha \in (0, 1)$.

Jungck's 1976 paper [4] initiative to introduce a pair of commuting maps in a complete metric space has given a new dimension among the group of fixed point theorist to rethink the generalization of Banach Contraction Condition for obtaining common fixed points not only in metric spaces,normed linear but also in other abstract spaces. Further Sessa [13] in 1982 introduced the concept of weaker condition of commutativity which is called as weakly pair of maps commuting maps in a metric space. Variety of weakly commuting pair of maps in PM spaces are Compatible maps [15], Compatible maps of type (A) [21], Weakly compatible maps [5], Biased maps [6] and altering distance function [17]. We also ask the readers to refer ([2, 3, 11]), we shall define a new class of implicit relations for obtaining common fixed points in the next section.

2. Preliminaries

Definition 1 ([1]). *A mapping $F : \mathbb{R} \to [0,1]$ is called a distribution function if it satisfies the following conditions:*

(a) F is nondecreasing;

(b) F is left continuous, with $\inf\{F(t) : t \in \mathbb{R}\} = 0$ *and* $\sup\{F(t) : t \in \mathbb{R}\} = 1$.

We shall denote by $\mathbb{D}$ the set of all distribution functions while H will always denote the specific distribution function defined by

$$[1] \;\; H(x) = \begin{cases} 0 & \text{if } x \leq 0 \\ 1 & \text{if } x > 0 \end{cases} .$$

Definition 2 ([1]). *A probabilistic metric space is a pair (X, F),where X is a non empty set and F is a function defined $F : X \times X \to \mathbb{D}$ (the set of all distribution function) satisfying the following properties hold:*

$(PM1)$ $F_{x,y}(0) = 0$,

$(PM2)$ $F_{x,y}(t) = H(t)$,*iff* $x = y$,

$(PM3)$ $F_{x,y}(t) = F_{y,x}(t)$, *and*

$(PM4)$ $F_{x,y}(s) = 1$ *and* $F_{y,z}(t) = 1$, *then* $F_{x,z}(s+t) = 1$ *for all* $x, y, z \in X$ *and* $s, t > 0$.

Definition 3 ([1]). *A mapping $\Delta : [0,1] \times [0,1] \to [0,1])$ is said to be a triangular norm (briefly t-norm) if for every $a, b, c \in [0,1]$,*

(a) $\Delta(a, 1) = a$ *for every* $a \in [0,1]$,

(b) $\Delta(0, 0) = 0$,

(*c*) $\Delta(a,b) = \Delta(b,a)$ *for every* $a,b \in [0,1]$, *and*

(*d*) $c \geq a$ *and* $d \geq b$, *then* $\Delta(c,d) \geq \Delta(a,b)$ $(a,b,c,d \in [0,1])$,

(*e*) $\Delta(a, \Delta(b,c)) = \Delta(\Delta(a,b), c)$.

Example of basic Δ*-norm :*

1). The minimum t-norm : $\Delta_{m(a,b)} = min\{a,b\}$,

2). The product t-norm :$\Delta_{p(a,b)} = a.b$

3). The Likasiewicz t-norm : $\Delta_L(a,b) = max\{a+b-1, 0\}$.

4). The weakest t-norm, the Drastic product:

$$\Delta_D(a,b) = \begin{cases} \min(x,y) & if \;\; \max(x,y) = 1, \\ 1 & if \;\; otherwise \end{cases}$$

.

Definition 4 ([1]). *A Menger PM-Space is a triplet* (X, F, Δ), *where* (X, F) *is a PM-space and* T *is a t-norm with the following condition:*

$$F_{x,z}(s+t) \geq \Delta(F_{x,y}(s), F_{y,z}(t)) \; for \; all \; x,y,z \in X \; and \; s,t > 0$$

This inequality is known as Menger's triangle inequality.

Definition 5 ([1]). *Let* (X, F, Δ) *be a PM-space. Then,*
(*a*) *a sequence* $\{x_n\}$ *in* X *is said to be convergent to a point* $x \in X$, *if for every* $\epsilon > 0$ *and* $0 < \lambda < 1$ *there exists a positive integer* $\mathbb{Z}^+$ *such that* $F_{x_n,x}(\epsilon) > 1 - \lambda$ *whenever* $n \geq \mathbb{Z}^+$;

(*b*) *a sequence* $\{x_n\}$ *in* X *is called a cauchy sequence if for every* $\epsilon > 0$ *and* $\lambda > 0$ *we can find a positive integer*$\mathbb{Z}^+$ *such that* $F_{x_n,x_m}(\epsilon) > 1 - \lambda$ *whenever* $n, m \geq \mathbb{Z}^+$;

(*c*) *a Menger PM-space is said to be complete if every Cauchy sequence is convergent to a point in* X:

In 1991, Mishra [15] introduced Compatible Mappings in PM-Space setting.

Definition 6 ([15]). *Let* (X, F, Δ) *be a Menger space such that the t-norm* Δ *is continuous and* A, S *ne mappings from* X *into itself. Then,* S *and* T *are said to be compatible if*

$$\lim_{n\to\infty} F_{AS_{x_n}, SA_{x_n}}(t) = 1,$$

for all $t > 0$, *whenever* $\{x_n\}$ *is a sequence in* X *such that*

$$\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = z,$$

for some $z \in X$.

In 1992, Cho, Murthy and Stojakovic [21] introduced the following.

Definition 7 ([21]). *Let (X, F, Δ) be a Menger space such that T-norm t is continuous and A, S be mapping X into itself compatible maps of type (A) if*

$$\lim_{n\to\infty} F_{SA_{x_n}, AA_{x_n}}(t) = 1,$$

and

$$\lim_{n\to\infty} F_{AS_{x_n}, SS_{x_n}}(t) = 1,$$

for all $t > 0$, whenever $\{x_n\}$ is a sequence in X such that

$$\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = z,$$

for some $z \in X$.

Recently, Amari and Moutawaki [9] introduced a generalization of non compatible maps as property (E.A).

Definition 8 ([9]). *Let A and S be two self-maps of a metric space (X, d). The pair (A, S) is said to satisfy property (E.A), if there exists a sequence $\{x_n\}$ in X such that*

$$\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = t,$$

for some $t \in X$.

In a similar way we state property (E.A) in Menger probabilistic metric spaces.

Definition 9 ([9]). *A pair of self-mappings(f, g) of a Manger probabilistic metric space (X, F, Δ) is said to satisfy property (E.A), if there exists a sequence $\{x_n\}$ in X such that*

$$\lim_{n\to\infty} F_{fx_n, gx_n}(t) = 1,$$

for all $t > 0$.

$$\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = t,$$

for some $t \in X$.

Inspired by Lie, Ali and Khan [20] M.Imdad,M.Tanveer and M.Hasan [10] introduce common property (E.A) in PM-space setting.

Definition 10 ([10]). *Two pairs (A, S) and (B, T) of self mappings of a Menger PM space (X, F, Δ) are said to satisfy the common property E.A. If there exist two sequence $\{x_n\}$, $\{x_y\}$ in X and some $t \in X$ such that,*

$$\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Ty_n = \lim_{n\to\infty} By_n = t.$$

In fact, Pathak and Jungck [6] introduced the Biased maps concept in Metric Space setting which is weaker notion of compatible and after long time P.P. Murthy, M.R. Singh and L.S. Singh [12] are generalization weakly compatible maps in metric space. Inspired by of this paper we shall introduce biased maps in PM-Space.

Definition 11. *Let A and S be self-maps of Menger space (X, F, Δ). Then the pair (A, S) is S-biased iff whenever $\{x_n\}$ is a sequence in X and Ax_n,$Sx_n \to t \in X$, then*

$$\alpha F_{SAx_n,Sx_n}(t) \geq \alpha F_{ASx_n,Ax_n}(t)$$

if $\alpha = \liminf$ *and if* $\alpha = \limsup$.

Definition 12. *Let A and S be self-maps of Menger space (X, F, Δ). The pair (A, S) is said to be S-biased of type (A) and A-biased of type (A) if,*

$$\alpha F_{SSx_n,Ax_n}(t) \geq \alpha F_{ASx_n,Sx_n}(t),$$

$$\alpha F_{AAx_n,Sx_n}(t) \geq \alpha F_{SAx_n,Ax_n}(t),$$

whenever $\{x_n\}$ is a sequence in X and Ax_n,$Sx_n \to t \in X$, then if $\alpha = \liminf$ *and if* $\alpha = \limsup$.

Definition 13. *Let A, S are self maps on PM space X. The pair (A, S) is said to be weakly S-biased iff $Aa = Sa$ implies $F_{SAa,Sa}(t) \geq F_{ASa,Aa}(t)$, for some $a \in X$.*

Similarly, if the roles of A and S are interchange in above definition, then the pair (A, S) is said to be a weakly A-biased .

Definition 14. *Let A, S are self maps on PM space X. The pair (A, S) is said to be weakly S-biased of type (A) if $Aa = Sa$ implies $F_{SSa,Aa}(t) \geq F_{ASa,Sa}(t)$, for some $a \in X$.*

Similarly, if the roles of A and S are replace in about definition,then the pair (A, S) is said to be a weakly A-biased of type(A).

The following example shows that weakly biased maps and weakly biased maps of type(A) are independent.

Example 1. Let A, S be self mappings on Menger Probabilistic Metric Space $X = [0, 1]$ with usual metric define by $d(x, y) = |x - y|$ and (X, F, Δ) be the induced Manger space with $F(x, y)t = F_{x,y}(t) = H(t - d(x, y)$ for all $x, y \in X$ and for all $t > 0$. Define mapping $A, S : X \to X$ by

$$S(x) = \begin{cases} 2 - 2x & \text{for } x \in [0, \frac{1}{2}] \\ 1 & \text{for } x \in (\frac{1}{2}, 1], \end{cases}$$

$$A(x) = \begin{cases} 2x & \text{for } x \in [0, \frac{1}{2}] \\ \frac{1}{2} & \text{for } x \in (\frac{1}{2}, 1]. \end{cases}$$

At $x = \frac{1}{2}$, we get $A(\frac{1}{2})$=$S(\frac{1}{2})$=1 and $SS(\frac{1}{2}) = S(1) = 1$, $AA(\frac{1}{2}) = A(1) = \frac{1}{2}$, $AS(\frac{1}{2}) = A(1) = \frac{1}{2}$, $SA(\frac{1}{2}) = S(1) = 1$,

Now, consider $x_n = \{\frac{1}{2}\}\ \forall\ n$.

$$\lim_{n\to\infty} F_{SAx_n, Sx_n}(t) = \lim_{n\to\infty} F_{1,1}(t) = 1,$$

$$\lim_{n\to\infty} F_{ASx_n, Ax_n}(t) = H(t - d(ASx_n - Ax_n) = H(t - \frac{1}{2}) \neq 1.$$

Therefore,

$$\lim_{n\to\infty} F_{ASx_n, Ax_n}(t) < 1.$$

Thus, (A, S) is weakly S-biased.

$$\lim_{n\to\infty} F_{SS_{x_n}, A_{x_n}}(t) = \lim_{n\to\infty} F_{1,\frac{1}{2}}(t) \neq 1.$$

Therefore,

$$\lim_{n\to\infty} F_{SS_{x_n}, A_{x_n}}(t) < 1,$$

and

$$\lim_{n\to\infty} F_{AS_{x_n}, S_{x_n}}(t) = \lim_{n\to\infty} F_{\frac{1}{2},1}(t) \neq 1.$$

Therefore,

$$\lim_{n\to\infty} F_{AS_{x_n}, SS_{x_n}}(t) < 1.$$

Then, (A, S) is not weakly S-biased of type (A).

Example 2. Let A, S be self mappings on Menger Probabilistic Metric Space $X = [0, 1]$ with usual metric define by $d(x, y) = |x - y|$ and (X, F, Δ) be the induced Manger space with $F(x, y)t = F_{x,y}(t) = H(t - d(x, y)$ for all $x, y \in X$ and for all $t > 0$. Define mapping $A, S : X \to X$ by

$$S(x) = \begin{cases} 1 - x & \text{for } x \in [0, \frac{1}{2}] \\ 1 & \text{for } x \in (\frac{1}{2}, 1], \end{cases}$$

$$A(x) = \begin{cases} \frac{1}{2} & \text{for } x \in [0, \frac{1}{2}] \\ x & \text{for } x \in (\frac{1}{2}, 1]. \end{cases}$$

Now, $Ax = Sx$ at $x = \frac{1}{2}$.

Since $A(\frac{1}{2}) = S(\frac{1}{2}) = \frac{1}{2}$.

$SS(\frac{1}{2}) = S(\frac{1}{2}) = \frac{1}{2}$, $AA(\frac{1}{2}) = A(\frac{1}{2}) = \frac{1}{2}$, $AS(\frac{1}{2}) = A(\frac{1}{2}) = 1$, $SA(\frac{1}{2}) = S(1) = 1$, we get,

$F_{SSx_n,Ax_n}(t) = \lim_{n\to\infty} F_{\frac{1}{2},\frac{1}{2}}(t) = 1,\ t > 0,$

$F_{ASx_n,Sx_n}(t) = \lim_{n\to\infty} F_{1,\frac{1}{2}}(t) < 1,\ t > 0.$

The pair (A, S) is weakly S-biased of type(A).

$$\lim_{n\to\infty} F_{SAx_n,Sx_n}(t) = \lim_{n\to\infty} F_{1,\frac{1}{2}}(t) \neq 1,$$

$$\lim_{n\to\infty} F_{ASx_n,Ax_n}(t) = \lim_{n\to\infty} F_{1,\frac{1}{2}}(t) \neq 1.$$

Then, (A, S) is not weakly S-biased maps.

3. A class of implicit relation

In this section,we introduce a new class of implicit function which is different from Popa [19] and furnish example to substantiate the worth of this definition.

Let F_5 be the set of all real continuous functions and increasing on its each variables. Let $f : [0, 1]^5 \to R$ satisfies the following conditions:

(f_1) $f(1, 1, u, 1, u) > u$ for all $u \in (0, 1)$,

(f_2) $f(1, u, 1, u, 1) > u$ for all $u \in (0, 1)$,

(f_3) $f(u, u, 1, u, u) > u$ for all $u \in (0, 1)$,

(f_4) $f(u, 1, u, u, u) > u$ for all $u \in (0, 1)$,

(f_5) $f(u, 1, 1, u, u) > u$ for all $u \in (0, 1)$.

Example 3. Define $f(t_1, t_2, t_3, t_4, t_5) : [0, 1]^5 \to \mathbb{R}$ as, $f(t_1, t_2, t_3, t_4, t_5) = t_1 + \psi(min\{t_2, t_3, t_4, t_5\})$, where $\psi : [0, 1] \to [0, 1]$ is increasing and continuous such that $\psi(t) \geq t$.

(f_1) $f(1, 1, \frac{1}{2}, 1, \frac{1}{2}) = 1 + \psi(\frac{1}{2}) = 1 + \frac{1}{2} = \frac{3}{2}$,

i.e;

$$f(1, 1, \frac{1}{2}, 1, \frac{1}{2}) > \frac{1}{2}.$$

(f_2) $f(1, \frac{1}{2}, 1, \frac{1}{2}, 1) = 1 + \psi(\frac{1}{2}) = 1 + \frac{1}{2} = \frac{3}{2}$,

i.e;

$$f(1, \frac{1}{2}, 1, \frac{1}{2}, 1) > \frac{1}{2}.$$

(f_3) $f(\frac{1}{2}, \frac{1}{2}, 1, \frac{1}{2}, \frac{1}{2}) = \frac{1}{2} + \psi(\frac{1}{2}) = \frac{1}{2} + \frac{1}{2} = 1,$

i.e;

$$f(\frac{1}{2},\frac{1}{2},1,\frac{1}{2},\frac{1}{2}) > \frac{1}{2}.$$

(f_4) $f(\frac{1}{2},1,\frac{1}{2},\frac{1}{2},\frac{1}{2}) = \frac{1}{2}+\psi(\frac{1}{2}) > \frac{1}{2}+\frac{1}{2} = 1,$

i.e;

$$f(\frac{1}{2},1,\frac{1}{2},\frac{1}{2} > \frac{1}{2}.$$

(f_5) $f(\frac{1}{2},1,1,\frac{1}{2},\frac{1}{2}) = \frac{1}{2}+\psi(\frac{1}{2}) > \frac{1}{2}+\frac{1}{2} = 1,$

i.e;

$$f(\frac{1}{2},1,1,\frac{1}{2},\frac{1}{2}) > \frac{1}{2}.$$

Clearly $f \in F_5$.

Lemma 1. *Let A, B, S and T be self mapping of a Menger space (X, F, Δ) satisfying the following:*

Either (i) $A(X) \subset T(X)$ and the pair (A, S) satisfies the property $(E.A)$;

or (ii) $B(X) \subset S(X)$ and the pair (B, T) satisfies the property $(E.A)$;

(iii) for any $x, y \in X$, $f \in F_5$ and for any $t > 0$;

(3) $F_{Ax,By}(t) \geq f(F_{Sx,Ty}(t), F_{Sx,Ax}(t), F_{Ty,By}(t), F_{Ax,Ty}(t), F_{Sx,By}(t)).$

Then the pairs (A, S) and (B, T) share common property (E.A).

Proof. Assume (i) holds: Suppose that the pair (A, S) has the property (E.A), then there exists a sequence $\{x_n\}$ in X such that

(4) $$\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = u, \text{ for some } u \in X.$$

Since $A(X) \subset T(X)$, hence for each $\{x_n\}$ in X there exists a sequence $\{y_n\} \in X$ such that $Ax_n = Ty_n$, from (4), we get,

(5) $$\lim_{n\to\infty} Ty_n = \lim_{n\to\infty} Ax_n = u, \text{ for some } u \in X.$$

Now, we have to prove that $By_n \to u$ as $n \to \infty$. Assume that $\lim_{n\to\infty} By_n \neq u$. On taking $x = x_n$ and $y = y_n$ in (3), we obtain that

$$f(F_{Sx_n,Ty_n}(t), F_{Sx_n,Ax_n}(t), F_{Ty_n,By_n}(t), F_{Ax_n,Ty_n}(t), F_{Sx_n,By_n}(t)) \\ \leq F_{Ax_n,By_n}(t).$$

On taking limits as $n \to \infty$, then we get

$$f(F_{u,u}(t), F_{u,u}(t), F_{u,\lim_{n\to\infty} By_n}(t), F_{u,u}(t), F_{u,\lim_{n\to\infty} By_n}(t)) \leq F_{u,\lim_{n\to\infty} By_n}(t),$$

$$f(1,1,F_{u,\lim_{n\to\infty}By_n}(t),1,F_{u,\lim_{n\to\infty}By_n}(t)) \le F_{u,\lim_{n\to\infty}By_n}(t)$$

which is a contradiction from the condition (f_1) and therefore

(6) $$\lim_{n\to\infty} By_n = u.$$

From (4) – (6).

Hence the pairs (A,S) and (B,T) enjoys common property (E.A).

Similarly, proof is same if condition (ii) holds. ■

4. Main results

Now we prove our main result.

Theorem 1. *Let A,B,S and T be self mapping of a Manger space (X,F,Δ) satisfying the condition* (3) *and*

Either (i) the pair (A,S) satisfies the property $(E.A)$ and $A(X)\subset T(X)$ and $S(X)$ is a closed subset of X.

Or (ii) the pair (B,T) satisfies the property $(E.A)$ and $B(X)\subset S(X)$ and $T(X)$ is a closed subset of X.

Then A, B, S and T have common coincidence point in X.

Proof. Assume (i) holds in the view of Lemma 1, the pairs (A,S) and (B,T) share the common property (E.A), that is there exist two sequence $\{x_n\}$ and $\{y_n\}$in X such that

(7) $$\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = \lim_{n\to\infty} By_n = \lim_{n\to\infty} Ty_n = u$$

for some $u\in X$.

Since $S(X)$ is closed subset of X then there is a $u\in X$ such that $Sa=u$ for some $a\in X$. Assume that $Aa\neq u$ for some $a\in X$ then putting $x=x_n$ and $y=y_n$ in (3), we obtain

(8) $$\begin{aligned} F_{Aa,By_n}(t) \ge f(&F_{Sa,Ty_n}(t),F_{Sa,Aa}(t),F_{Ty_n,By_n}(t),\\ &F_{Aa,Ty_n}(t),F_{Sa,By_n}(t)). \end{aligned}$$

On taking limits as $n\to\infty$ and using (7), we get

$$\begin{aligned} F_{Aa,u}(t) &\ge f(F_{u,u}(t),F_{u,Aa}(t),F_{u,u}(t),F_{Aa,u}(t),F_{u,u}(t)),\\ &\ge f(1,F_{u,Aa}(t),1,F_{u,Aa}(t),1), \end{aligned}$$

a contradiction from (f_2).
Thus

(9) $$Aa = Sa = u.$$

Since $A(X) \subset T(X)$, there exist $b \in X$, such that

$$Aa = Tb = u. \tag{10}$$

Now, we have to show that $Bb = u$, assume that $Bb \neq u$, then putting $x = a$ and $y = b$ in (3), we obtain that

$$F_{Aa,Bb}(t) \geq f(F_{Sa,Tb}(t), F_{Sa,Aa}(t), F_{Tb,Bb}(t), F_{Aa,Tb}(t), F_{Sa,Bb}(t)). \tag{11}$$

Now using (9) and (10) in (11), we get

$$\begin{aligned} F_{u,Bb}(t) &\geq f(F_{u,u}(t), F_{u,u}(t), F_{u,Bb}(t), F_{u,u}(t), F_{u,Bb}(t)), \\ &\geq f(1, 1, F_{u,Bb}(t), 1, F_{u,Bb}(t)), \end{aligned}$$

a contradiction from (f_1). Therefore

$$Bb = Aa = Sa = Tb = u.$$

Therefore, A, B, S and T have common coincidence point in X. ■

Theorem 2. *In addition to the Theorem 1, if the pairs (A, S) and (B, T) are satisfying weakly S-biased and weakly T-biased respectively. Then A, B, S and T have a unique common fixed point in X.*

Proof. From Theorem 1, we get

$$Bb = Aa = Sa = Tb = u. \tag{12}$$

First, we show that $Au = u$, assume that $Au \neq u$. On taking $x = u$ and $y = y_n$ in (3), we get

$$\begin{aligned} F_{Au,By_n}(t) \geq f(&F_{Su,Ty_n}(t), F_{Su,Au}(t), F_{Ty_n,By_n}(t), \\ &F_{Au,Ty_n}(t), F_{Su,By_n}(t)), \end{aligned} \tag{13}$$

On taking limits as $n \to \infty$ and using (12) in (13), we have

$$\begin{aligned} F_{Au,u}(t) &\geq f(F_{Su,u}(t), F_{Su,Au}(t), F_{u,u}(t), F_{Au,u}(t), F_{Su,u}(t)) \\ &\geq f(F_{Su,u}(t), \min\{F_{Au,u}(t), F_{Su,u}(t)\}, \\ &\qquad 1, F_{Au,u}(t), F_{Su,u}(t)). \end{aligned} \tag{14}$$

Since the pair (A,S) is weakly S-biased type of (A), then

$$\begin{aligned} Aa = Sa &\Rightarrow F_{SSa,Aa}(t) \geq F_{AAa,Sa}(t), \\ &\Rightarrow F_{Su,u}(t) \geq F_{Au,u}(t). \end{aligned} \tag{15}$$

Substitute (15) in (14), we have

$$F_{Au,u}(t) \geq f(F_{Au,u}(t), F_{Au,u}(t), 1, F_{Au,u}(t), F_{Au,u}(t)).$$

a contradiction from (f_3), hence

$$Au = u. \tag{16}$$

From (15) and (16), we get,

$$Su = u.$$

Therefore,

$$Au = Su = u. \tag{17}$$

Now, we have to show that $Bu = u$ Assume that $Bu \neq u$, then we have on taking $x = a$ and $y = u$ in (3) we get

$$\begin{aligned} F_{Aa,Bu}(t) \geq f(&F_{Sa,Tu}(t), F_{Sa,Aa}(t), F_{Tu,Bu}(t), \\ &F_{Aa,Tu}(t), F_{Sa,Bu}(t)) \end{aligned} \tag{18}$$

Since the pair (B, T) is weakly T-biased map,then

$$\begin{aligned} Bb = Tb \Rightarrow\ & F_{TBb,Tb}(t) \geq F_{BTb,Bb}(t), \\ \Rightarrow\ & F_{Tu,u}(t) \geq F_{Bu,u}(t). \end{aligned} \tag{19}$$

From (12), (19) and using tiangular inequality, we have

$$\begin{aligned} F_{u,Bu}(t) &\geq f(F_{u,Tu}(t), F_{u,u}(t), F_{u,Bu}(t), F_{u,Tu}(t), F_{u,Bu}(t)), \\ F_{u,Bu}(t) &\geq f(F_{u,Tu}(t), 1, F_{u,Tu}(t), F_{u,Bu}(t), F_{u,Bu}(t)), \end{aligned} \tag{20}$$

a contradiction from (f_4). Therefore

$$F_{Bu,u}(t) \geq 1 \Rightarrow Bu = u. \tag{21}$$

Since the pair (B, T) is weakly T-biased map, then we have

$$F_{Tu,u}(t) \geq F_{Bu,u}(t) = F_{u,u}(t).$$

This leads to

$$F_{Tu,u}(t) \geq 1 \Rightarrow Tu = u. \tag{22}$$

From (21) and (22), we get

$$Bu = Tu = u.$$

Hence, A, B, S and T have a common fixed point in X. ∎

Theorem 3. *In addition to the Theorem* 1*, if the pairs* (A, S) *and* (B, T) *are satisfying weakly S-biased of type* (A) *and weakly T-biased of type* (A) *respectively. Then* A*,* B*,* S *and* T *have a unique common fixed point in* X

Proof. From Theorem 1, we get

$$Bb = Aa = Sa = Tb = u. \tag{23}$$

Now, we have to show that $Au = u$, if assume that $Au \neq u$.

On taking $x = u$ and $y = y_n$ in (3), we get

$$\begin{aligned} F_{Au,By_n}(t) \geq f(&F_{Su,Ty_n}(t), F_{Su,Au}(t), F_{Ty_n,By_n}(t), \\ &F_{Au,Ty_n}(t), F_{Su,By_n}(t)). \end{aligned} \tag{24}$$

On taking limits as $n \to \infty$ and using (7) in (24), we have

$$\begin{aligned} F_{Au,u}(t) &\geq f(F_{Su,u}(t), F_{Su,Au}(t), F_{u,u}(t), F_{Au,u}(t), F_{Su,u}(t)), \\ &\geq f(F_{Su,u}(t), \min\{F_{Au,u}(t), F_{Su,u}(t)\}, \\ &\qquad 1, F_{Au,u}(t), F_{Su,u}(t)). \end{aligned} \tag{25}$$

Since the pair (A, S) is weakly S-biased type of (A), then

$$\begin{aligned} Aa = Sa(= u) &\Rightarrow F_{SSa,Aa}(t) \geq F_{AAa,Sa}(t), \\ &\Rightarrow F_{Su,u}(t) \geq F_{Au,u}(t). \end{aligned} \tag{26}$$

Substitute (26) in (25), we have

$$F_{Au,u}(t) \geq f(F_{Au,u}(t), F_{Au,u}(t), 1, F_{Au,u}(t), F_{Au,u}(t)),$$

a contradiction from (f_3). Hence

$$Au = u,$$

and from (26),

$$Su = u.$$

Therefore,

$$Au = Su = u. \tag{27}$$

Now, we have to show that $Bu = u$ if Assume that $Bu \neq u$. On taking $x = a$ and $y = u$ in (3), we get

$$\begin{aligned} F_{Aa,Bu}(t) \geq f(&F_{Sa,Tu}(t), F_{Sa,Aa}(t), F_{Tu,Bu}(t), \\ &F_{Aa,Tu}(t), F_{Sa,Bu}(t)). \end{aligned} \tag{28}$$

Since the pair (B,T) is weakly T-biased type of (A),then

$$Bb = Tb \Rightarrow F_{TTb,Bb}(t) \geq F_{BBb,Tb}(t) \Rightarrow F_{Tu,u}(t) \geq F_{Bu,u}(t).$$

From (23), (27) and (28) and using tiangular inequality, we have

$$\begin{aligned} F_{u,Bu}(t) &\geq f(F_{u,Tu}(t), F_{u,u}(t), F_{u,Bu}(t), F_{u,Tu}(t), F_{u,Bu}(t)), \\ &\geq f(F_{u,Tu}(t), 1, F_{u,Tu}(t), F_{u,Bu}(t), F_{u,Bu}(t)), \end{aligned}$$

a contradiction from (f_4). Therefore,

$$F_{Bu,u}(t) \geq 1 \Rightarrow Bu = u.$$

Since the pair (B,T) is weakly T-biased of type(A),

$$F_{Tu,u}(t) \geq F_{Bu,u}(t) = F_{u,u}(t).$$

This leads to

$$F_{Tu,u}(t) \geq 1 \Rightarrow Tu = u.$$

Therefore,

$$Bu = Tu = u. \tag{29}$$

Hence from (27) and (29) A, B, S and T have common fixed point in X. ■

Uniqueness. Let u and w be two the fixed points such that

$$Au = Su = Tu = Bu = u,$$

and

$$Aw = Sw = Tw = Bw = w.$$

Putting $x = u$ and $y = w$ in (3), we obtain

$$F_{Au,Bw}(t) \geq f(F_{Su,Tw}(t), F_{Su,Au}(t), F_{Tw,Bw}(t), F_{Au,Tw}(t), F_{Su,Bu}(t)),$$

$$\begin{aligned} F_{u,w}(t) &\geq f(F_{u,w}(t), F_{u,u}(t), F_{w,w}(t), F_{u,w}(t), F_{u,w}(t)), \\ &\geq f(F_{u,w}(t), 1, 1, F_{u,w}(t), F_{u,w}(t)), \end{aligned}$$

a contradiction (f_5), therefore $F_{u,w}(t) \geq 1$, then $u = w$.

This completes the proof.

Now we are ready to four Corollaries based on our main theorem:

5. Corollaries and example

Corollary 1 ([10]). *Let A, B, S and T be self-mappings on a Menger PM space (X, F, δ) satisfying inequality*

$$f(F_{Ax,By}(t), F_{Sx,Ty}(t), F_{Sx,Ax}(t), F_{Ty,By}(t), F_{Sx,By}(t), F_{Ax,Ty}(t)) \geq 0,$$

where $f \in F_6$(page no.5 in [7]). Suppose that

i) the pair (A, S) (or (B, T)) enjoys the property (E.A),

ii) $AX \subset TX$ (or $BX \subset SX$),

iii) SX (or TX)is a closed subset of X.

Then the pairs A, S and B, T have a point of coincidence each. Moreover, A, B, S and T have a unique common fixed point provided that both the pairs (A, S) and (B, T) are weakly compatible.

Corollary 2. *Let A and S be self mappings of a Manger space (X, F, Δ) satisfying the following:*

The pair (A, S) which is weakly-S biased maps and satisfies the property $(E.A)$ and $A(X) \subset S(X)$ and $S(X)$ is a closed subset of X. For any $x, y \in X$, $f \in F_5$ and for all $t > 0$,

$$F_{Ax,Ay}(t) \geq f(F_{Sx,Sy}(t), F_{Sx,Ay}(t), F_{Sy,Ay}(t), F_{Ax,Sy}(t), F_{Sx,Ay}(t)).$$

Then A and S have a point of coincidence each. Moreover, A and S have unique common fixed point.

Corollary 3. *Let A, B, S and T be self mappings of a Manger space (X, F, Δ) satisfying the following:*

(i) $A(X) \subset T(X)$ and $B(X) \subset S(X)$,

(ii) the pairs (A, S) and (B, T) satisfy common property $(E.A)$ and compatible,

(iii) $S(X)$ and $T(X)$ are closed subspace of X,

(iv) for any $x, y \in X$, $f \in F_5$ and for all $t > 0$.

$$F_{Ax,By}(t) \geq f(F_{Sx,Ty}(t), F_{Sx,Ay}(t), F_{Ty,By}(t), F_{Ax,Ty}(t), F_{Sx,By}(t)).$$

Then the pairs (A, S) and (B, T) have a point of coincidence point. Moreover, A, B, S and T have unique common fixed point.

Corollary 4. *Let A and S be self mappings of a Manger space (X, F, Δ) satisfying the following:*

The pair (A, S) which is weakly-S biased maps type (A) and satisfies the property $(E.A)$ and $A(X) \subset S(X)$ and $S(X)$ is a closed subset of X.

For any $x, y \in X$, $f \in F_5$ and for all $t > 0$,

$$F_{Ax,Ay}(t) \geq f(F_{Sx,Sy}(t), F_{Sx,Ay}(t), F_{Sy,Ay}(t), F_{Ax,Sy}(t), F_{Sx,Ay}(t)).$$

Then A and S have a point of coincidence point. Moreover, A and S have unique common fixed point.

Corollary 5. *Let A, B, S and T be self mappings of a Manger space (X, F, Δ) satisfying the following:*

(i) $A(X) \subset T(X)$ and $B(X) \subset S(X)$,

(ii) the pairs (A, S) and (B, T) satisfy common property (E.A) and compatible maps of type (A),

(iii) $S(X)$ and $T(X)$ are closed subspace of X,

(iv) for any $x, y \in X$, $f \in F_5$ and for all $t > 0$.

$$F_{Ax,By}(t) \geq f(F_{Sx,Ty}(t), F_{Sx,Ay}(t), F_{Ty,By}(t), F_{Ax,Ty}(t), F_{Sx,By}(t)).$$

Then the pairs (A, S) and (B, T) have a point of coincidence point. Moreover, A, B, S and T have unique common fixed point.

Example 4. Let $X = R^+$ and F be defined by

$$F_{x,y}(t) = \begin{cases} \frac{t}{t+|x-y|}, & \text{if } t > 0 \\ 0, & \text{if } t = 0. \end{cases}$$

Then (X, F) is a PM Space. Let A, B, S and T be self maps on X and defined by

$$Ax = \begin{cases} 0, & \text{if } x = 0; \\ 1, & \text{if } x > 0. \end{cases} \quad \text{and} \quad Bx = \begin{cases} 0, & \text{if } x = 0 \text{ or } x > 6; \\ 1, & \text{if } x \in (0, 6]. \end{cases}$$

$$Sx = \begin{cases} 0, & \text{if } x = 0; \\ 2, & \text{if } x > 0. \end{cases} \quad \text{and} \quad Tx = \begin{cases} 0, & \text{if } x = 0; \\ 1, & \text{if } x \in (0, 6]; \\ x - 6, & \text{if } x > 6. \end{cases}$$

The pair (A, S) is satisfies (E.A) property at $x_n = 0$, for all n. $A(X) = \{0, 1\} \subset [0, \infty) = T(X)$ and the pairs (A, S) and (B, T) are also satisfied weakly S-biased and weakly T-biased maps at $x = 0$. We can also observe that the condition (3) satisfies for all possible cases and $f \in F_5$. The hypothesis of Theorem 1 satisfied, and 0 is the unique common fixed point of A, B, S and T in X.

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References

[1] Schweizer B., Sklar A., Probabilistic Metric Spaces, *North-Holland Series in Probability and Applied Mathimatics,* 5(1983).
[2] Patel D.K., Kumam P., Gopal D., Some discussion on the existence of common fixed points for a pair of maps, *Fixed point theory and applications,* 187(2013).
[3] Gopal D., Abbas M., Vetro C., Some new fixed point theorems in Menger PM-spaces with applications to Volterra type integral equation, *Applied Mathematics Computation,* 232(2014), 955-967.
[4] Jungck G., Commuting Mappings And Fixed Points, *American Mathematical Monthly,* 83(1976), 261-263.
[5] Jungck G., Rhoades B.E., Fixed points for set valued functions without continuity, *Indian Journal Pure Applied Mathematics,* 29(3)(1998), 227-238.
[6] Jungck G., Pathak H.K., Fixed Points via 'biased maps, *Proceedings of the American Mathematical Society,* 123(1995), 2049-2060.
[7] Jungck G., Common fixed points of non-metric spaces, *Far East Journal of mathematical Science,* 4(2)(1996), 199-215.
[8] Menger K., Statistical metrics, *Proceedings of the National Academy of Science USA,* 28(12)(1942), 535-537.
[9] Aamri M., El Moutawaki D., Some new common fixed point theorems under strict contractive conditions, *Journal of Mathematocal Analysis and Application,* 270(2002), 181-188.
[10] Imdad M, M.Tanveer M., Hasan M., Some common fixed point theorems in Menger PM space, *Fixed Point Theory and Applications,* Volume 2010, Article ID 819269, 14 pages.
[11] Tanveer M., Imdad M., Gopal D., Patel D.K., Common fixed point theorems in modified intuitionistic fuzzy metric spaces with common property (E.A), *Fixed Point Theory and Applications,* 2012(2012), 36.
[12] Murthy P.P., Singh M.R., Singh L.S., Weakly biased maps as a generalization of occasionally weakly compatible maps, *International Journal of Pure and Applied Mathematics,* (2014), 143-153.
[13] Sessa S., On a weak commutativity condition of mappings in fixed point considerations, *Publications Del'llnstitut Mathematique* (*Beograd*) (*N.S.*)*,* 32 (46)(1982), 149-153.
[14] Beloul S., A common fixed point theorem for weakly subsequentially sontinuous mappings satisfying implicit relation in Menger space, *Facta Universitis,* 5(2015),719-729.
[15] Mishra S.N., Common fixed points of compatible mappings in probabilistic metric spaces, *Mathamatica Japonica,* 36(1991), 283-289.
[16] Kumar S., Rani A., Some common fixed point theorems in Menger space, *Applied Mathamatics,* 3(2012), 235-245.

[17] Doenovi T., Kumam P., Gopal D., Patel D.K., Takai A., On fixed point theorems involving altering distances in Menger probabilistic metric spaces, *Journal of Ineqqualities and Application,* (2013), 567.

[18] Sehgal V.M., Bharucha-Reid A.T., Fixed points of contraction mappings on probabilistic metric spaces, *Proceedings of the American Mathematical Society,* 13(1972), 97-102.

[19] Popa V., A General Fixed Point Theorem for Weakly Compatible Mappings in Compact Metric Spaces, *Turkish J. Math.,* 25(2)(2015), 465-474.

[20] Lie Y., Ali J., Khan L., Coincidence and fixed points in symmetric spaces under strict contractions, *Journal of Mathematical Anaiysis and Application,* 32(2008), 117-124.

[21] Cho Y.J., Murthy P.P., Stojakovic M., Compatible mappings of type (A) and common fixed point in Menger spaces, *Communications of the Korean Mathematical Society,* 7(2), 325-339.

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