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FIXED POINT RESULTS VIA GENERALIZED RATIONAL AND CONVEX TYPE CONTRACTIONS IN MODULAR METRIC SPACES

Abstract. In the current manuscript, some fixed point results for generalized rational and convex type contractions in the context of modular metric spaces are established. The derived results generalizes some well known results from the existing literature.

Key words: rational type contraction, generalize convex contraction in modular metric space.

AMS Mathematics Subject Classification: 47H10, 54H25.

1. Introduction and preliminaries

Metric Fixed point theory is one of the most effective research subject in the development of non-linear analysis. Banach proved a significant result known as Banach contraction Principle. This principle has been commonly adopted in different directions either varying by the contractive condition or by altering the underlying space. Dass and Gupta [1] brought to the light the concept of rational type contraction in a metric space. In the same way some rational type contraction in $dq-$metric space was introduced in [18]. Sessa, introduced the concept of weakly commuting in [13], Jungck generalized this idea, to compatible [10] and then to weakly compatible mappings [9]. One can verify with ease that the two mappings which are commute will be compatible but not conversely. The notion of weakly compatible was introduced by Junck and Rhoades [8] and showed that compatible maps are weakly compatible but not conversely. Istratescu introduced the notion of convex contraction mappings in metric spaces in [19] and obtained some fixed point results which was generalized by some authors in the setting of metric, ordered metric, orthogonal metric, cone metric, b-metric and 2-metric spaces [20, 21, 22, 23, 24, 25, 26, 27, 28, 29].

Chityaskov introduced the notion of modular metric spaces [2], and present a few interesting applications for superposition operators by exercising the theory of modular metric spaces [3]. After that researchers analyzed and broaden fixed point problems in modular metric spaces. Some of them can be noted [4, 5, 7, 11, 12, 13]. Rahimpoor proved some coincident and common fixed point theorems for contractive mapping in modular metric spaces [17].

In the current manuscript, using generalized rational and convex type contraction, some unique and common fixed point results in the setting of modular metric spaces are established. The presented work generalizes some well known results particularly the results of [11] and [17] from the exiting literature in the setting of modular metric spaces.

Definition 1 ([10]). *A metric modular on a non-empty set X is a function $\omega_\sigma(x,y) : (0,\infty) \times X \times X \to [0,\infty)$ that will be written as $\omega_\sigma(x,y) = \omega(\sigma,x,y)$; for all $x,y,z \in X$ and for all $\sigma > 0$, satisfies the following three conditions:*

(i) $\omega_\sigma(x,y) = 0$ if and only if $x = y$, for all $\sigma > 0$ and $x,y \in X$;
(ii) $\omega_\sigma(x,y) = \omega_\sigma(y,x)$, for all $\sigma > 0$ and $x,y \in X$;
(iii) $\omega_{\sigma+\vartheta}(x,y) \leq \omega_\sigma(x,z) + \omega_\vartheta(z,y)$ for all $\sigma,\vartheta > 0$ and all $x,y,z \in X$.

If instead of (i) we have only the condition
(iv) $\omega_\sigma(x,x) = 0$,then ω is said to be a (metric) pseudomodular on X.
And if ω satisfies (iv) and
(v) given $x,y \in X$, if there exist $\sigma > 0$ possibly depending on x and y, such that $\omega_\sigma(x,y) = 0$ then $x = y$, with this condition ω is called a strick modular on X.

If instead of (iii) we replace the following condition;

$$\omega_{\sigma+\vartheta}(x,y) \leq \frac{\sigma}{\sigma+\vartheta}\omega_\sigma(x,y) + \frac{\vartheta}{\sigma+\vartheta}\omega_\vartheta(x,y).$$

Then ω is called a convex modular on X.

Remark 1. Let ω be a modular on a set X, then for given $x,y \in X$, the function $0 < \sigma \to \omega_\sigma(x,y) \in [0,\infty]$ is non increasing on $(0,\infty)$.

In fact if $0 < \sigma < \vartheta$, then by above definition

$$\omega_\vartheta(x,y) \leq \omega_{\vartheta-\sigma}(x,x) + \omega_\sigma(x,y) = \omega_\sigma(x,y) \ \ \forall x,y \in X.$$

Definition 2 ([17]). *Given a modular ω on X. A sequence $\{x_n\} \equiv \{x_n\}_{n=1}^\infty$ in $X_\omega(X^*_\omega)$ is said to be modular convergent to an element $x \in X$ if there exist a number $\sigma > 0$, possibly depending on $\{x_n\}$ and x, such that* $\lim_{n\to\infty} \omega_\sigma(x_n,x) = 0$. *i.e $x_n \to x$ as $n \to \infty$.*

Definition 3 ([11]). *Given a modular ω on X. A sequence $\{x_n\} \subset X_\omega$ in $X_\omega(X^*_\omega)$ is said to be modular Cauchy (ω-Cauchy) if for all $\sigma > 0$,* $\lim_{m,n\to\infty} \omega_\sigma(x_n,x_m) = 0$.

Definition 4 ([17]). *A modular space X_ω is said to be modular complete if each Cauchy sequence in X_ω is modular convergent. More incisively; if $\{x_n\} \subset X_\omega$ and there exist $\sigma = \sigma\{x_n\} > 0$ such that*

$$\lim_{m,n\to\infty} \omega_\sigma(x_n, x_m) = 0$$

then there exists $x \in X_\omega$ such that $\lim_{n\to\infty} \omega_\sigma(x_n, x) = 0$.

Definition 5 ([7]). *A modular ω on X is said to satisfy the Δ_2-condition if $\lim_{n\to\infty} \omega_\sigma(x_n, x) = 0$, for some $\sigma > 0$ implies that $\lim_{n\to\infty} \omega_\sigma(x_n, x) = 0$ for all $\sigma > 0$.*

2. Main results

Definition 6. *Let X_ω be a modular metric space and $P, Q : X_\omega \to X_\omega$. A point $x \in X_\omega$ is said to be a coincident point of P and Q if $Px = Qx$. The mapping* P,Q *are said to be weakly compatible if they commute at their coincident point (i.e., $QPx = PQx$ whenever $Qx = Px$).*

Definition 7. *Let X_ω be a modular metric space. A self mapping T on X is said to be a generalized convex contraction if there exist a mapping $\alpha : X \times X \to [0,\infty)$ and $\alpha_1, \alpha_2 \in [0,1)$ with $\alpha_1 + \alpha_2 < 1$ such that*

$$\alpha(x,y)\omega_\sigma(T^2x, T^2y) \le \alpha_1\omega_\sigma(x,y) + \alpha_2\omega_\sigma(Tx,Ty).$$

Definition 8. *Let X_ω be a modular metric space. A self mapping T on X is said to be a generalized convex contraction of order-2, if there exist a mapping $\alpha : X \times X \to [0,\infty)$ and $\beta_1, \beta_2, \gamma_1, \gamma_2 \in [0,1)$ with $(\beta_1 + \beta_2 + \gamma_1 + \gamma_2) < 1$ such that*

$$\begin{aligned}\alpha(x,y)\omega_\sigma(T^2x, T^2y) \le\ & \beta_1\omega_\sigma(x,Tx) + \beta_2\omega_\sigma(Tx,T^2x)\\ & + \gamma_1(y,Ty) + \gamma_2\omega_\sigma(Ty,T^2y).\end{aligned}$$

Definition 9. *Let X_ω be a modular metric space. A self mapping T on X is said to be a generalized convex contraction of type-2, if there exist a mapping $\alpha : X \times X \to [0,\infty)$ and $\alpha_i, \beta_i, \gamma_i \ge 0$ with $\sum_{i=1,2}(\alpha_i + \beta_i + \gamma_i) < 1$ such that*

$$\begin{aligned}\alpha(x,y)\omega_\sigma(T^2x, T^2y) \le\ & \alpha_1\omega_\sigma(x,y) + \alpha_2\omega_\sigma(Tx,Ty) + \beta_1\omega_\sigma(x,Tx)\\ & + \beta_2\omega_\sigma(Tx,T^2x) + \gamma_1\omega_\sigma(y,Ty)\\ & + \gamma_2\omega_\sigma(Ty,T^2y).\end{aligned}$$

Definition 10. *Let X_ω be a modular metric space. A self mapping T on X is said to be a generalized convex contraction of rational type, if there exist* *a mapping* $\alpha : X\times X \to [0,\infty)$ *and* $\alpha_i, \beta_i, \gamma_i \geq 0$ *with* $\sum_{i=1,2}(\alpha_i+\beta_i+\gamma_i) < 1$ *such that*

$$\begin{aligned}\alpha(x,y)\omega_\sigma(T^2x,T^2y) \ \leq\ & \alpha_1\omega_\sigma(x,y)+\alpha_2\omega_\sigma(Tx,Ty)\\ &+\beta_1\frac{\omega_\sigma(x,Tx)[1+\omega_{2\sigma}(x,Ty)]}{1+\omega_\sigma(x,y)+\omega_\sigma(y,Ty)}\\ &+\beta_2\frac{\omega_\sigma(Tx,T^2x)\omega_\sigma(x,Tx)}{[1+\omega_\sigma(x,y)]}\\ &+\gamma_1\frac{\omega_\sigma(y,Ty)[1+\omega_\sigma(x,Tx)]}{1+\omega_\sigma(x,y)}\\ &+\gamma_2\frac{\omega_\sigma(Ty,T^2y)+\omega_\sigma(y,Tx)}{1+\omega_\sigma(y,Ty)\omega_\sigma(y,Tx)}.\end{aligned}$$

Example 1. Let $X_\omega = \{1,3,5\}$, $\omega_\sigma(x,y) = \frac{d(x,y)}{\sigma}$ where $d(x,y) = |x-y|$ and T be a self map on X_ω defined by $T1 = 3$, $T3 = 1$ and $T5 = 5$. Then by putting $\alpha_1 = \alpha_2 = \beta_1 = \beta_2 = \gamma_1 = \gamma_2 = \frac{1}{12}$,$x = 1$ and $y = 3$, we have

$$\begin{aligned}\frac{2}{\sigma} \ =\ & \omega_\sigma(T^21,T^23) > \alpha_1\omega_\sigma(1,3)+\alpha_2\omega_\sigma(T1,T3)\\ &+\beta_1\omega_\sigma(1,T1)+\beta_2\omega_\sigma(T1,T^21)\\ &+\gamma_1\omega_\sigma(3,T3)+\gamma_2\omega_\sigma(T3,T^23) = \frac{1}{\sigma}.\end{aligned}$$

Thus, T is not a generalized convex contraction of type-2, while by putting $\alpha(x,y) = \frac{1}{6}$ whenever $x \leq y$ and $\alpha(x,y) = 0$, otherwise and $\alpha_1 = \alpha_2 = \beta_1 = \beta_2 = \gamma_1 = \gamma_2 = \frac{1}{12}$. Then it is easy to see that T is a generalized convex contraction of type-2.

To prove the main results we needed the following in sequel.

Suppose $Q(X_\omega) \subseteq P(X_\omega)$. Let x_0 be an arbitrary point in X_ω. Now as $Q(X_\omega) \subseteq P(X_\omega)$ then there exist a point $x_1 \in X_\omega$ such that $Qx_0 = Px_1$. With the help of which we construct a sequence x_n in X_ω such that $Px_n = Qx_{n-1}$ for all $n \geq 1$.

Theorem 1. *Let* X_ω *be a modular metric space and* $P, Q : X_\omega \to X_\omega$. *be two mappings such that* $Q(X_\omega) \subseteq P(X_\omega)$ *and* $P(X_\omega)$ *be a* ω*-complete subspace of* X_ω. *Suppose there exist mappings* $\alpha_i : X_\omega \times X_\omega \to [0,1)$, $i = 1,2,...7$ *such that the following assertion for all* $x, y \in X_\omega$ *and* $\sigma > 0$ *hold.*

(*i*) $\alpha(Px,Qy) \leq \alpha(Px,Py)$; *and* $\alpha(Qx,Py) \leq \alpha(Px,Py)$;

$$\begin{aligned}(ii)\ \ \omega_\sigma(Qx,Qy) \ \leq\ & \alpha_1(Px,Py)\omega_\sigma(Px,Py)\\ &+\alpha_2(Px,Py)[\omega_\sigma(Px,Qx)+\omega_\sigma(Py,Qy)]\\ &+\alpha_3(Px,Py)[\omega_{2\sigma}(Px,Qy)+\omega_\sigma(Py,Qx)]\end{aligned}$$

$$+ \alpha_4(Px,Py)\frac{\omega_\sigma(Py,Qy)[1+\omega_\sigma(Px,Qx)]}{1+\omega_\sigma(Px,Py)}$$
$$+ \alpha_5(Px,Py)\frac{\omega_{2\sigma}(Px,Qy)\omega_\sigma(Px,Qx)}{[1+\omega_\sigma(Px,Py)]}$$
$$+ \alpha_6(Px,Py)\frac{\omega_\sigma(Py,Qy)+\omega_\sigma(Py,Qx)}{1+\omega_\sigma(Py,Qy)\omega_\sigma(Py,Qx)}$$
$$+ \alpha_7(Px,Py)\frac{\omega_\sigma(Px,Qx)[1+\omega_{2\sigma}(Px,Qy)]}{1+\omega_\sigma(Px,Py)+\omega_\sigma(Py,Qy)}.$$

(iii) $\Sigma_{i=1}^{3}\alpha_i(Px,Py)+\alpha_5(Px,Py)+\Sigma_{i=2}^{7}\alpha_i(Px,Py)<1$.

(iv) $\omega_\sigma(Px,Qy)<\infty$.

Then Q and P have a coincident point.

Proof. Let x_0 be an arbitrary point in X_ω since $Q(X_\omega)\subseteq P(X_\omega)$ so there exist sequence $\{Qx_n\}$ in X_ω such that

$$Px_n = Qx_{n-1}, \text{ for all } n \geq 1.$$

Now putting $x=x_n$ and $y=x_{n+1}$ in (ii) we have

$$\omega_\sigma(Qx_n,Qx_{n+1}) \leq \alpha_1(Px_n,Px_{n+1})\omega_\sigma(Px_n,Px_{n+1})$$
$$+ \alpha_2(Px_n,Px_{n+1})[\omega_\sigma(Px_n,Qx_n)+\omega_\sigma(Px_{n+1},Qx_{n+1})]$$
$$+ \alpha_3(Px_n,Px_{n+1})[\omega_{2\sigma}(Px_n,Qx_{n+1})+\omega_\sigma(Px_{n+1},Qx_n)]$$
$$+ \alpha_4(Px_n,Px_{n+1})\frac{\omega_\sigma(Px_{n+1},Qx_{n+1})[1+\omega_\sigma(Px_n,Qx_n)]}{1+\omega_\sigma(Px_n,Px_{n+1})}$$
$$+ \alpha_5(Px_n,Px_{n+1})\frac{\omega_{2\sigma}(Px_n,Qx_{n+1})\omega_\sigma(Px_n,Qx_n)}{[1+\omega_\sigma(Px_n,Px_{n+1})]}$$
$$+ \alpha_6(Px_n,Px_{n+1})\frac{\omega_\sigma(Px_{n+1},Qx_{n+1})+\omega_\sigma(Px_{n+1},Qx_n)}{1+\omega_\sigma(Px_{n+1},Qx_{n+1})\omega_\sigma(Px_{n+1},Qx_n)}$$
$$+ \alpha_7(Px_n,Px_{n+1})\frac{\omega_\sigma(Px_n,Qx_n)[1+\omega_{2\sigma}(Px_n,Qx_{n+1})]}{1+\omega_\sigma(Px_n,Px_{n+1})+\omega_\sigma(Px_{n+1},Qx_{n+1})},$$

using (3) we get

$$\omega_\sigma(Qx_n,Qx_{n+1}) \leq \alpha_1(Px_n,Px_{n+1})\omega_\sigma(Px_n,Px_{n+1})$$
$$+ \alpha_2(Px_n,Px_{n+1})[\omega_\sigma(Px_n,Px_{n+1})+\omega_\sigma(Qx_n,Qx_{n+1})]$$
$$+ \alpha_3(Px_n,Px_{n+1})[\omega_{2\sigma}(Px_n,Px_{n+2})+\omega_\sigma(Qx_n,Qx_n)]$$
$$+ \alpha_4(Px_n,Px_{n+1})\frac{\omega_\sigma(Qx_n,Qx_{n+1})[1+\omega_\sigma(Px_n,Px_{n+1})]}{1+\omega_\sigma(Px_n,Px_{n+1})}$$
$$+ \alpha_5(Px_n,Px_{n+1})\frac{\omega_{2\sigma}(Px_n,Px_{n+2})\omega_\sigma(Px_n,Px_{n+1})}{[1+\omega_\sigma(Px_n,Px_{n+1})]}$$

$$+ \alpha_6(Px_n, Px_{n+1})\frac{\omega_\sigma(Qx_n, Qx_{n+1}) + \omega_\sigma(Px_{n+1}, Px_{n+1})}{1 + \omega_\sigma(Qx_n, Qx_{n+1})\omega_\sigma(Qx_n, Qx_n)}$$
$$+ \alpha_7(Px_n, Px_{n+1})\frac{\omega_\sigma(Px_n, Px_{n+1})[1 + \omega_{2\sigma}(Px_n, Px_{n+2})]}{1 + \omega_\sigma(Px_n, Px_{n+1}) + \omega_\sigma(Px_{n+1}, Px_{n+2})},$$

with the help of condition (i) of Theorem 1, we get

$$\begin{aligned}
\omega_\sigma(Qx_n, Qx_{n+1}) &\le \alpha_1(Px_0, Px_0)\omega_\sigma(Px_n, Px_{n+1}) \\
&\quad + \alpha_2(Px_0, Px_0)[\omega_\sigma(Px_n, Px_{n+1}) + \omega_\sigma(Qx_n, Qx_{n+1})] \\
&\quad + \alpha_3(Px_0, Px_0)[\omega_{2\sigma}(Px_n, Px_{n+2}) + \omega_\sigma(Qx_n, Qx_n)] \\
&\quad + \alpha_4(Px_0, Px_0)\frac{\omega_\sigma(Qx_n, Qx_{n+1})[1 + \omega_\sigma(Px_n, Px_{n+1})]}{1 + \omega_\sigma(Px_n, Px_{n+1})} \\
&\quad + \alpha_5(Px_0, Px_0)\frac{\omega_{2\sigma}(Px_n, Px_{n+2})\omega_\sigma(Px_n, Px_{n+1})}{[1 + \omega_\sigma(Px_n, Px_{n+1})]} \\
&\quad + \alpha_6(Px_0, Px_0)\frac{\omega_\sigma(Qx_n, Qx_{n+1}) + \omega_\sigma(Px_{n+1}, Px_{n+1})}{1 + \omega_\sigma(Qx_n, Qx_{n+1})\omega_\sigma(Qx_n, Qx_n)} \\
&\quad + \alpha_7(Px_0, Px_0)\frac{\omega_\sigma(Px_n, Px_{n+1})[1 + \omega_{2\sigma}(Px_n, Px_{n+2})]}{1 + \omega_\sigma(Px_n, Px_{n+1}) + \omega_\sigma(Px_{n+1}, Px_{n+2})} \\
&\le \alpha_1(Px_0, Px_0)\omega_\sigma(Px_n, Px_{n+1}) \\
&\quad + \alpha_2(Px_0, Px_0)[\omega_\sigma(Px_n, Px_{n+1}) + \omega_\sigma(Qx_n, Qx_{n+1})] \\
&\quad + \alpha_3(Px_0, Px_0)[\omega_{2\sigma}(Px_n, Px_{n+2}) + \omega_\sigma(Qx_n, Qx_n)] \\
&\quad + \alpha_4(Px_0, Px_0)\omega_\sigma(Qx_n, Qx_{n+1}) + \alpha_5(Px_0, Px_0)\omega_{2\sigma}(Px_n, Px_{n+2}) \\
&\quad + \alpha_6(Px_0, Px_0)\omega_\sigma(Qx_n, Qx_{n+1}) + \alpha_7(Px_0, Px_0)\omega_\sigma(Px_n, Px_{n+1}),
\end{aligned}$$

also we have

$$\begin{aligned}
\omega_{2\sigma}(Px_n, Px_{n+2}) &\le \omega_\sigma(Px_n, Px_{n+1}) + \omega_\sigma(Px_{n+1}, Px_{n+2}) \\
&= \omega_\sigma(Px_n, Px_{n+1}) + \omega_\sigma(Qx_n, Qx_{n+1}),
\end{aligned}$$

so we obtain

$$\begin{aligned}
\omega_\sigma(Qx_n, Qx_{n+1}) &\le \alpha_1(Px_0, Px_0)\omega_\sigma(Px_n, Px_{n+1}) \\
&\quad + \alpha_2(Px_0, Px_0)[\omega_\sigma(Px_n, Px_{n+1}) + \omega_\sigma(Qx_n, Qx_{n+1})] \\
&\quad + \alpha_3(Px_0, Px_0)[\omega_\sigma(Px_n, Px_{n+1}) + \omega_\sigma(Qx_n, Qx_{n+1})] \\
&\quad + \alpha_4(Px_0, Px_0)\omega_\sigma(Qx_n, Qx_{n+1}) \\
&\quad + \alpha_5(Px_0, Px_0)[\omega_\sigma(Px_n, Px_{n+1}) + \omega_\sigma(Qx_n, Qx_{n+1})] \\
&\quad + \alpha_6(Px_0, Px_0)\omega_\sigma(Qx_n, Qx_{n+1}) + \alpha_7(Px_0, Px_0)\omega_\sigma(Px_n, Px_{n+1}).
\end{aligned}$$

This implies that

$$\omega_\sigma(Qx_n, Qx_{n+1}) \le k\omega_\sigma(Px_n, Px_{n+1}) \text{ for all } n \in N,$$

where

$$k = \frac{\Sigma_{i=1}^{3}\alpha_i(Px_0,Px_0)+\alpha_5(Px_0,Px_0)+\alpha_7(Px_0,Px_0)}{1-\Sigma_{i=2}^{6}\alpha_i(Px_0,Px_0)} < 1,$$

so by induction we get

$$\omega_\sigma(Qx_n,Qx_{n+1}) \leq k^n\omega_\sigma(Qx_0,Qx_1) \text{ for all } n\in N. \tag{1}$$

From this we can easily imply that $\{Qx_n\}$ is a $\omega-$Cauchy sequence. Also $P(X_\omega)$ is $\omega-$complete, So there exists $u,v\in X_\omega$ such that $u=P(v)$ and $Qx_n\to u$ as $n\to\infty$. Since ω satisfy the Δ_2- condition on X we get $\lim_{n\to\infty}\omega_\sigma(Qx_n,u)=0$ for all $\sigma>0$ therefore

$$\lim_{n\to\infty}\omega_\sigma(Qx_n,u)=\lim_{n\to\infty}\omega_\sigma(Px_n,u)=0,\ \ \forall\ \sigma>0. \tag{2}$$

Now by taking $x=x_n$ and $y=v$ in condition (ii), of the theorem we obtain

$$\begin{aligned}\omega_\sigma(Qx_n,Qv)\ \leq\ &\alpha_1(Px_n,Pv)\omega_\sigma(Px_n,Pv)\\
&+\alpha_2(Px_n,Pv)[\omega_\sigma(Px_n,Qx_n)+\omega_\sigma(Pv,Qv)]\\
&+\alpha_3(Px_n,Pv)[\omega_{2\sigma}(Px_n,Qv)+\omega_\sigma(Pv,Qx_n)]\\
&+\alpha_4(Px_n,Pv)\frac{\omega_\sigma(Pv,Qv)[1+\omega_\sigma(Px_n,Qx_n)]}{1+\omega_\sigma(Px_n,Pv)}\\
&+\alpha_5(Px_n,Pv)\frac{\omega_{2\sigma}(Px_n,Qv)\omega_\sigma(Px_n,Qx_n)}{[1+\omega_\sigma(Px_n,Pv)]}\\
&+\alpha_6(Px_n,Pv)\frac{\omega_\sigma(Pv,Qv)+\omega_\sigma(Pv,Qx_n)}{1+\omega_\sigma(Pv,Qv)\omega_\sigma(Pv,Qx_n)}\\
&+\alpha_7(Px_n,Pv)\frac{\omega_\sigma(Px_n,Qx_n)[1+\omega_{2\sigma}(Px_n,Qv)]}{1+\omega_\sigma(Px_n,Pv)+\omega_\sigma(Pv,Qv)},\end{aligned}$$

by Remark 1 that the function $\sigma\to\omega_\sigma(x,y)$ is non-increasing and by condition (i) we have

$$\begin{aligned}&\omega_\sigma(Qx_n,Qv)\leq\alpha_1(Px_0,Pv)\omega_\sigma(Px_n,Pv)\\
&\quad+\alpha_2(Px_0,Pv)[\omega_\sigma(Px_n,Qx_n)+\omega_\sigma(Pv,Qv)]\\
&\quad+\alpha_3(Px_0,Pv)[\omega_\sigma(Px_n,Qx_n)+\omega_\sigma(Qx_n,Qv)+\omega_\sigma(Pv,Qx_n)]\\
&\quad+\alpha_4(Px_0,Pv)\frac{\omega_\sigma(Pv,Qv)[1+\omega_\sigma(Px_n,Qx_n)]}{1+\omega_\sigma(Px_n,Pv)}\\
&\quad+\alpha_5(Px_0,Pv)\frac{[\omega_\sigma(Px_n,Qx_n)+\omega_\sigma(Qx_n,Qv)]\omega_\sigma(Px_n,Qx_n)}{[1+\omega_\sigma(Px_n,Pv)]}\\
&\quad+\alpha_6(Px_0,Pv)\frac{\omega_\sigma(Pv,Qv)+\omega_\sigma(Pv,Qx_n)}{1+\omega_\sigma(Pv,Qv)\omega_\sigma(Pv,Qx_n)}\\
&\quad+\alpha_7(Px_0,Pv)\frac{\omega_\sigma(Px_n,Qx_n)[1+\omega_\sigma(Px_n,Qx_n)+\omega_\sigma(Qx_n,Qv)]}{1+\omega_\sigma(Px_n,Pv)+\omega_\sigma(Pv,Qv)}.\end{aligned}$$

By using (5) and letting $n \to \infty$ in the above inequality we get

$$\begin{aligned}\omega_\sigma(Pv, Qv) \ \le\ & \alpha_1(Px_0, Pv)\omega_\sigma(Pv, Pv)\\ &+ \alpha_2(Px_0, Pv)[\omega_\sigma(Pv, Qv) + \omega_\sigma(Pv, Qv)]\\ &+ \alpha_3(Px_0, Pv)[\omega_\sigma(Pv, Qv) + \omega_\sigma(Pv, Qv)]\\ &+ \alpha_4(Px_0, Pv)\omega_\sigma(Pv, Qv) + \alpha_5(Px_0, Pv)[\omega_\sigma(Pv, Qv)\\ &+ \alpha_6(Px_0, Pv)\omega_\sigma(Pv, Qv) + \alpha_7(Px_0, Pv)\omega_\sigma(Pv, Qv),\end{aligned}$$

so $[1 - \alpha_2(Px_0, Pv) - \alpha_3(Px_0, Pv) - \sum_{i=2}^{7} \alpha_i(Px_0, Pv)]\omega_\sigma(Pv, Qv) \le 0$, for all $\sigma > 0$ and hence

$$Pv = Qv = u.$$

Thus we have proved that P and Q have a coincident point. ■

Theorem 2. *In addition to the hypotheses of Theorem 1, suppose that P and Q are weakly compatible, then P and Q have a unique common fixed point. Further for any $x_o \in X_\omega$, the sequence $\{Qx_n\}$ with initial point x_0 modular converges to the common fixed point.*

Proof. Let P, Q are weakly compatible then

$$Pu = PQv = QPv = Qu,$$

we will show that $Qu = u = Qv$. Let suppose $\omega_\sigma(Qu, Qv) > 0$ for all $\sigma > 0$, by taking $x = u$ and $y = v$ in condition (ii) of the previous theorem we get

$$\begin{aligned}\omega_\sigma(Qu, Qv) \ \le\ & \alpha_1(Pu, Pv)\omega_\sigma(Pu, Pv)\\ &+ \alpha_2(Pu, Pv)[\omega_\sigma(Pu, Qu) + \omega_\sigma(Pv, Qv)]\\ &+ \alpha_3(Pu, Pv)[\omega_{2\sigma}(Pu, Qv) + \omega_\sigma(Pv, Qu)]\\ &+ \alpha_4(Pu, Pv)\frac{\omega_\sigma(Pv, Qv)[1 + \omega_\sigma(Pu, Qu)]}{1 + \omega_\sigma(Pu, Pv)}\\ &+ \alpha_5(Pu, Pv)\frac{\omega_{2\sigma}(Pu, Qv)\omega_\sigma(Pu, Qu)}{[1 + \omega_\sigma(Pu, Pv)]}\\ &+ \alpha_6(Pu, Pv)\frac{\omega_\sigma(Pv, Qv) + \omega_\sigma(Pv, Qu)}{1 + \omega_\sigma(Pv, Qv)\omega_\sigma(Pv, Qu)}\\ &+ \alpha_7(Pu, Pv)\frac{\omega_\sigma(Pu, Qu)[1 + \omega_{2\sigma}(Pu, Qv)]}{1 + \omega_\sigma(Pu, Pv) + \omega_\sigma(Pv, Qv)}\\ =\ & \alpha_1(Pu, Pv)\omega_\sigma(Qu, Qv)\\ &+ \alpha_3(Pu, Pv)[\omega_{2\sigma}(Qu, Qv) + \omega_\sigma(Qv, Qu)]\\ &+ \alpha_6(Pu, Pv)\omega_\sigma(Qv, Qu),\end{aligned}$$

for all $u, v \in X_\omega$ and using Remark 1 we get

$$\begin{aligned}\omega_\sigma(Qu, Qv) \ \leq\ & \alpha_1(Pu, Pv)\omega_\sigma(Qu, Qv)\\ & + \alpha_3(Pu, Pv)[\omega_\sigma(Qu, Qv) + \omega_\sigma(Qv, Qu)]\\ & + \alpha_6(Pu, Pv)\omega_\sigma(Qv, Qu),\end{aligned}$$

for all $\sigma > 0$ this implies that

$$[1 - \alpha_1(Pu, Pv) - 2\{\alpha_3(Pu, Pv)\} - \alpha_6(Pu, Pv)]\omega_\sigma(Qu, Qv) \leq 0,$$

which is a contradiction by assumption. Therefore $Pu = Qu = Qv = v$ and hence P,Q have common fixed point. Now to show the uniqueness of the common fixed point, suppose that u,z be two common fixed point, i.e.,

$$Qu = Pu = u \ \text{ and } \ Qz = Pz = z.$$

By letting $x = u$ and $y = z$ in (ii) we get

$$\begin{aligned}\omega_\sigma(Qu, Qz) \ \leq\ & \alpha_1(Pu, Pz)\omega_\sigma(Pu, Pz)\\ & + \alpha_2(Pu, Pz)[\omega_\sigma(Pu, Qu) + \omega_\sigma(Pz, Qz)]\\ & + \alpha_3(Pu, Pz)[\omega_{2\sigma}(Pu, Qz) + \omega_\sigma(Pz, Qu)]\\ & + \alpha_4(Pu, Pz)\frac{\omega_\sigma(Pz, Qz)[1 + \omega_\sigma(Pu, Qu)]}{1 + \omega_\sigma(Pu, Pz)}\\ & + \alpha_5(Pu, Pz)\frac{\omega_{2\sigma}(Pu, Qz)\omega_\sigma(Pu, Qu)}{[1 + \omega_\sigma(Pu, Pz)]}\\ & + \alpha_6(Pu, Pz)\frac{\omega_\sigma(Pz, Qz) + \omega_\sigma(Pz, Qu)}{1 + \omega_\sigma(Pz, Qz)\omega_\sigma(Pz, Qu)}\\ & + \alpha_7(Pu, Pz)\frac{\omega_\sigma(Pu, Qu)[1 + \omega_{2\sigma}(Pu, Qz)]}{1 + \omega_\sigma(Pu, Pz) + \omega_\sigma(Pz, Qz)},\end{aligned}$$

for all $\sigma > 0$ which implies that

$$[1 - \alpha_1(Pu, Pz) - 2\{\alpha_3(Pu, Pz)\} - \alpha_6(Pu, Pz)]\omega_\sigma(Qu, Qz) \leq 0,$$

for all $\sigma > 0$. Which is contradiction. Therefore $\omega_\sigma(u, z) = 0$ for all $\sigma > 0$ and so $u = z$. Clearly, for any $x_0 \in X_\omega$, the Q-P sequence $\{Qx_n\}$ with initial point x_0 converges to the unique common fixed point. ■

By setting $P = I_{X_\omega}$, we deduce the following result of fixed point for one self-mapping from Theorem 1

Corollary 1. *Let X_ω be a modular metric space and $Q : X_\omega \to X_\omega$. Suppose there exist mappings $\alpha_i : X \times X \to [0, 1)$ such that the following assertion for all $x, y \in X_\omega$ and $\sigma > 0$ hold.*

(i) $\alpha(x, Qy) \leq \alpha(x, y)$; *and* $\alpha(Qx, y) \leq \alpha(x, y)$;

$$\begin{aligned}(ii)\quad \omega_\sigma(Qx, Qy) \leq\; & \alpha_1(x, y)\omega_\sigma(x, y) \\ & + \alpha_2(x, y)[\omega_\sigma(x, Qx) + \omega_\sigma(y, Qy)] \\ & + \alpha_3(x, y)[\omega_{2\sigma}(x, Qy) + \omega_\sigma(y, Qx)] \\ & + \alpha_4(x, y)\frac{\omega_\sigma(y, Qy)[1 + \omega_\sigma(x, Qx)]}{1 + \omega_\sigma(x, y)} \\ & + \alpha_5(x, y)\frac{\omega_{2\sigma}(x, Qy)\omega_\sigma(x, Qx)}{[1 + \omega_\sigma(x, y)]} \\ & + \alpha_6(x, y)\frac{\omega_\sigma(y, Qy) + \omega_\sigma(y, Qx)}{1 + \omega_\sigma(y, Qy)\omega_\sigma(y, Qx)} \\ & + \alpha_7(x, y)\frac{\omega_\sigma(x, Qx)[1 + \omega_{2\sigma}(x, Qy)]}{1 + \omega_\sigma(x, y) + \omega_\sigma(y, Qy)}.\end{aligned}$$

(iii) $\Sigma_{i=1}^{3}\alpha_i(x, y) + \alpha_5(x, y) + \Sigma_{i=2}^{7}\alpha_i(x, y) < 1.$

(iv) $\omega_\sigma(x, Qy) < \infty$.

Then Q *has a unique fixed point. Further for any* $x_0 \in X_\omega$, *the Picard sequence* $\{Qx_n\}$ *with initial point* x_0 *modular converges to the fixed point.*

Corollary 2. *Let* X_ω *be a modular metric space and* $Q : X_\omega \to X_\omega$. *Suppose there exist mappings* $\alpha_2 : X \times X \to [0, \frac{1}{2})$ *such that the following assertion for all* $x, y \in X_\omega$ *and* $\sigma > 0$ *hold.*

I). $\omega_\sigma(Qx, Qy) \leq \alpha_2(x, y)[\omega_\sigma(x, Qx) + \omega_\sigma(y, Qy)]$;

II). $\omega_\sigma(x, Qy) < \infty$.

Then Q *has a unique fixed point*

Corollary 3. *Let* X_ω *be a modular metric space and* $Q : X_\omega \to X_\omega$. *Suppose there exist mappings* $\alpha_1 : X \times X \to [0, 1)$ *such that the following assertion for all* $x, y \in X_\omega$ *and* $\sigma > 0$ *hold.*

I). $\omega_\sigma(Qx, Qy) \leq \alpha_1(x, y)\omega_\sigma(x, y)$

II). $\omega_\sigma(x, Qy) < \infty$.

Then Q *has a unique fixed point.*

Example 2. Let $X_\omega = \{0, 1, 2\}$ and $\omega_\lambda = \frac{d(x,y)}{\lambda}$, with

$$d(x, y) = \begin{cases} x + y, & \text{if } x \neq y, \\ 0, & x = y. \end{cases}$$

Define $Q : X_\omega \to X_\omega$ by

$$Q(x) = \begin{cases} x - 1, & \text{if } x \neq 0, \\ 0, & x = y. \end{cases}$$

$\alpha : X_\omega \times X_\omega \longrightarrow [0,1)$ is defined as:

$$\alpha(x,y) = \frac{x+y}{21} + \frac{2}{3},$$

where $0 \leq \alpha(x,y) < 1$ for all $x, y \in X_\omega$.
Which shows that

$$\omega_\lambda(Qx, Qy) \leq \alpha(x,y)\omega_\lambda(x,y)$$

for all $x, y \in X_\omega$ and $\lambda > 0$. Thus all the conditions of Corollary 3 are hold and 0 is a unique fixed point of Q.

Remark 2. The Theorems 1 and 2 generalizes the results of Rahimpoor et al [17]. Moreover the Corollaries 2 and 3, are the generalization of the results of Mongkolkeha, Sintunavarat, and P. Kumam [11]

Here we extend the notion of generalized convex contraction mapping generalized convex contraction mapping of order-2, convex contraction mapping of type-2 and convex contraction of rational type and prove some fixed point results in the setting of modular metric spaces.

Lemma 1. *Let X_ω be a modular metric space and T is an asymptotically regular self mapping at a point $x \in X_\omega$ i.e.,$\omega_\sigma(T^n x, T^{n+1}x) \to 0$ as $n \to \infty$, for all $\sigma > 0$ then T has the approximate fixed point property.*

Proof. Let $x_0 \in X$, Then:

$$\begin{aligned}
&\omega_\sigma(T^n x_0, T^{n+1} x_0) \to 0 \text{ as } n \to \infty \Leftrightarrow \\
&\forall \epsilon > 0,\ \exists\, n_0(\epsilon) \in N \text{ such that } \forall n \geq n_0(\epsilon), \omega_\sigma(T^n x_0, T^{n+1}x_0) < \epsilon \Leftrightarrow \\
&\forall \epsilon > 0,\ \exists\, n_0(\epsilon) \in N \text{ such that } \forall\, n \geq n_0(\epsilon), \omega_\sigma(T^n x_0, T(T^n x_0)) < \epsilon.
\end{aligned}$$

Denoting

$$y_0 = T^n(x_0).$$

It follows that:

$$\forall\, \epsilon > 0,\ \exists\, y_0 \in X \text{ such that } \omega_\sigma(y_0, T(y_0)) < \epsilon.$$

So for each $\epsilon > 0$ there exist an $\epsilon-$fixed point of T in X namely y_0.
This means that T has approximate fixed point property. ■

Now, we are equipped to prove our main result:

Theorem 3. *Let X_ω be a modular metric space and $T : X \to X$ be a generalized convex contraction of rational type. Suppose that T is $\alpha-$admissible and there exists $x_0 \in X$ such that $\alpha(Tx_0, x_0) \geq 1$. Then T has an approximate fixed point. Further T has a fixed point if T is continuous and X_ω is a complete modular metric space. Moreover, if for all $x, y \in Fix(T)$ we have $\alpha(x,y) \geq 1$, then T has a unique fixed point in X.*

Proof. Let $x_0 \in X$ be such that $\alpha(Tx_0, x_0) \geq 1$. Now we define a sequence $\{x_n\}$ by $x_{n+1} = T^{n+1}x_0$, for all $n \geq 1$. If $x_n = x_{n+1}$, i.e., $T^n x_0 = T(T^n x_0)$ for some n then the the conclusion of the theorem follows immediately. Let we assume that $x_n \neq x_{n+1}$ for all $n \geq 0$. Since T is $\alpha-$*admissible*. Therefore we obtain $\alpha(T^{n+1}x_0, T^n x_0) \geq 1$, for all $n \geq 0$. Now we put $\upsilon = \max\{\omega_\sigma(x_0, Tx_0), \omega_\sigma(Tx_0, T^2x_0)\}$, $\mu = \sum_{i=1,2}(\alpha_i + \beta_i + \gamma_i) - \gamma_2$ and $\lambda = 1 - \gamma_2$. Now by taking $x = x_0$ and $y = Tx_0$, in (2) we have

$$\begin{aligned}
\omega_\sigma(T^2x_0, T^3x_0) &\leq \alpha(x_0, Tx_0)\omega_\sigma(T^2x_0, T^3x_0) \\
&\leq \alpha_1\omega_\sigma(x_0, Tx_0) + \alpha_2\omega_\sigma(Tx_0, T^2x_0) \\
&\quad + \beta_1\frac{\omega_\sigma(x_0, Tx_0)[1 + \omega_{2\sigma}(x_0, T^2x_0)]}{1 + \omega_\sigma(x_0, Tx_0) + \omega_\sigma(Tx_0, T^2x_0)} \\
&\quad + \beta_2\frac{\omega_\sigma(Tx_0, T^2x_0)\omega_\sigma(x_0, Tx_0)}{[1 + \omega_\sigma(x_0, Tx_0)]} \\
&\quad + \gamma_1\frac{\omega_\sigma(Tx_0, T^2x_0)[1 + \omega_\sigma(x_0, Tx_0)]}{1 + \omega_\sigma(x_0, Tx_0)} \\
&\quad + \gamma_2\frac{\omega_\sigma(T^2x_0, T^3x_0) + \omega_\sigma(Tx_0, Tx_0)}{1 + \omega_\sigma(Tx_0, T^2x_0)\omega_\sigma(Tx_0, Tx_0)} \\
&\leq \alpha_1\omega_\sigma(x_0, Tx_0) + \alpha_2\omega_\sigma(Tx_0, T^2x_0) \\
&\quad + \beta_1\frac{\omega_\sigma(x_0, Tx_0)[1 + \omega_\sigma(x_0, Tx_0) + \omega_\sigma(Tx_0, T^2x_0)]}{1 + \omega_\sigma(x_0, Tx_0) + \omega_\sigma(Tx_0, T^2x_0)} \\
&\quad + \beta_2\frac{\omega_\sigma(Tx_0, T^2x_0)\omega_\sigma(x_0, Tx_0)}{[1 + \omega_\sigma(x_0, Tx_0)]} \\
&\quad + \gamma_1\frac{\omega_\sigma(Tx_0, T^2x_0)[1 + \omega_\sigma(x_0, Tx_0)]}{1 + \omega_\sigma(x_0, Tx_0)} \\
&\quad + \gamma_2\frac{\omega_\sigma(T^2x_0, T^3x_0) + \omega_\sigma(Tx_0, Tx_0)}{1 + \omega_\sigma(Tx_0, T^2x_0)\omega_\sigma(Tx_0, Tx_0)} \\
&\leq \alpha_1\omega_\sigma(x_0, Tx_0) + \alpha_2\omega_\sigma(Tx_0, T^2x_0) \\
&\quad + \beta_1\omega_\sigma(x_0, Tx_0) + \beta_2\omega_\sigma(Tx_0, T^2x_0) \\
&\quad + \gamma_1\omega_\sigma(Tx_0, T^2x_0) + \gamma_2\omega_\sigma(T^2x_0, T^3x_0) \\
&= (\alpha_1 + \beta_1)\omega_\sigma(x_0, Tx_0) + (\alpha_2 + \beta_2 + \gamma_1)\omega_\sigma(Tx_0, T^2x_0) \\
&\quad + \gamma_2\omega_\sigma(T^2x_0, T^3x_0) \\
&\leq (\alpha_1 + \alpha_2 + \beta_1 + \beta_2 + \gamma_1)\upsilon + \gamma_2\omega_\sigma(T^2x_0, T^3x_0) \\
&= \mu\upsilon + \gamma_2\omega_\sigma(T^2x_0, T^3x_0).
\end{aligned}$$

Therefore we get $(1 - \gamma_2)\omega_\sigma(T^2x_0, T^3x_0) \leq \mu\upsilon$, that is $\omega_\sigma(T^2x_0, T^3x_0) \leq (\frac{\mu}{\lambda})\upsilon$, where $(\frac{\mu}{\lambda}) < 1$ as $\sum_{i=1,2}(\alpha_i + \beta_i + \gamma_i) < 1$.

Again by taking $x = Tx_0$ and $y = T^2x_0$, in (2) we obtain

$$\begin{aligned}
\omega_\sigma(T^3x_0, T^4x_0) &\leq \alpha(Tx_0, T^2x_0)\omega_\sigma(T^3x_0, T^4x_0 \\
&\leq \alpha_1\omega_\sigma(Tx_0, T^2x_0) + \alpha_2\omega_\sigma(T^2x_0, T^3x_0) \\
&\quad + \beta_1\frac{\omega_\sigma(Tx_0, T^2x_0)[1 + \omega_{2\sigma}(Tx_0, T^3x_0)]}{1 + \omega_\sigma(Tx_0, T^2x_0) + \omega_\sigma(T^2x_0, T^3x_0)} \\
&\quad + \beta_2\frac{\omega_\sigma(T^2x_0, T^3x_0)\omega_\sigma(Tx_0, T^2x_0)}{[1 + \omega_\sigma(Tx_0, T^2x_0]} \\
&\quad + \gamma_1\frac{\omega_\sigma(T^2x_0, T^3x_0)[1 + \omega_\sigma(Tx_0, T^2x_0)]}{1 + \omega_\sigma(Tx_0, T^2x_0)} \\
&\quad + \gamma_2\frac{\omega_\sigma(T^3x_0, T^4x_0) + \omega_\sigma(T^2x_0, T^2x_0)}{1 + \omega_\sigma(T^2x_0, T^3x_0)\omega_\sigma(T^2x_0, T^2x_0)} \\
&\leq \alpha_1\omega_\sigma(Tx_0, T^2x_0) + \alpha_2\omega_\sigma(T^2x_0, T^3x_0) \\
&\quad + \beta_1\frac{\omega_\sigma(Tx_0, T^2x_0)[1 + \omega_\sigma(Tx_0, T^2x_0) + \omega_\sigma(T^2x_0, T^3x_0)]}{1 + \omega_\sigma(Tx_0, T^2x_0) + \omega_\sigma(T^2x_0, T^3x_0)} \\
&\quad + \beta_2\frac{\omega_\sigma(T^2x_0, T^3x_0)\omega_\sigma(Tx_0, T^2x_0)}{[1 + \omega_\sigma(Tx_0, T^2x_0]} \\
&+ \gamma_1\frac{\omega_\sigma(T^2x_0, T^3x_0)[1 + \omega_\sigma(Tx_0, T^2x_0)]}{1 + \omega_\sigma(Tx_0, T^2)} \\
&\quad + \gamma_2\frac{\omega_\sigma(T^3x_0, T^4x_0) + \omega_\sigma(T^2x_0, T^2x_0)}{1 + \omega_\sigma(T^2x_0, T^3x_0)\omega_\sigma(T^2x_0, T^2x_0)} \\
&\leq \alpha_1\omega_\sigma(Tx_0, T^2x_0) + \alpha_2\omega_\sigma(T^2x_0, T^3x_0) \\
&\quad + \beta_1\omega_\sigma(Tx_0, T^2x_0) + \beta_2\omega_\sigma(T^2x_0, T^3x_0) \\
&\quad + \gamma_1\omega_\sigma(T^2x_0, T^3x_0) + \gamma_2\omega_\sigma(T^3x_0, T^4x_0) \\
&= (\alpha_1 + \beta_1)\omega_\sigma(Tx_0, T^2x_0) + (\alpha_2 + \beta_2 + \gamma_1)\omega_\sigma(T^2x_0, T^3x_0) \\
&\quad + \gamma_2\omega_\sigma(T^3x_0, T^4x_0) \\
&\leq (\alpha_1 + \beta_1)\upsilon + (\alpha_2 + \beta_2 + \gamma_1)(\frac{\mu}{\lambda})\upsilon + \gamma_2\omega_\sigma(T^3x_0, T^4x_0) \\
&\leq (\alpha_1 + \alpha_2 + \beta_1 + \beta_2 + \gamma_1)\upsilon + \gamma_2\omega_\sigma(T^3x_0, T^4x_0).
\end{aligned}$$

Therefore $\omega_\sigma(T^3x_0, T^4x_0) \leq (\frac{\mu}{\lambda})\upsilon$ and

$$\begin{aligned}
\omega_\sigma(T^4x_0, T^5x_0) &\leq \alpha(T^2x_0, T^3x_0)\omega_\sigma(T^4x_0, T^5x_0) \\
&\leq \alpha_1\omega_\sigma(T^2x_0, T^3x_0) + \alpha_2\omega_\sigma(T^3x_0, T^4x_0) \\
&\quad + \beta_1\frac{\omega_\sigma(T^2x_0, T^3x_0)[1 + \omega_{2\sigma}(T^2x_0, T^4x_0)]}{1 + \omega_\sigma(T^2x_0, T^3x_0 + \omega_\sigma(T^3x_0, T^4x_0)} \\
&\quad + \beta_2\frac{\omega_\sigma(T^3x_0, T^4x_0)\omega_\sigma(T^2x_0, T^3x_0)}{[1 + \omega_\sigma(T^2x_0, T^3x_0)]}
\end{aligned}$$

$$
\begin{aligned}
&+ \gamma_1 \frac{\omega_\sigma(T^3x_0, T^4x_0)[1 + \omega_\sigma(T^2x_0, T^3x_0)]}{1 + \omega_\sigma(T^2x_0, T^3x_0)} \\
&+ \gamma_2 \frac{\omega_\sigma(T^4x_0, T^5x_0) + \omega_\sigma(T^3x_0, T^3x_0)}{1 + \omega_\sigma(T^3x_0, T^4x_0)\omega_\sigma(T^3x_0, T^3x_0)} \\
&\leq \alpha_1\omega_\sigma(T^2x_0, T^3x_0) + \alpha_2\omega_\sigma(T^3x_0, T^4x_0) \\
&+ \beta_1 \frac{\omega_\sigma(T^2x_0, T^3x_0)[1 + \omega_\sigma(T^2x_0, T^3x_0 + \omega_\sigma(T^3x_0, T^4x_0)]}{1 + \omega_\sigma(T^2x_0, T^3x_0 + \omega_\sigma(T^3x_0, T^4x_0)} \\
&+ \beta_2 \frac{\omega_\sigma(T^3x_0, T^4x_0)\omega_\sigma(T^2x_0, T^3x_0)}{[1 + \omega_\sigma(T^2x_0, T^3x_0)]} \\
&+ \gamma_1 \frac{\omega_\sigma(T^3x_0, T^4x_0)[1 + \omega_\sigma(T^2x_0, T^3x_0)]}{1 + \omega_\sigma(T^2x_0, T^3x_0)} \\
&+ \gamma_2 \frac{\omega_\sigma(T^4x_0, T^5x_0) + \omega_\sigma(T^3x_0, T^3x_0)}{1 + \omega_\sigma(T^3x_0, T^4x_0)\omega_\sigma(T^3x_0, T^3x_0)} \\
&\leq \alpha_1\omega_\sigma(T^2x_0, T^3x_0) + \alpha_2\omega_\sigma(T^3x_0, T^4x_0) \\
&+ \beta_1\omega_\sigma(T^2x_0, T^3x_0) + \beta_2\omega_\sigma(T^3x_0, T^4x_0) \\
&+ \gamma_1\omega_\sigma(T^3x_0, T^4x_0) + \gamma_2\omega_\sigma(T^4x_0, T^5x_0) \\
&\leq (\alpha_1 + \beta_1)(\frac{\mu}{\lambda})\upsilon + (\alpha_2 + \beta_2 + \gamma_1)(\frac{\mu}{\lambda})\upsilon \\
&+ \gamma_2\omega_\sigma(T^4x_0, T^5x_0) \\
&= (\alpha_1 + \alpha_2 + \beta_1 + \beta_2 + \gamma_1)(\frac{\mu}{\lambda})\upsilon + \gamma_2\omega_\sigma(T^4x_0, T^5x_0).
\end{aligned}
$$

It follows

$$
\begin{aligned}
\omega_\sigma(T^4x_0, T^5x_0) &\leq \frac{(\alpha_1 + \alpha_2 + \beta_1 + \beta_2 + \gamma_1)}{(1 - \gamma_2)}(\frac{\mu}{\lambda})\upsilon \\
&= (\frac{\mu}{\lambda})^2\upsilon.
\end{aligned}
$$

Also we have

$$
\begin{aligned}
\omega_\sigma(T^5x_0, T^6x_0) &\leq \alpha(T^3x_0, T^4x_0)\omega_\sigma(T^5x_0, T^6x_0) \\
&\leq \alpha_1\omega_\sigma(T^3x_0, T^4x_0) + \alpha_2\omega_\sigma(T^4x_0, T^5x_0) \\
&+ \beta_1 \frac{\omega_\sigma(T^3x_0, T^4x_0)[1 + \omega_{2\sigma}(T^3x_0, T^5x_0)]}{1 + \omega_\sigma(T^3x_0, T^4x_0) + \omega_\sigma(T^4x_0, T^5x_0)} \\
&+ \beta_2 \frac{\omega_\sigma(T^4x_0, T^5x_0)\omega_\sigma(T^3x_0, T^4x_0)}{[1 + \omega_\sigma(T^3x_0, T^4x_0)]} \\
&+ \gamma_1 \frac{\omega_\sigma(T^4x_0, T^5x_0)[1 + \omega_\sigma(T^3x_0, T^4x_0)]}{1 + \omega_\sigma(T^3x_0, T^4x_0)} \\
&+ \gamma_2 \frac{\omega_\sigma(T^5x_0, T^6x_0) + \omega_\sigma(T^4x_0, T^4x_0)}{1 + \omega_\sigma(T^4x_0, T^5x_0)\omega_\sigma(T^4x_0, T^4x_0)}
\end{aligned}
$$

$$
\begin{aligned}
&\le \alpha_1\omega_\sigma(T^3x_0,T^4x_0)+\alpha_2\omega_\sigma(T^4x_0,T^5x_0)\\
&\quad+\beta_1\frac{\omega_\sigma(T^3x_0,T^4x_0)[1+\omega_\sigma(T^3x_0,T^4x_0)+\omega_\sigma(T^4x_0,T^5x_0)]}{1+\omega_\sigma(T^3x_0,T^4x_0)+\omega_\sigma(T^4x_0,T^5x_0)}\\
&\quad+\beta_2\frac{\omega_\sigma(T^4x_0,T^5x_0)\omega_\sigma(T^3x_0,T^4x_0)}{[1+\omega_\sigma(T^3x_0,T^4x_0)]}\\
&\quad+\ +\gamma_1\frac{\omega_\sigma(T^4x_0,T^5x_0)[1+\omega_\sigma(T^3x_0,T^4x_0)]}{1+\omega_\sigma(T^3x_0,T^4x_0)}\\
&\quad+\gamma_2\frac{\omega_\sigma(T^5x_0,T^6x_0)+\omega_\sigma(T^4x_0,T^4x_0)}{1+\omega_\sigma(T^4x_0,T^5x_0)\omega_\sigma(T^4x_0,T^4x_0)}\\
&\le \alpha_1\omega_\sigma(T^3x_0,T^4x_0)+\alpha_2\omega_\sigma(T^4x_0,T^5x_0)\\
&\quad+\beta_1\omega_\sigma(T^3x_0,T^4x_0)+\beta_2\omega_\sigma(T^4x_0,T^5x_0)\\
&\quad+\gamma_1\omega_\sigma(T^4x_0,T^5x_0)+\gamma_2\omega_\sigma(T^5x_0,T^6x_0)\\
&\quad+\ \le(\alpha_1+\beta_1)\omega_\sigma(T^3x_0,T^4x_0)+(\alpha_2\\
&\quad+\beta_2+\gamma_1)\omega_\sigma(T^4x_0,T^5x_0)+\gamma_2\omega_\sigma(T^5x_0,T^6x_0)\\
&\le(\alpha_1+\beta_1)(\frac{\mu}{\lambda})\upsilon+(\alpha_2+\beta_2+\gamma_1)(\frac{\mu}{\lambda})\upsilon+\gamma_2\omega_\sigma(T^5x_0,T^6x_0),
\end{aligned}
$$

which follows

$$
\begin{aligned}
\omega_\sigma(T^5x_0,T^6x_0)\ &\le\ \frac{(\alpha_1+\alpha_2+\beta_1+\beta_2+\gamma_1)}{(1-\gamma_2)}(\frac{\mu}{\lambda})\upsilon\\
&=(\frac{\mu}{\lambda})^2\upsilon.
\end{aligned}
$$

By continuing this process and following the similar argument as in [19], we get $\omega_\sigma(T^mx_0,T^{m+1}x_0)\le(\frac{\mu}{\lambda})^l\upsilon$, whenever $m=2l$ or $m=2l+1$, for $l\ge1$ or $\omega_\sigma(T^mx_0,T^{m+1}x_0)\le(\frac{\mu}{\lambda})^{l-1}\upsilon$ whenever $m=2l$ or $m=2l-1$, for $l\ge2$. Therefore, $\omega_\sigma(T^mx_0,T^{m+1}x_0)\to0$ as $m\to\infty$,i.e., T is asymptotically regular at x_0. Thus by lemma 1, we can say that T has an approximate fixed point.

Now suppose that T is continuous X_ω is a complete modular metric space. In order to show that $\{x_n\}$ is a Cauchy in X, we choose m,n as non-zero positive integers such that $m<n$ with the following cases.

Case (i). For $m=2l$ with $l,q\ge1$, then

$$
\begin{aligned}
\omega_\sigma(T^mx_0,T^{m+q}x_0)\ &=\ \omega_\sigma(T^{2l}x_0,T^{2l+q}x_0)\\
&\le\ \omega_{\frac{\sigma}{q}}(T^{2l}x_0,T^{2l+1}x_0)+\omega_{\frac{\sigma}{q}}(T^{2l+1}x_0,T^{2l+2}x_0)\\
&\qquad+\omega_{\frac{\sigma}{q}}(T^{2l+2}x_0,T^{2l+3}x_0)+\omega_{\frac{\sigma}{q}}(T^{2l+3}x_0,T^{2l+4}x_0)+...\\
&\le\ (\frac{\mu}{\lambda})^l\upsilon+(\frac{\mu}{\lambda})^l\upsilon+(\frac{\mu}{\lambda})^{l+1}\upsilon+(\frac{\mu}{\lambda})^{l+1}\upsilon+....
\end{aligned}
$$

$$
\begin{aligned}
&= 2(\frac{\mu}{\lambda})^l \upsilon + (\frac{\mu}{\lambda})^{l+1} \upsilon + (\frac{\mu}{\lambda})^{l+2} \upsilon + ... \\
&= 2(\frac{\mu}{\lambda})^l \{1 + (\frac{\mu}{\lambda})^l + (\frac{\mu}{\lambda})^{l+1} + ...\} \upsilon \\
&= 2(\frac{\mu}{\lambda})^l \frac{1}{1-\frac{\mu}{\lambda}} \upsilon.
\end{aligned}
$$

Similarly,

Case (ii). For $m = 2l + 1$ with $l, q \geq 1$, we have

$$
\begin{aligned}
\omega_\sigma(T^m x_0, T^{m+q} x_0) &= \omega_\sigma(T^{2l+1} x_0, T^{2l+q+1} x_0) \\
&\leq \omega_{\frac{\sigma}{q}}(T^{2l+1} x_0, T^{2l+2} x_0) + \omega_{\frac{\sigma}{q}}(T^{2l+2} x_0, T^{2l+3} x_0) \\
&\quad + \omega_{\frac{\sigma}{q}}(T^{2l+3} x_0, T^{2l+4} x_0) + \omega_{\frac{\sigma}{q}}(T^{2l+4} x_0, T^{2l+5} x_0) + ... \\
&\leq (\frac{\mu}{\lambda})^l \upsilon + (\frac{\mu}{\lambda})^l \upsilon + (\frac{\mu}{\lambda})^{l+1} \upsilon + (\frac{\mu}{\lambda})^{l+1} \upsilon + ... \\
&= 2(\frac{\mu}{\lambda})^l \upsilon + (\frac{\mu}{\lambda})^{l+1} \upsilon + (\frac{\mu}{\lambda})^{l+2} \upsilon + ... \\
&= 2(\frac{\mu}{\lambda})^l \{1 + (\frac{\mu}{\lambda})^l + (\frac{\mu}{\lambda})^{l+1} + ...\} \upsilon \\
&= 2(\frac{\mu}{\lambda})^l \frac{1}{1-\frac{\mu}{\lambda}} \upsilon.
\end{aligned}
$$

Taking $l \to \infty$ in all cases since $(\frac{\mu}{\lambda}) < 1$, therefore we obtain $\omega_\sigma(T^m x_0, T^n x_0) \to 0$. Therefore $\{x_n\}$ is a Cauchy sequence in X. Since X is complete, therefore there exist a point $z \in X$ such that $x_n = T^n x_0 \to z \in X$ as $n \to \infty$. And by the continuity of T, we have $z = lim_{n\to\infty} T(T^n x_0) = Tz$. This shows that z is a fixed point of T.

Now, we will show that T has a unique fixed point in X. For that we assume that $z, z^* \in Fix(T)$ such that $z \neq z^*$. By hypothesis $\alpha(z, z^*) \geq 1$ and from (2) taking $x = z$ and $y = z^*$

$$
\begin{aligned}
\omega_\sigma(z, z^*) &= \omega_\sigma(T^2 z, T^2 z^*) \\
&\leq \alpha(z, z^*) \omega_\sigma(T^2 z, T^2 z^*) \\
&\leq \alpha_1 \omega_\sigma(z, z^*) + \alpha_2 \omega_\sigma(Tz, Tz^*) + \beta_1 \frac{\omega_\sigma(z, Tz)[1 + \omega_{2\sigma}(z, Tz^*)]}{1 + \omega_\sigma(z, z^*) + \omega_\sigma(z^*, Tz^*)} \\
&\quad + \beta_2 \frac{\omega_\sigma(Tz, T^2 z) \omega_\sigma(z, Tz)}{[1 + \omega_\sigma(z, z^*)]} + \gamma_1 \frac{\omega_\sigma(z^*, Tz^*)[1 + \omega_\sigma(z, Tz)]}{1 + \omega_\sigma(z, z^*)} \\
&\quad + \gamma_2 \frac{\omega_\sigma(Tz^*, T^2 z^*) + \omega_\sigma(z^*, Tz)}{1 + \omega_\sigma(z^*, Tz^*) \omega_\sigma(z^*, Tz)} \\
&\leq \alpha_1 \omega_\sigma(z, z^*) + \alpha_2 \omega_\sigma(Tz, Tz^*) + \beta_1 \omega_\sigma(z, Tz) \\
&\quad + \beta_2 \omega_\sigma(Tz, T^2 z) + \gamma_1 \omega_\sigma(z^*, Tz^*) + \gamma_2 \omega_\sigma(Tz^*, T^2 z^*) \\
&\leq (\alpha_1 + \alpha_2) \omega_\sigma(z, z^*).
\end{aligned}
$$

Which follows that $(1-\alpha_1-\alpha_2)\omega_\sigma(z,z^*) \leq 0$, which in turn gives $\omega_\sigma(z,z^*) = 0$, a contradiction. Thus T has a unique fixed point in X. ■

Corollary 4. *Let X_ω be a modular metric space and $T : X \to X$ be a generalized convex contraction. Suppose that T is $\alpha-$admissible and there exists $x_0 \in X$ such that $\alpha(Tx_0, x_0) \geq 1$. Then T has an approximate fixed point. Further T has a fixed point if T is continuous and X_ω is a complete modular metric space. Moreover, if for all $x, y \in Fix(T)$ we have $\alpha(x,y) \geq 1$, then T has a unique fixed point in X.*

Proof. Since $T : X \to X$ is a generalized convex contraction, so we obtain

$$\begin{aligned}
\alpha(x,y)\omega_\sigma(T^2x, T^2y) &\leq \alpha_1\omega_\sigma(x,y) + \alpha_2\omega_\sigma(Tx,Ty) \\
&\leq \alpha_1\omega_\sigma(x,y) + \alpha_2\omega_\sigma(Tx,Ty) \\
&\quad + \beta_1 \frac{\omega_\sigma(x,Tx)[1+\omega_{2\sigma}(x,Ty)]}{1+\omega_\sigma(x,y)+\omega_\sigma(y,Ty)} \\
&\quad + \beta_2 \frac{\omega_\sigma(Tx,T^2x)\omega_\sigma(x,Tx)}{[1+\omega_\sigma(x,y)]} \\
&\quad + \gamma_1 \frac{\omega_\sigma(y,Ty)[1+\omega_\sigma(x,Tx)]}{1+\omega_\sigma(x,y)} \\
&\quad + \gamma_2 \frac{\omega_\sigma(Ty,T^2y)+\omega_\sigma(y,Tx)}{1+\omega_\sigma(y,Ty)\omega_\sigma(y,Tx)}.
\end{aligned}$$

Which shows that T is a generalized convex contraction of rational type. Thus by Theorem3 the conclusion follows. ■

Theorem 4. *Let X_ω be a modular metric space and $T : X \to X$ be a generalized convex contraction of type-2. Suppose that T is $\alpha-$admissible and there exists $x_0 \in X$ such that $\alpha(Tx_0, x_0) \geq 1$. Then T has an approximate fixed point. Further T has a fixed point if T is continuous and X_ω is a complete modular metric space. Moreover, if for all $x, y \in Fix(T)$ we have $\alpha(x,y) \geq 1$, then T has a unique fixed point in X.*

Proof. Let $x_0 \in X$ be such that $\alpha(Tx_0, x_0) \geq 1$. Now we define a sequence $\{x_n\}$ by $x_{n+1} = T^{n+1}x_0$, for all $n \geq 1$. If $x_n = x_{n+1}$, i.e., $T^n x_0 = T(T^n x_0)$ for some n then the the conclusion of the theorem follows immediately. Let we assume that $x_n \neq x_{n+1}$ for all $n \geq 0$. Since T is α-admissible. Therefore we obtain $\alpha(T^{n+1}x_0, T^n x_0) \geq 1$, for all $n \geq 0$.

Now we put $\upsilon = \max\{\omega_\sigma(x_0, Tx_0), \omega_\sigma(Tx_0, T^2x_0)\}$, $\mu = \sum_{i=1,2}(\alpha_i + \beta_i + \gamma_i) - \gamma_2$ and $\lambda = 1 - \gamma_2$.

Now by taking $x = x_0$ and $y = Tx_0$, in (1) we have

$$\begin{aligned}\omega_\sigma(T^2x_0, T^3x_0) &\leq \alpha(x_0, Tx_0)\omega_\sigma(T^2x_0, T^3x_0)\\ &\leq \alpha_1\omega_\sigma(x_0, Tx_0) + \alpha_2\omega_\sigma(Tx_0, T^2x_0)\\ &\quad + \beta_1\omega_\sigma(x_0, Tx_0) + \beta_2\omega_\sigma(Tx_0, T^2x_0)\\ &\quad + \gamma_1\omega_\sigma(Tx_0, T^2x_0) + \gamma_2\omega_\sigma(T^2x_0, T^3x_0)\\ &= (\alpha_1 + \beta_1)\omega_\sigma(x_0, Tx_0)\\ &\quad + (\alpha_2 + \beta_2 + \gamma_1)\omega_\sigma(Tx_0, T^2x_0) + \gamma_2\omega_\sigma(T^2x_0, T^3x_0)\\ &\leq (\alpha_1 + \alpha_2 + \beta_1 + \beta_2 + \gamma_1)\upsilon + \gamma_2\omega_\sigma(T^2x_0, T^3x_0)\\ &= \mu\upsilon + \gamma_2\omega_\sigma(T^2x_0, T^3x_0).\end{aligned}$$

Therefore we get $(1 - \gamma_2)\omega_\sigma(T^2x_0, T^3x_0) \leq \mu\upsilon$, that is $\omega_\sigma(T^2x_0, T^3x_0) \leq (\frac{\mu}{\lambda})\upsilon$, where $(\frac{\mu}{\lambda}) < 1$ as $\sum_{i=1,2}(\alpha_i + \beta_i + \gamma_i) < 1$.

Again by taking $x = Tx_0$ and $y = T^2x_0$, in (1) we obtain

$$\begin{aligned}\omega_\sigma(T^3x_0, T^4x_0) &\leq \alpha(Tx_0, T^2x_0)\omega_\sigma(T^3x_0, T^4x_0)\\ &\leq \alpha_1\omega_\sigma(Tx_0, T^2x_0) + \alpha_2\omega_\sigma(T^2x_0, T^3x_0)\\ &\quad + \beta_1\omega_\sigma(Tx_0, T^2x_0) + \beta_2\omega_\sigma(T^2x_0, T^3x_0)\\ &\quad + \gamma_1\omega_\sigma(T^2x_0, T^3x_0) + \gamma_2\omega_\sigma(T^3x_0, T^4x_0)\\ &= (\alpha_1 + \beta_1)\omega_\sigma(Tx_0, T^2x_0)\\ &\quad + (\alpha_2 + \beta_2 + \gamma_1)\omega_\sigma(T^2x_0, T^3x_0) + \gamma_2\omega_\sigma(T^3x_0, T^4x_0)\\ &\leq (\alpha_1 + \beta_1)\upsilon + (\alpha_2 + \beta_2 + \gamma_1)(\frac{\mu}{\lambda})\upsilon + \gamma_2\omega_\sigma(T^3x_0, T^4x_0)\\ &\leq (\alpha_1 + \alpha_2 + \beta_1 + \beta_2 + \gamma_1)\upsilon + \gamma_2\omega_\sigma(T^3x_0, T^4x_0).\end{aligned}$$

Therefore $\omega_\sigma(T^3x_0, T^4x_0) \leq (\frac{\mu}{\lambda})\upsilon$ and

$$\begin{aligned}\omega_\sigma(T^4x_0, T^5x_0) &\leq \alpha(T^2x_0, T^3x_0)\omega_\sigma(T^4x_0, T^5x_0)\\ &\leq \alpha_1\omega_\sigma(T^2x_0, T^3x_0) + \alpha_2\omega_\sigma(T^3x_0, T^4x_0)\\ &\quad + \beta_1\omega_\sigma(T^2x_0, T^3x_0) + \beta_2\omega_\sigma(T^3x_0, T^4x_0)\\ &\quad + \gamma_1\omega_\sigma(T^3x_0, T^4x_0) + \gamma_2\omega_\sigma(T^4x_0, T^5x_0)\\ &\leq (\alpha_1 + \beta_1)(\frac{\mu}{\lambda})\upsilon + (\alpha_2 + \beta_2 + \gamma_1)(\frac{\mu}{\lambda})\upsilon\\ &\quad + \gamma_2\omega_\sigma(T^4x_0, T^5x_0)\\ &= (\alpha_1 + \alpha_2 + \beta_1 + \beta_2 + \gamma_1)(\frac{\mu}{\lambda})\upsilon + \gamma_2\omega_\sigma(T^4x_0, T^5x_0).\end{aligned}$$

It follows

$$\begin{aligned}\omega_\sigma(T^4x_0, T^5x_0) &\leq \frac{(\alpha_1 + \alpha_2 + \beta_1 + \beta_2 + \gamma_1)}{(1 - \gamma_2)}(\frac{\mu}{\lambda})\upsilon\\ &= (\frac{\mu}{\lambda})^2\upsilon.\end{aligned}$$

Also we have

$$\begin{aligned}\omega_\sigma(T^5x_0,T^6x_0) &\le \alpha(T^3x_0,T^4x_0)\omega_\sigma(T^5x_0,T^6x_0)\\ &\le \alpha_1\omega_\sigma(T^3x_0,T^4x_0)+\alpha_2\omega_\sigma(T^4x_0,T^5x_0)\\ &\quad+\beta_1\omega_\sigma(T^3x_0,T^4x_0)+\beta_2\omega_\sigma(T^4x_0,T^5x_0)\\ &\quad+\gamma_1\omega_\sigma(T^4x_0,T^5x_0)+\gamma_2\omega_\sigma(T^5x_0,T^6x_0)\\ &\le (\alpha_1+\beta_1)\omega_\sigma(T^3x_0,T^4x_0)+(\alpha_2+\beta_2+\gamma_1)\omega_\sigma(T^4x_0,T^5x_0)\\ &\quad+\gamma_2\omega_\sigma(T^5x_0,T^6x_0)\\ &\le (\alpha_1+\beta_1)(\frac{\mu}{\lambda})\upsilon+(\alpha_2+\beta_2+\gamma_1)(\frac{\mu}{\lambda})\upsilon+\gamma_2\omega_\sigma(T^5x_0,T^6x_0),\end{aligned}$$

which follows

$$\begin{aligned}\omega_\sigma(T^5x_0,T^6x_0) &\le \frac{(\alpha_1+\alpha_2+\beta_1+\beta_2+\gamma_1)}{(1-\gamma_2)}(\frac{\mu}{\lambda})\upsilon\\ &= (\frac{\mu}{\lambda})^2\upsilon.\end{aligned}$$

By continuing this process and following the similar argument as in [19], we get $\omega_\sigma(T^mx_0,T^{m+1}x_0)\le(\frac{\mu}{\lambda})^l\upsilon$, whenever $m=2l$ or $m=2l+1$, for $l\ge1$ or $\omega_\sigma(T^mx_0,T^{m+1}x_0)\le(\frac{\mu}{\lambda})^{l-1}\upsilon$ whenever $m=2l$ or $m=2l-1$, for $l\ge2$. Therefore, $\omega_\sigma(T^mx_0,T^{m+1}x_0)\to0$ as $m\to\infty$, i.e., T is asymptotically regular at x_0. Thus by lemma 1, we can say that T has an approximate fixed point.

Now suppose that T is continuous X_ω is a complete modular metric space. In order to show that $\{x_n\}$ is a cauchy in X, we choose m,n as non-zero positive integers such that $m<n$ with the following cases.

Case (i). For $m=2l$ with $l,q\ge1$, then

$$\begin{aligned}\omega_\sigma(T^mx_0,T^{m+q}x_0) &= \omega_\sigma(T^{2l}x_0,T^{2l+q}x_0)\\ &\le \omega_{\frac{\sigma}{q}}(T^{2l}x_0,T^{2l+1}x_0)+\omega_{\frac{\sigma}{q}}(T^{2l+1}x_0,T^{2l+2}x_0)\\ &\quad+\omega_{\frac{\sigma}{q}}(T^{2l+2}x_0,T^{2l+3}x_0)+\omega_{\frac{\sigma}{q}}(T^{2l+3}x_0,T^{2l+4}x_0)+...\\ &\le (\frac{\mu}{\lambda})^l\upsilon+(\frac{\mu}{\lambda})^l\upsilon+(\frac{\mu}{\lambda})^{l+1}\upsilon+(\frac{\mu}{\lambda})^{l+1}\upsilon+....\\ &= 2(\frac{\mu}{\lambda})^l\upsilon+(\frac{\mu}{\lambda})^{l+1}\upsilon+(\frac{\mu}{\lambda})^{l+2}\upsilon+...\\ &= 2(\frac{\mu}{\lambda})^l\{1+(\frac{\mu}{\lambda})^l+(\frac{\mu}{\lambda})^{l+1}+...\}\upsilon\\ &= 2(\frac{\mu}{\lambda})^l\frac{1}{1-\frac{\mu}{\lambda}}\upsilon.\end{aligned}$$

Similarly,

Case (ii). For $m = 2l+1$ with $l, q \geq 1$, we have

$$\begin{aligned}
\omega_\sigma(T^m x_0, T^{m+q}x_0) &= \omega_\sigma(T^{2l+1}x_0, T^{2l+q+1}x_0) \\
&\leq \omega_{\frac{\sigma}{q}}(T^{2l+1}x_0, T^{2l+2}x_0) + \omega_{\frac{\sigma}{q}}(T^{2l+2}x_0, T^{2l+3}x_0) \\
&\quad + \omega_{\frac{\sigma}{q}}(T^{2l+3}x_0, T^{2l+4}x_0) + \omega_{\frac{\sigma}{q}}(T^{2l+4}x_0, T^{2l+5}x_0) + ... \\
&\leq (\frac{\mu}{\lambda})^l \upsilon + (\frac{\mu}{\lambda})^l \upsilon + (\frac{\mu}{\lambda})^{l+1} \upsilon + (\frac{\mu}{\lambda})^{l+1} \upsilon + ... \\
&= 2(\frac{\mu}{\lambda})^l \upsilon + (\frac{\mu}{\lambda})^{l+1} \upsilon + (\frac{\mu}{\lambda})^{l+2} \upsilon + ... \\
&= 2(\frac{\mu}{\lambda})^l \{1 + (\frac{\mu}{\lambda})^l + (\frac{\mu}{\lambda})^{l+1} + ...\} \upsilon \\
&= 2(\frac{\mu}{\lambda})^l \frac{1}{1-\frac{\mu}{\lambda}} \upsilon.
\end{aligned}$$

Taking $l \to \infty$ in all cases since $(\frac{\mu}{\lambda}) < 1$, therefore we obtain $\omega_\sigma(T^m x_0, T^n x_0) \to 0$. Therefore $\{x_n\}$ is a Cauchy sequence in X. Since X is complete, therefore there exist a point $z \in X$ such that $x_n = T^n x_0 \to z \in X$ as $n \to \infty$. And by the continuity of T, we have $z = lim_{n\to\infty} T(T^n x_0) = Tz$. This shows that z is a fixed point of T.

Now, we will show that T has a unique fixed point in X. For that we assume that $z, z^* \in Fix(T)$ such that $z \neq z^*$. By hypothesis $\alpha(z, z^*) \geq 1$ and from (1) taking $x = z$ and $y = z^*$

$$\begin{aligned}
\omega_\sigma(z, z^*) &= \omega_\sigma(T^2 z, T^2 z^*) \\
&\leq \alpha_1 \omega_\sigma(z, z^*) + \alpha_2 \omega_\sigma(Tz, Tz^*) + \beta_1 \omega_\sigma(z, Tz) \\
&\quad + \beta_2 \omega_\sigma(Tz, T^2 z) + \gamma_1 \omega_\sigma(z^*, Tz^*) + \gamma_2 \omega_\sigma(Tz^*, T^2 z^*) \\
&\leq (\alpha_1 + \alpha_2)\omega_\sigma(z, z^*).
\end{aligned}$$

Which follows that $(1-\alpha_1-\alpha_2)\omega_\sigma(z, z^*) \leq 0$, which in turn gives $\omega_\sigma(z, z^*) = 0$, a contradiction. Thus T has a unique fixed point in X. ■

Corollary 5. *Let X_ω be a modular metric space and $T : X \to X$ be a generalized convex contraction. Suppose that T is $\alpha-$admissible and there exists $x_0 \in X$ such that $\alpha(Tx_0, x_0) \geq 1$. Then T has an approximate fixed point. Further T has a fixed point if T is continuous and X_ω is a complete modular metric space. Moreover, if for all $x, y \in Fix(T)$ we have $\alpha(x, y) \geq 1$, then T has a unique fixed point in X.*

Proof. Since $T : X \to X$ is a generalized convex contraction, so we obtain

$$\alpha(x, y)\omega_\sigma(T^2 x, T^2 y) \leq \alpha_1 \omega_\sigma(x, y) + \alpha_2 \omega_\sigma(Tx, Ty)$$

$$\begin{aligned}\leq\ & \alpha_1\omega_\sigma(x,y)+\alpha_2\omega_\sigma(Tx,Ty)+\beta_1\omega_\sigma(x,Tx)\\ &+\beta_2\omega_\sigma(Tx,T^2x)+\gamma_1\omega_\sigma(y,Ty)\\ &+\gamma_2\omega_\sigma(Ty,T^2y).\end{aligned}$$

Which shows that T is a generalized convex contraction of type-2. Thus by Theorem 4 the conclusion follows. ■

Corollary 6. *Let X_ω be a metric space and $T : X \to X$ be a generalized convex contraction of order-2. Suppose that T is $\alpha-$admissible and there exists $x_0 \in X$ such that $\alpha(Tx_0, x_0) \geq 1$. Then T has an approximate fixed point. Further T has a fixed point if T is continuous and X_ω is a complete modular metric space. Moreover, if for all $x, y \in Fix(T)$ we have $\alpha(x, y) \geq 1$, then T has a unique fixed point in X.*

Proof. Since $T : X \to X$ is a generalized convex contraction of order-2, we have

$$\begin{aligned}\alpha(x,y)\omega_\sigma(T^2x,T^2y)\ \leq\ & \beta_1\omega_\sigma(x,Tx)+\beta_2\omega_\sigma(Tx,T^2x)+\gamma_1\omega_\sigma(y,Ty)\\ &+\gamma_2\omega_\sigma(Ty,T^2y)\\ \leq\ & \alpha_1\omega_\sigma(x,y)+\alpha_2\omega_\sigma(Tx,Ty)+\beta_1\omega_\sigma(x,Tx)\\ &+\beta_2\omega_\sigma(Tx,T^2x)+\gamma_1\omega_\sigma(y,Ty)\\ &+\gamma_2\omega_\sigma(Ty,T^2y).\end{aligned}$$

Which shows that T is a generalized convex contraction of type-2. Thus by Theorem 4 the conclusion follows. ■

Example 3. Let $X_\omega = [0, 1]$, and $\omega_\lambda(x, y) = \frac{d(x,y)}{\lambda}$ where $d(x, y) = |x-y|$ and L be a self map on X_ω defined by $Lx = \frac{x^2}{3} + \frac{1}{6}$, taking $\alpha(x, y) = 1$. Then $\alpha(Lx, Ly) = 1$ for all $x, y \in X_\omega$. therefore, L is continuous and $\alpha - admissible$.

Thus, we have

$$\begin{aligned}\alpha(x,y)\omega_\lambda(L^2x,L^2y) &= \frac{|L^2x-L^2y|}{\lambda}\\ &= \frac{1}{\lambda}\left|\frac{x^4+x^2+\frac{19}{4}}{27}-\frac{y^4+y^2+\frac{19}{4}}{27}\right|\\ &= \frac{1}{27\lambda}|(x^4-y^4)+(x^2-y^2)|\\ &\leq \frac{1}{6\lambda}|(x^4-y^4)+(x^2-y^2)|\\ &\leq \frac{1}{6\lambda}|(x^2-y^2)+(x-y)|\end{aligned}$$

$$\begin{aligned}&\leq \frac{1}{3\lambda}|\frac{x^2-y^2}{2}|+\frac{1}{6\lambda}|(x-y)|\\&=\frac{1}{3\lambda}d(Lx,Ly)+\frac{1}{6\lambda}d(x,y)\\&=\frac{1}{3}\omega_\lambda(Lx,Ly)+\frac{1}{6}\omega_\lambda(x,y).\end{aligned}$$

which shows that the mapping L is a generalized convex contraction with $\alpha_1=\frac{1}{6}$ and $\alpha_2=\frac{1}{3}$. Thus the conditions of Corollary 5 are all satisfied and $x=\frac{3-\sqrt{7}}{2}$ is the unique fixed point of L in X_ω.

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