F A S C I C U L I M A T H E M A T I C I

Nr 63 2020

DOI: 10.21008/j.0044-4413.2020.0006

Harina P. Waghamore and Ramya Maligi

SOME RESULTS ON UNIQUENESS AND VALUE DISTRIBUTION FOR q-SHIFT DIFFERENCE DIFFERENTIAL POLYNOMIALS

Abstract. In this paper, we investigate the uniqueness and value distribution of q-shift difference differential polynomials of entire and meromorphic functions with zero order and obtain some results which improve and generalizes the previous results of Harina P. Waghamore and Sangeetha Anand [1].

Key words: entire functions, meromorphic functions, sharing values, difference-differential polynomials.

AMS Mathematics Subject Classification: 30D35.

1. Introduction and main results

In this article, we use some basic results and symbols of Nevanlinna's value distribution theory of meromorphic functions in $\mathbb{C}$ such as the first and second main theorems, and the common notations such as the characteristic function $T(r, f)$, the proximity function $m(r, f)$ and the counting functions $N(r, f)$ (with multiplicities) and $\overline{N}(r, f)$ (without multiplicities); $S(r, f)$ denotes any quantity satisfying $S(r, f) = o(T(r, f))$ as $r \to \infty$ except possibly on a set of finite Lebesgue measure, not necessarily the same at each occurence.

Let f and g be two non-constant meromorphic functions defined in the open complex plane $\mathbb{C}$. If for some $a \in \mathbb{C} \cup \{\infty\}$, $f - a$ and $g - a$ have the same set of zeros with the same multiplicities, we say that f and g share the value a CM (counting multiplicities), and if we do not consider the multiplicities then f and g are said to share the value a IM (ignoring multiplicities).

Definition 1. *Linear differential polynomial is defined as*

$$L(f) = \sum_{i=0}^{k} b_i(z) f^{(i)}(z),$$

where $b_1(z), b_2(z), ..., b_k(z)$ *are small functions of* $f(z)$.

In 2011, Liu et al. [2] considered uniqueness of difference polynomials of meromorphic functions, corresponding to uniqueness theorems of meromorphic functions sharing values (see, e.g.,[5]) and obtained the following results.

Theorem A. *Let* $f(z)$ *and* $g(z)$ *be two transcendental meromorphic functions with finite order. Suppose that* c *is a non-zero complex constant and* n *is an integer. If* $n \geq 14$ *and* $f^n(z)f(z+c)$ *and* $g^n(z)g(z+c)$ *share 1 CM, then* $f(z) \equiv tg(z)$ *or* $f(z)g(z) = t$*, where* $t^{n+1} = 1$.

Theorem B. *Under the conditions of Theorem A, if* $n \geq 26$ *and* $f^n(z)f(z+c)$ *and* $g^n(z)g(z+c)$ *share 1 IM, then* $f(z) \equiv tg(z)$ *or* $f(z)g(z) = t$*, where* $t^{n+1} = 1$.

In 2013, Liu et al. [4], considered the case of q-shift difference polynomials and extended the Theorem A as follows:

Theorem C. *Let* $f(z)$ *and* $g(z)$ *be two transcendental meromorphic functions with* $\rho(f) = \rho(g) = 0$*. Suppose that* q *and* c *are two non-zero complex constants and* n *is an integer . If* $n \geq 14$ *and* $f^n(z)f(qz+c)$ *and* $g^n(z)g(qz+c)$ *share 1 CM, then* $f(z) \equiv tg(z)$ *or* $f(z)g(z) = t$*, where* $t^{n+1} = 1$.

Theorem D. *Under the conditions of Theorem C, if* $n \geq 26$ *and* $f^n(z)f(qz+c)$ *and* $g^n(z)g(qz+c)$ *share 1 IM, then* $f(z) \equiv tg(z)$ *or* $f(z)g(z) = t$*, where* $t^{n+1} = 1$.

Theorem E. *Let* $f(z)$ *and* $g(z)$ *be two transcendental entire functions with* $\rho(f) = \rho(g) = 0$ *, let* q *and* c *be two non-zero complex constants, let* $P(z) = a_n z^n + a_{n-1} z^{n-1} + ... + a_1 z + a_0$ *be a non-zero polynomial, where* $a_n (\neq 0), a_{n-1}, ..., a_0$*, are complex constants and* k *denotes the number of the distinct zero of* $P(z)$*. If* $n > 2k+1$ *and* $P(f(z))f(qz+c)$ *and* $P(g(z))g(qz+c)$ *share 1 CM, then one of the following results holds:*

a). $f(z) \equiv tg(z)$ for a constant t such that $t^d = 1$, where $d = GCD\{\lambda_0, \lambda_1, ..., \lambda_n\}$ and

$$\lambda_j = \begin{cases} n+1, & a_j = 0, \\ j+1, & a_j \neq 0, \end{cases}$$

b). $f(z)$ and $g(z)$ satisfy the algebraic equation $R(f(z), g(z)) = 0$, where

$$R(w_1, w_2) = P(w_1)w_1(qz+c) - P(w_2)w_2(qz+c).$$

In 2016, Harina P. Waghamore and Sangeetha Anand [1] investigated the value distribution for q-shift polynomials of transcendental meromorphic and entire functions with zero order and obtained the following results.

Theorem F. *Let $f(z)$ and $g(z)$ be two transcendental meromorphic functions with $\rho(f) = \rho(g) = 0$. Let q and c be two non-zero complex constants, n an integer and $P_m(z) = a_m z^m + a_{m-1} z^{m-1} + ... + a_1 z + a_0$. If $n \geq 5m + 19$ and $f^n(z)P_m(f(qz+c))f^{'}(z)$ and $g^n(z)P_m(g(qz+c))g^{'}(z)$ share 1 CM, then $f(z) \equiv tg(z)$ or $f(z)g(z) = t$, where $t^d = 1, d = GCD(n+m+1, n+m, ..., n+m+1-i, ..., n+1)$,$a_{m-i} \neq 0$, for some $i = 0, 1, ...m$.*

Theorem G. *Under the conditions of Theorem F, if $n \geq 11m + 31$, $f^n(z)P_m(f(qz+c))f^{'}(z)$ and $g^n(z)P_m(g(qz+c))g^{'}(z)$ share 1 IM, then conclusion of Theorem F still holds.*

Theorem H. *Let $f(z)$ and $g(z)$ be two transcendental entire functions with $\rho(f) = \rho(g) = 0$, q and c are two non-zero complex constants and k denote the number of distinct zeros of $P_m(z)$. If $m > n+2k+4$, $f^n(z)P_m(f(qz+c))f^{'}(z)$ and $g^n(z)P_m(g(qz+c))g^{'}(z)$ share 1 CM, then one of the following results holds:*

a). $f(z) \equiv tg(z)$ for a constant t such that $t^d = 1$, where

$$d = GCD(n+m+1, n+m, ..., n+m+1-i, ..., n+1).$$

$a_{m-i} \neq 0$, for some $i = 0, 1, ..., m$.

b). $f(z)$ and $g(z)$ satisfy the algebraic equation $R(f, g) \equiv 0$ where

$$\begin{aligned} R(w_1, w_2) &= w_1^{n+1}\left[\frac{a_m w_1^m}{n+m+1} + \frac{a_{m-1} w_1^{m-1}}{n+m} + ... + \frac{a_0}{n+1}\right] \\ &- w_2^{n+1}\left[\frac{a_m w_2^m}{n+m+1} + \frac{a_{m-1} w_2^{m-1}}{n+m} + ... + \frac{a_0}{n+1}\right]. \end{aligned}$$

By considering Definition 1, we obtain results on the uniqueness and value distribution of q-shift difference differential polynomials of transcendental entire and meromorphic functions of the form $f^n(z)P_m(f(qz+c))L(f)$ and $g^n(z)P_m(g(qz+c))L(g)$. Our results improve and generalize the results due to [1].

Theorem 1. *Let $f(z)$ and $g(z)$ be two transcendental meromorphic functions with $\rho(f) = \rho(g) = 0$. Let q and c be two non-zero complex constants, n an integer and $P_m(z) = a_m z^m + a_{m-1} z^{m-1} + ... + a_1 z + a_0$. If $n \geq 3m + 5k + 14$ and $f^n(z)P_m(f(qz+c))L(f)$ and $g^n(z)P_m(g(qz+c))L(g)$ share 1 CM, then $f(z) \equiv tg(z)$ or $f(z)g(z) = t$, where $t^d = 1$, $d = GCD(\{n+m, n+m-1, ..., n+m-i, ..., n\}$, $\{n+m+1, n+m, ..., n+m+1-i, ..., n+1\}$,..., $\{n+m+k, n+m+k-1, ..., n+m+k-i, , ..., n+k\})$, $a_{m-i} \neq 0$, for some $i = 0, 1, ..., m$.*

Theorem 2. *Under the conditions of Theorem 1, if $n \geq 9m + 11k + 20$, $f^n(z)P_m(f(qz+c))L(f)$ and $g^n(z)P_m(g(qz+c))L(g)$ share 1 IM, then conclusion of Theorem 1 still holds.*

Theorem 3. *Let $f(z)$ and $g(z)$ be two transcendental entire functions with $\rho(f)=\rho(g)=0$, q and c are two non-zero complex constants and t_m denote the number of distinct zeros of $P_m(z)$. If $m > n+2t_m+k+1$, $f^n(z)P_m(f(qz+c))L(f)$ and $g^n(z)P_m(g(qz+c))L(g)$ share 1 CM, then one of the following results holds:*

a). $f(z)\equiv tg(z)$ for a constant t such that $t^d=1$, where

$$\begin{aligned} d &= GCD(\{n+m+2, n+m+1, ..., n+m+2-i, ..., n+2\},\\ &\quad \{n+m+1, ..., n+m, ..., n+m+1-i, ..., n+1\}, ...,\\ &\quad \{n+m+k-1, n+m+k-2, ..., n+m+k-i, ..., n+k-1\}).\end{aligned}$$

$a_{m-i}\neq 0$, for some $i=0,1,...,m$.

b). $f(z)$ and $g(z)$ satisfy the algebraic equation $R(f,g)\equiv 0$ where

$$\begin{aligned} R(w_1,w_2) &= w_1^{n+2}\left[\frac{a_m b_0 w_1^m}{n+m+2}+...+\frac{a_0b_0}{n+2}\right]\\ &\quad + w_1^{n+1}\left[\frac{a_m b_1 w_1^m}{n+m+1}+...+\frac{a_0b_1}{n+1}\right]+...\\ &\quad + w_1^{n+k-1}\left[\frac{a_m b_k w_1^m(k-1)}{n+m+k-1}+...+\frac{a_0b_k(k-1)}{n+k-1}\right]\\ &\quad - \left\{w_2^{n+2}\left[\frac{a_m b_0 w_2^m}{n+m+2}+...+\frac{a_0b_0}{n+2}\right]\right.\\ &\quad + w_2^{n+1}\left[\frac{a_m b_1 w_2^m}{n+m+1}+...+\frac{a_0b_1}{n+1}\right]+...\\ &\quad \left.+\ w_2^{n+k-1}\left[\frac{a_m b_k w_2^m(k-1)}{n+m+k-1}+...+\frac{a_0b_k(k-1)}{n+k-1}\right]\right\}.\end{aligned}$$

Remark 1. For $i=0,1,..,k$, if $b_i(z)=0$ for $i\neq 1$ and $b_1(z)=1$ in $L(f)$ of Theorems 1, 2 and 3, then Theorems 1, 2 and 3 reduces to Theorems F, G and H.

Remark 2. If $k=1$ in Theorems 1 and 2, then Theorems 1 and 2 improve and generalize Theorems F and G.

Remark 3. If $k=1$ and $t_m=k$ in Theorem 3, then Theorem 3 improve and generalize Theorem H.

The following example shows that the conditions in Theorem 1 cannot be removed.

Example 1. Let $f(z)=\sin z$,$g(z)=\cos z$, $q=1$, $k=0$, $c=2\pi$, $n=17$ and $m=1$. Hence we have $n\geq 17$ and $f^n(z)P_m(f(qz+c))L(f)=g^n(z)P_m(g(qz+c))L(g)$. Therefore $f^n(z)P_m(f(qz+c))L(f)$ and $g^n(z)P_m(g(qz+c))L(g)$ share 1 CM. Clearly, we get $f=tg$.

Example 2. Let $f(z) = e^z$, $g(z) = -e^z$, $q = 1$, $k = 3$, $c = 1$, $n = 32$ and $m = 1$. Hence we have $n > 31$. Here $L(f) = f + f^{(1)} + f^{(2)} + f^{(3)} = 4e^z$, $L(g) = g + g^{(1)} + g^{(2)} + g^{(3)} = -4e^z$ and $f^n(z)P_m(f(qz + c))L(f) = g^n(z)P_m(g(qz + c))L(g)$. Therefore $f^n(z)P_m(f(qz + c))L(f)$ and $g^n(z)P_m(g(qz+c))L(g)$ share 1 CM. Then we get $f = tg$, where t is d^{th} root of unity.

Example 3. Let $f(z) = e^z$, $g(z) = e^{-z}$, $q = 1$, $k = 4$, $c = 1$, $n = 37$ and $m = 1$. Hence we have $n > 36$. Here $L(f) = f^{(4)} + f^{(3)} - f^{(2)} - f^{(1)} - f = -e^z$, $L(g) = g^{(4)} + g^{(3)} - g^{(2)} - g^{(1)} - g = -e^{-z}$ and $f^n(z)P_m(f(qz + c))L(f) = g^n(z)P_m(g(qz + c))L(g)$. Therefore $f^n(z)P_m(f(qz + c))L(f)$ and $g^n(z)P_m(g(qz + c))L(g)$ share 1 CM. Then we get $f(z)g(z) = t$, where t is d^{th} root of unity.

2. Some lemmas

For the proof of our main results, we need the following lemmas.

Lemma 1 ([8]). *Let $f(z)$ be a non-constant meromorphic function and $a_n(\neq 0), a_{n-1}, ..., a_0$ be small functions with respect to $f(z)$. Then*

$$T(r, a_n f^n + a_{n-1} f^{n-1} + ... + a_1 f + a_0) = nT(r, f) + S(r, f)$$

Lemma 2 ([6]). *Let $f(z)$ be a transcendental meromorphic function of finite logarithmic order and q, η be two non-zero complex constants. Then we have*

$$\begin{aligned} T(r, f(qz + \eta)) &= T(r, f) + S(r, f), \\ N(r, f(qz + \eta)) &= N(r, f) + S(r, f), \\ N\left(r, \frac{1}{f(qz + \eta)}\right) &= N(r, \frac{1}{f}) + S(r, f). \end{aligned}$$

Lemma 3 ([8]). *Let $f(z)$ be a non-constant meromorphic function in the complex plane. Then*

1. $m\left(r, \frac{L(f)}{f}\right) = S(r, f)$.
2. $T(r, L(f)) \leq T(r, f) + k\overline{N}(r, f) + S(r, f)$.

Lemma 4 ([3]). *Let $f(z)$ be a non-constant meromorphic function and p, k be positive integers. Then*

1. $T(r, f^k) \leq T(r, f) + k\overline{N}(r, f) + S(r, f)$
2. $N_p\left(r, \frac{1}{f^{(k)}}\right) \leq T(r, f^k) - T(r, f) + N_{p+k}\left(r, \frac{1}{f}\right) + S(r, f)$
3. $N_p\left(r, \frac{1}{f^{(k)}}\right) \leq N_{p+k}\left(r, \frac{1}{f}\right) + k\overline{N}(r, f) + S(r, f)$
4. $N\left(r, \frac{1}{f^{(k)}}\right) \leq N\left(r, \frac{1}{f}\right) + k\overline{N}(r, f) + S(r, f)$.

Lemma 5 ([7]). *Let $F(z)$ and $G(z)$ be two non-constant meromorphic functions. If $F(z)$ and $G(z)$ share 1 CM, then one of the following three cases holds:*

1. $T(r,F)+T(r,G)\leq 2\left\{N_2\left(r,\frac{1}{F}\right)+N_2(r,F)N_2\left(r,\frac{1}{G}\right)+N_2(r,G)\right\}$
$+S(r,F)+S(r,G),$
2. $F\equiv G,$
3. $FG\equiv 1.$

Lemma 6 ([5]). *Let F and G be two non-constant meromorphic functions. Let F and G share 1 IM and*

$$H=\frac{F''}{F'}-2\frac{F'}{F-1}-\frac{G''}{G'}+2\frac{G'}{G-1}$$

If $H\not\equiv 0$, then

$$\begin{aligned}T(r,F)+T(r,G) \ \leq \ & 2\left(N_2\left(r,\frac{1}{F}\right)+N_2(r,F)+N_2\left(r,\frac{1}{G}\right)+N_2(r,G)\right)\\ &+3\left(\overline{N}(r,F)+\overline{N}(r,G)+\overline{N}\left(r,\frac{1}{F}\right)+\overline{N}\left(r,\frac{1}{G}\right)\right)\\ &+S(r,F)+S(r,G).\end{aligned}$$

Lemma 7 ([4]). *Let $f(z)$ be an entire function with $\rho(f)=0$, let c and q be two fixed non-zero complex constants, let $P(z)=a_nz^n+a_{n-1}z^{n-1}+...+a_1z+a_0$ be a non-zero polynomial , where $a_n(\neq 0),a_{n-1},...,a_0$, are complex constants, then*

$$T(r,P(f(z))f(qz+c))=T(r,P(f(z))f(z))+S(r,f).$$

3. Proof of Theorem 1

Let $F(z)=f^n(z)P_m(f(qz+c))L(f)$ and $G(z)=g^n(z)P_m(g(qz+c))L(g)$.

Since $f(z)$ is a transcendental meromorphic function of zero order, by Lemma 1, Lemma 2 and Lemma 3, we get

$$\begin{aligned}T(r,F) \ &\leq \ T(r,f^n(z))+T(r,P_m(f(qz+c)))+T(r,L(f))\\ &\leq \ (n+m+k+1)T(r,f)+S(r,f)\end{aligned} \tag{1}$$

On the otherhand from Lemma 1, Lemma 2, Lemma 4 and Lemma 3, we deduce that

$$\begin{aligned}(n+m)T(r,f) \ &\leq \ T(r,f^n(z)P_m(f(qz+c)))\\ &\leq \ T\left(r,\frac{F}{L(f)}\right)+S(r,f)\\ &\leq \ T(r,F)+(k+1)T(r,f)+S(r,f).\end{aligned}$$

Therefore

$$(n+m-k-1)T(r,f)+S(r,f)\leq T(r,F). \tag{2}$$

From (1) and (2), we obtain

$$\begin{aligned}(3)\ (n+m-k-1)T(r,f)+S(r,f) &\leq T(r,F)\\ &\leq (n+m+k+1)T(r,f)+S(r,f).\end{aligned}$$

Similarly, we have

$$\begin{aligned}(4)\ (n+m-k-1)T(r,g)+S(r,g) &\leq T(r,G)\\ &\leq (n+m+k+1)T(r,g)+S(r,g).\end{aligned}$$

Also, we have

$$\begin{aligned}N_2\left(r,\frac{1}{F}\right) &\leq 2\bar{N}\left(r,\frac{1}{f}\right)+N\left(r,\frac{1}{P_m(f(qz+c))}\right)\\ &\quad +N\left(r,\frac{1}{L(f)}\right)+S(r,f)\\ &\leq (k+m+3)T(r,f)+S(r,f)\end{aligned} \tag{5}$$

Similarly,

$$N_2\left(r,\frac{1}{G}\right)\leq (k+m+3)T(r,g)+S(r,g), \tag{6}$$

$$N_2(r,F)\leq (k+m+3)T(r,f)+S(r,f), \tag{7}$$

$$N_2(r,G)\leq (k+m+3)T(r,g)+S(r,g). \tag{8}$$

Since F and G share 1 CM, let us assume (1) of Lemma 5 holds and hence

$$\begin{aligned}T(r,F)+T(r,G) &\leq 2\left[N_2\left(r,\frac{1}{F}\right)+N_2(r,F)+N_2\left(r,\frac{1}{G}\right)+N_2(r,G)\right]\\ &\quad +S(r,F)+S(r,G).\end{aligned}$$

Substituting (3)-(8), we obtain

$$\begin{aligned}T(r,F)+T(r,G) &\leq 2(2(k+m+3)(T(r,f)+T(r,g)))\\ &\quad +S(r,f)+S(r,g)\\ &\leq (4k+4m+12)(T(r,f)+T(r,g))\\ &\quad +S(r,f)+S(r,g).\end{aligned} \tag{9}$$

From (3), (4) and (9), we get

$$(n+m-k-1-4k-4m-12)(T(r,f)+T(r,g)) \leq S(r,f)+S(r,g)$$

Therefore,

$$(n-3m-5k-13)(T(r,f)+T(r,g)) \leq S(r,f)+S(r,g) \tag{10}$$

which is a contradiction, since $n \geq 3m+5k+14$. Thus by Lemma 5, we have $F \equiv G$ or $FG \equiv 1$. If $F \equiv G$, that is,

$$f^n(z)P_m(f(qz+c))L(f) \equiv g^n(z)P_m(g(qz+c))L(g)$$

Set $H(z) = f(z)/g(z)$. Suppose that $H(z)$ is not a constant. Then, we obtain

$$\frac{f^n(z)P_m(f(qz+c))L(f)}{g^n(z)P_m(g(qz+c))L(g)} = 1$$

$$H^n(z)P_m(H(qz+c))L(H) = 1 \tag{11}$$

From Lemma 2 and (11), we get

$$\begin{aligned} nT(r,H) &= T\left(r, \frac{1}{P_m(H(qz+c))L(H)}\right) \\ &\leq T(r, P_m(H(qz+c))L(H)) + S(r,H) \\ &\leq (k+m+1)T(r,H(z)) + S(r,H) \end{aligned} \tag{12}$$

Hence, $H(z)$ must be non-zero constant, since $n \geq 3m+5k+14$. Set $H(z) = t$. By (11), we have $t^d = 1$. Thus $f(z) = tg(z)$, where

$$\begin{aligned} d &= GCD(\{n+m, n+m-1, ..., n+m-i, ..., n\}, \\ &\quad \{n+m+1, n+m, ..., n+m+1-i, ..., n+1\}, ..., \\ &\quad \{n+m+k, n+m+k-1, ..., n+m+k-i, ..., n+k\}, \end{aligned}$$

$a_{m-i} \neq 0$, for some $i = 0, 1, ..., m$. If $FG \equiv 1$, that is,

$$f^n(z)P_m(f(qz+c))L(f).g^n(z)P_m(g(qz+c))L(g) = 1$$

Let $L(z) = f(z).g(z)$. Using similar method as above, we obtain that $L(z)$ must also be a non-zero constant. Thus we have $fg = t$, where $t^d = 1$,

$$\begin{aligned} d &= GCD(\{n+m, n+m-1, ..., n+m-i, ..., n\}, \\ &\quad \{n+m+1, n+m, ..., n+m+1-i, ..., n+1\}, ..., \\ &\quad \{n+m+k, n+m+k-1, ..., n+m+k-i, ..., n+k\}) \end{aligned}$$

$a_{m-i} \neq 0$ for some $i = 0, 1, ..., m$.

4. Proof of Theorem 2

Let $F(z) = f^n(z)P_m(f(qz+c))L(f)$ and $G(z) = g^n(z)P_m(g(qz+c))L(g)$, then F and G share 1 IM. If $H \not\equiv 0$ then by Lemma 6, we have

$$\begin{aligned}(13)\qquad T(r,F)+T(r,G) \;\le\; & 2\left(N_2\left(r,\frac{1}{F}\right)+N_2(r,F)\right.\\ & \left.+N_2\left(r,\frac{1}{G}\right)+N_2(r,G)\right)\\ & +3\left(\overline{N}(r,F)+\overline{N}(r,G)+\overline{N}\left(r,\frac{1}{F}\right)\right.\\ & \left.+\;\overline{N}\left(r,\frac{1}{G}\right)\right)+\;S(r,F)+S(r,G).\end{aligned}$$

By Lemma 2, we obtain

$$\begin{aligned}(14)\qquad \overline{N}(r,F(z)) \;=\; & \overline{N}(r,f^n(z)P_m(f(qz+c)L(f))+S(r,f)\\ \le\; & (k+m+1)T(r,f)+S(r,f),\end{aligned}$$

Similarly,

$$(15)\qquad \overline{N}\left(r,\frac{1}{F(z)}\right)\le (k+m+1)T(r,f)+S(r,f),$$

$$(16)\qquad \overline{N}(r,G(z))\le (k+m+1)T(r,g)+S(r,g),$$

$$(17)\qquad \overline{N}\left(r,\frac{1}{G(z)}\right)\le (k+m+1)T(r,g)+S(r,g).$$

Together Lemma 6 with (5)-(8) and (14)-(17), we have

$$\begin{aligned}(18)\qquad T(r,F)+T(r,G) \;\le\; & 2(2(k+m+3))\big(T(r,f)+T(r,g)\big)\\ & +3(2(k+m+1))\big(T(r,f)+T(r,g)\big)\\ & +S(r,f)+S(r,g).\end{aligned}$$

By (3), (4) and (18)

$$\begin{aligned}(19)\qquad & (n+m-k-1)\big(T(r,f)+T(r,g)\big)\\ & \le (10k+10m+18)\big(T(r,f)+T(r,g)\big)+S(r,f)+S(r,g)\end{aligned}$$

which is impossible, since $n \ge 9m+11k+20$. Hence, we have $H \equiv 0$.

By integrating H twice, we have

$$F = \frac{(b+1)G + (a-b-1)}{bG + (a-b)} \tag{20}$$

which yields $T(r,F) = T(r,G) + O(1)$. From (3), (4), we obtain

$$(n+m-k-1)T(r,f) \le (n+m+k+1)T(r,g) + S(r,f) + S(r,g) \tag{21}$$

$$(n+m-k-1)T(r,g) \le (n+m+k+1)T(r,f) + S(r,f) + S(r,g). \tag{22}$$

Next, we will prove that $F \equiv G$ or $FG \equiv 1$.

Case 1. $(b \ne 0, -1)$. If $a-b-1 \ne 0$, by (20), we obtain

$$\overline{N}\left(r, \frac{1}{F}\right) = \overline{N}\left(r, \frac{1}{G - (a-b-1)/(b+1)}\right). \tag{23}$$

Combining the Nevanlinna second main theorem with Lemma 2, (3),(4) and (22), we obtain

$$\begin{aligned}
(n+m-k-1)T(r,g) &\le T(r,G) + S(r,g) \\
&\le \overline{N}\left(r, \frac{1}{G}\right) + \overline{N}(r,G) + \overline{N}\left(r, \frac{1}{G - (a-b-1)/(b+1)}\right) + S(r,g) \\
&\le \overline{N}\left(r, \frac{1}{G}\right) + \overline{N}(r,G) + \overline{N}\left(r, \frac{1}{F}\right) + S(r,g) \\
&\le \overline{N}\left(r, \frac{1}{g}\right) + \overline{N}\left(r, \frac{1}{P_m(g(qz+c))}\right) + \overline{N}\left(r, \frac{1}{L(g)}\right) \\
&\quad + \overline{N}(r,g) + \overline{N}(r, P_m(g(qz+c))) + \overline{N}(r, L(g)) \\
&\quad + \overline{N}\left(r, \frac{1}{f}\right) + \overline{N}\left(r, \frac{1}{P_m(f(qz+c))}\right) + \overline{N}\left(r, \frac{1}{L(f)}\right) + S(r,g) \\
&\le (3+2k+m)T(r,g) + (1+k+m)T(r,f) + S(r,g)
\end{aligned} \tag{24}$$

By simple calculation, we get contradiction, since $n \ge 9m + 11k + 20$. Hence we obtain, $a-b-1 = 0$, so

$$F = \frac{(b+1)G}{bG+1} \tag{25}$$

Using the similar method as above, we obtain

$$\begin{aligned}
(n+m-k-1)T(r,g) &\le T(r,G) + S(r,g) \\
&\le \overline{N}\left(r, \frac{1}{G}\right) + \overline{N}(r,G) + \overline{N}\left(r, \frac{1}{G+1/b}\right) + S(r,g) \\
&\le \overline{N}\left(r, \frac{1}{G}\right) + \overline{N}(r,G) + \overline{N}\left(r, \frac{1}{F}\right) + S(r,g) \\
&\le (3+2k+m)T(r,g) + (1+k+m)T(r,f) + S(r,g)
\end{aligned}$$

which is impossible.

Case 2. If $b=-1$ and $a=-1$, then $FG\equiv 1$ follows trivially. Therefore, consider $b=-1$ and $a\neq -1$. By (20), we have

$$F=\frac{a}{a+1-G}. \tag{26}$$

Similarly, as above we get contradiction.

Case 3. If $b=0$, $a=1$, then $F\equiv G$ follows trivially. Therefore, consider $b=0$ and $a\neq 1$. By (20), we have

$$F=\frac{G+a-1}{a}. \tag{27}$$

Similarly, as above we get contradiction.

5. Proof of Theorem 3

Let $f(z)$ and $g(z)$ be two transcendental entire functions. Since $f^n(z)P_m(f(qz+c))L(f)$ and $g^n(z)P_m(g(qz+c))L(g)$ share 1 CM, we have

$$\frac{f^n(z)P_m(f(qz+c))L(f)-1}{g^n(z)P_m(g(qz+c))L(g)-1}=e^{l(z)} \tag{28}$$

where $l(z)$ is an entire function, by $\rho(f)=0$ and $\rho(g)=0$, we have $e^{l(z)}\equiv\eta$ a constant. Rewriting (28),

$$\eta g^n(z)P_m(g(qz+c))L(g)=f^n(z)P_m(f(qz+c))L(f)+\eta-1. \tag{29}$$

If $\eta\neq 1$, by the first main theorem, the second main theorem and Lemma 2, we have

$$\begin{aligned} T(r,f^n(z)P_m(f(qz+c))L(f)) &\le \overline{N}(r,f^n(z)P_m(f(qz+c))L(f)) \\ &\quad+\overline{N}\left(r,\frac{1}{f^n(z)P_m(f(qz+c))L(f))}\right) \\ &\quad+\overline{N}\left(r,\frac{1}{f^n(z)P_m(f(qz+c))L(f)-1}\right)+S(r,F) \\ &\le \overline{N}\left(r,\frac{1}{f^n(z)}\right)+\sum_{j=1}^{t_m}\overline{N}\left(r,\frac{1}{f(qz+c)-\gamma_j}\right) \\ &\quad+\overline{N}\left(r,\frac{1}{L(f)}\right)+\overline{N}\left(r,\frac{1}{G}\right)+S(r,f)+S(r,g) \\ &\le (n+t_m+k+1)T(r,f)+(n+t_m+k+1)T(r,g) \\ &\quad+S(r,f)+S(r,g) \end{aligned} \tag{30}$$

By Lemma 7 and (30), we have

$$\begin{aligned}(n+m+k+1)T(r,f) &= T(r,f^n(z)P_m(f(qz+c))L(f))\\ &\leq (n+t_m+k+1)T(r,f)\\ &\quad + (n+t_m+k+1)T(r,g)+S(r,f)+S(r,g)\end{aligned}$$

$$(m-t_m)T(r,f) \leq (n+t_m+k+1)T(r,g)+S(r,f)+S(r,g) \tag{31}$$

Similarly,

$$(m-t_m)T(r,g) \leq (n+t_m+k+1)T(r,f)+S(r,f)+S(r,g) \tag{32}$$

Equations (31) and (32) imply that

$$(m-n-2t_m-k-1)\big(T(r,f)+T(r,g)\big) \leq S(r,f)+S(r,g) \tag{33}$$

which is impossible, since $m > n+2t_m+k+1$.

Hence we have $\eta = 1$. Rewriting (28),

$$f^n(z)P_m(f(qz+c))L(f) = g^n(z)P_m(g(qz+c))L(g) \tag{34}$$

Set $h(z) = f(z)/g(z)$

Case 1. Suppose that $h(z)$ is a constant. Integrating (34), we get

$$\begin{aligned} & f^{n+2}\left[\frac{a_m b_0 f^m(qz+c)}{n+m+2} + \frac{a_{m-1} b_0 f^{m-1}(qz+c)}{n+m+1} + ... + \frac{a_0 b_0}{n+2}\right]\\ &+ f^{n+1}\left[\frac{a_m b_1 f^m(qz+c)}{n+m+1} + \frac{a_{m-1} b_1 f^{m-1}(qz+c)}{n+m} + ... + \frac{a_0 b_1}{n+1}\right]\\ &+ ... + f^{n+k-1}\left[\frac{a_m b_k f^m(qz+c)(k-1)}{n+m+k-1}\right.\\ &\left.+ \frac{a_{m-1} b_k f^{m-1}(qz+c)(k-1)}{n+m+k-2} + ... + \frac{a_0 b_k (k-1)}{n+k-1}\right]\\ &= g^{n+2}\left[\frac{a_m b_0 g^m(qz+c)}{n+m+2} + \frac{a_{m-1} b_0 g^{m-1}(qz+c)}{n+m+1} + ... + \frac{a_0 b_0}{n+2}\right]\\ &+ g^{n+1}\left[\frac{a_m b_1 g^m(qz+c)}{n+m+1} + \frac{a_{m-1} b_1 g^{m-1}(qz+c)}{n+m} + ... + \frac{a_0 b_1}{n+1}\right]\\ &+ ... + g^{n+k-1}\left[\frac{a_m b_k g^m(qz+c)(k-1)}{n+m+k-1}\right.\\ &\left.+ \frac{a_{m-1} b_k g^{m-1}(qz+c)(k-1)}{n+m+k-2} + ... + \frac{a_0 b_k (k-1)}{n+k-1}\right]\end{aligned} \tag{35}$$

By substituting $f = gh$ in (35), we obtain

$$g^{n+2}h^{n+2}\Bigg[\frac{a_m b_0 g^m(qz+c)h^m}{n+m+2} + \frac{a_{m-1}b_0 g^{m-1}(qz+c)h^{m-1}}{n+m+1} + ... + \frac{a_0 b_0}{n+2}\Bigg]$$
$$+ g^{n+1}h^{n+1}\Bigg[\frac{a_m b_1 g^m(qz+c)h^m}{n+m+1} + \frac{a_{m-1}b_1 g^{m-1}(qz+c)h^{m-1}}{n+m} + ... + \frac{a_0 b_0}{n+1}\Bigg]$$
$$+ ... + g^{n+k-1}h^{n+k-1}\Bigg[\frac{a_m b_k g^m(qz+c)h^m(k-1)}{n+m+k-1} + \frac{a_{m-1}b_k g^{m-1}(qz+c)h^{m-1}(k-1)}{n+m+k-2} + ... + \frac{a_0 b_k(k-1)}{n+k-1}\Bigg]$$
$$= g^{n+2}\Bigg[\frac{a_m b_0 g^m(qz+c)}{n+m+2} + \frac{a_{m-1}b_0 g^{m-1}(qz+c)}{n+m+1} + ... + \frac{a_0 b_0}{n+2}\Bigg]$$
$$+ g^{n+1}\Bigg[\frac{a_m b_1 g^m(qz+c)}{n+m+1} + \frac{a_{m-1}b_1 g^{m-1}(qz+c)}{n+m} + ... + \frac{a_0 b_1}{n+1}\Bigg]$$
$$+ ... + g^{n+k-1}\Bigg[\frac{a_m b_k g^m(qz+c)(k-1)}{n+m+k-1} + \frac{a_{m-1}b_k g^{m-1}(qz+c)(k-1)}{n+m+k-2} + ... + \frac{a_0 b_k(k-1)}{n+k-1}\Bigg].$$

This implies

$$g^{n+2}\Bigg[\frac{a_m b_0 g^m(qz+c)}{n+m+2}(h^{n+m+2}-1) + \frac{a_{m-1}b_0 g^{m-1}(qz+c)}{n+m+1}(h^{n+m+1}-1)$$
$$+ ... + \frac{a_0 b_0}{n+2}(h^{n+2}-1) + \frac{a_m b_1 g^{m-1}(qz+c)}{n+m+1}(h^{n+m+1}-1)$$
$$+ \frac{a_{m-1}b_1 g^{m-2}(qz+c)}{n+m}(h^{n+m}-1) + ... + \frac{a_0 b_1 g^{-1}(qz+c)}{n+1}(h^{n+1}-1)$$
$$+ ... + \frac{a_m b_k g^{m+k-3}(qz+c)}{n+m+k-1}(h^{n+m+k-1}-1)(k-1)$$
$$+ \frac{a_{m-1}b_k(k-1)g^{m+k-4}(qz+c)}{n+m+k-2}(h^{n+m+k-2}-1) \tag{36}$$

$$+\ ...+\frac{a_0b_k(k-1)g^{k-3}(qz+c)}{n+k-1}(h^{n+k-1}-1)\Bigg]\equiv 0$$

Since g is a transcendental entire function, we have $g^{n+2}(z)\neq 0$. Hence, we obtain

$$\begin{aligned}(37)\quad &\frac{a_mb_0g^m(qz+c)}{n+m+2}(h^{m+n+2}-1)+\frac{a_{m-1}b_0g^{m-1}(qz+c)}{n+m+1}(h^{n+m+1}-1)\\ &+...+\frac{a_0b_0}{n+2}(h^{n+2}-1)+\frac{a_mb_1g^{m-1}(qz+c)}{n+m+1}(h^{n+m+1}-1)\\ &+\frac{a_{m-1}b_1g^{m-2}(qz+c)}{n+m}(h^{n+m}-1)+...+\frac{a_0b_1g^{-1}(qz+c)}{n+1}(h^{n+1}-1)\\ &+...+\frac{a_mb_kg^{m+k-3}(qz+c)}{n+m+k-1}(h^{n+m+k-1}-1)(k-1)\\ &+\frac{a_{m-1}b_k(k-1)g^{m+k-4}(qz+c)}{n+m+k-2}(h^{n+m+k-2}-1)\\ &+...+\frac{a_0b_k(k-1)g^{k-3}(qz+c)}{n+k-1}(h^{n+k-1}-1)\equiv 0\end{aligned}$$

Equation (37) implies that $h^d=1$, where

$$\begin{aligned}d\ =\ &GCD(\{n+m+2,n+m+1,...,n+m+2-i,...,n+2\},\\ &\{n+m+1,...,n+m,...,n+m+1-i,...,n+1\},...,\\ &\{n+m+k-1,n+m+k-2,...,n+m+k-i,...,n+k-1\}),\end{aligned}$$

$a_{m-i}\neq 0$, for some $i=0,1,...,m$.

Thus $f=tg$ for a constant t, such that $t^d=1$, where

$$\begin{aligned}d\ =\ &GCD(\{n+m+2,n+m+1,...,n+m+2-i,...,n+2\},\\ &\{n+m+1,...,n+m,...,n+m+1-i,...,n+1\},...,\\ &\{n+m+k-1,n+m+k-2,...,n+m+k-i,...,n+k-1\}),\end{aligned}$$

$a_{m-i}\neq 0$, for some $i=0,1,...,m$.

Case 2. Suppose that $h(z)$ is not a constant. Then by (37) $f(z)$ and $g(z)$ satisfy the algebraic equation $R(f,g)\equiv 0$, where

$$\begin{aligned}R(w_1,w_2)\ =\ &w_1^{n+2}\left[\frac{a_mb_0w_1^m}{n+m+2}+...+\frac{a_0b_0}{n+2}\right]\\ &+w_1^{n+1}\left[\frac{a_mb_1w_1^m}{n+m+1}+...+\frac{a_0b_1}{n+1}\right]\\ &+...+w_1^{n+k-1}\left[\frac{a_mb_kw_1^m}{n+m+k-1}+...+\frac{a_0b_k(k-1)}{n+k-1}\right]\end{aligned}$$

$$- \left\{ w_2^{n+2} \left[\frac{a_m b_0 w_2^m}{n+m+2} + ... + \frac{a_0 b_0}{n+2} \right] + w_2^{n+1} \left[\frac{a_m b_1 w_2^m}{n+m+1} + ... + \frac{a_0 b_1}{n+1} \right] \right.$$
$$\left. + ... + w_2^{n+k-1} \left[\frac{a_m b_k w_2^m}{n+m+k-1} + ... + \frac{a_0 b_k (k-1)}{n+k-1} \right] \right\}$$

6. Open question

Question. Whether the Theorem 3. hold for Meromorphic functions?

References

[1] Harina P. Waghamore; Sangeetha Anand, Uniqueness and value distribution for q-shift difference polynomials, *International J. of Math. Sci. and Engg. Appls.(IJMSEA),* 10(2016), 1-13.

[2] Liu, Kai; Liu, Xinling; Cao, TingBin, Value distributions and uniqueness of difference polynomials, *Adv. Difference Equ.* 2011*,* Art. ID 234215, 12 pp.

[3] Liu, Kai; Liu, Xinling; Cao, TingBin, Some results on zeros distributions and uniqueness of derivatives of difference, *...,* (2011) http.//arxiv.org/abs/1107.0773vl.

[4] Liu, Yong; Cao, Yinhong; Qi, Xiaoguang; Yi, Hongxun, Value sharing results for q-shifts difference polynomials, *Discrete Dyn. Nat. Soc.* 2013*,* Art. ID 152069, 6 pp.

[5] Xu, Junfeng; Yi, Hongxun, Uniqueness of entire functions and differential polynomials, *Bull. Korean Math. Soc.,* 44(4)(2007), 623-629.

[6] Xu, Junfeng; Zhang, Xiaobin, The zeros of q-shift difference polynomials of meromorphic functions, *Adv. Difference Equ.,* 200(2012), 10 pp..

[7] Yang, Chung-Chun; Hua, Xinhou, Uniqueness and value-sharing of meromorphic functions, *Ann. Acad. Sci. Fenn. Math.,* 22(2)(1997), 395-406.

[8] Yang, Chung-Chun; Yi, Hong-Xun, Uniqueness Theory of Meromorphic Functions, *Mathematics and its Applications, 557. Kluwer Academic Publishers Group,* Dordrecht, 2003.

Harina P. Waghamore
Department of Mathematics
Jnanabharathi Campus
Bangalore University
Bengaluru-560056, India
e-mail: harinapw@gmail.com

Ramya Maligi
Department of Mathematics
Jnanabharathi Campus
Bangalore University
Bengaluru-560056, India
e-mail: ramyamalgi@gmail.com

Received on 03.10.2019 and, in revised form, on 10.03.2020.