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FIXED POINT THEOREMS FOR INTEGRAL TYPE CONTRACTION CONDITION IN 2-METRIC SPACE

Abstract. In this paper we have introduced a new type of contraction mapping F_α-contraction which is the generalization of F-contraction and proved some common fixed point theorems using integral type contraction condition.

Key words: 2-metric space, $\alpha-$admissible, $F-$contraction.

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1. Introduction

Banach [2] investigated a fixed point theorem in metric space which is known as "Banach contraction principle". After that many researchers have investigated and improved this theorem on the extension and generalization of metric space such as B_v metric, generalized metric, cone metric etc. In 2002, Branciari [3] first proved Banach fixed point theorem using integral type contraction in metric space as below:

Let (X, d) be a complete metric space, $c \in (0, 1)$, and let $f : X \to X$ such that for each $x, y \in X$,

$$\int_0^{d(fx,fy)} \phi(t)dt \leq c \int_0^{d(x,y)} \phi(t)dt$$

where $\phi : [0,\infty) \to [0,\infty)$ is a Lebesgue integrable map which is summable (i.e., with finite integral) on each compact subset of $[0,\infty)$, non-negative, and such that for each $\epsilon > 0$, $\int_0^\epsilon \phi(t)dt > 0$; then f has a unique fixed point $a \in X$ such that for each $x \in X$, $\lim\limits_{n\to\infty} f^n x = a$.

In 2003, Rhoades[14] extended the Branciari's Theorem by replacing the condition as:

$$\int_0^{d(Sx,Sy)} \phi(t)dt \leq \alpha \int_0^{\max\{d(x,y),d(x,Sx),d(y,Sy),\frac{d(x,Sy)+d(y,Sx)}{2}\}} \phi(t)dt.$$

In 2009, Moradi and Biranvand [cf.[9]] extended the Rhoades theorem by replacing the condition as bellow:

$$\int_0^{d(TSx,TSy)} \phi(t)dt \leq \alpha \int_0^{\max\{d(Tx,Ty),d(Tx,TSx),d(Ty,TSy),\frac{d(Tx,TSy)+d(Ty,TSy)}{2}\}} \phi(t)dt.$$

Thereafter many researchers, Badehian and Asgari [1], Gupta et al. [5], Vats et al. [18], Sarwar et al. [16], Shoaib et al. [17] have used integral type contraction to prove their results in various metric spaces. In 1969, Gahler [4] has introduced the notion of 2-metric space. Many research workers such as Gupta et al. [5], Prajapati et al. [13] have established fixed point theorems using integral type contraction conditions in 2-Banach space. We have introduced a new contraction and have given some examples in support of this contraction. Also we have proved some theorems and have given some corollaries.

2. Definition

Gahler [4] has given the definition of 2-metric space as follows:

2-Metric space: Let X be a non-empty set and $d : X \times X \times X \to [0, +\infty)$ be a real valued function which satisfied the following conditions:
(i) for every distinct points x, y there is a point z in X such that $d(x, y, z) \neq 0$;
(ii) $d(x, y, z) = 0$ if any two of three of x, y, z are equal;
(iii) $d(x, y, z) = d(p(x, y, z))$ for all $x, y, z \in X$ and for all permutations $p(x, y, z)$ of x, y, z;
(iv) $d(x, y, z) \leq d(x, y, w) + d(x, w, z) + d(w, y, z)$ for all $x, y, z, w \in X$.
The mapping d satisfying the above properties is called a 2-metric and (X, d) is called a 2-metric space.

Note: Suppose, X be a non-empty set and ρ be a metric on X and d be defined on X by $d(x, y, z) = \rho(x, y)\rho(y, z)\rho(z, x)$. Then d is a 2-metric. So in this case, 2-metric space is the generalization of a metric space.

It is remarcable to know that every convergence sequence in metric space is Cauchy. But by an example it has been shown in [10] that in a 2-metric space a convergence sequence may not be a Cauchy sequence. This is a basic difference between metric space and 2-metric space.

Metric $\rho(x, y)$ means distance between two points x and y and 2-metric $d(x, y, z)$ means area of a triangle formed by the points x, y and z.

F-contraction: Wardowski[19] has defined F-contraction as follows:

Let $\mathbf{F} = \{F : \mathbb{R}_+ \to \mathbb{R}\}$ satisfying the following conditions:
(i) F is strictly increasing;
(ii) for all sequence $\{\alpha_n\} \in \mathbb{R}$, $\lim_{n\to+\infty} \alpha_n = 0$ if and only if $\lim_{n\to+\infty} F(\alpha_n) = -\infty$;
(iii) there exists $0 < k < 1$ such that $\lim_{\alpha\to 0_+} \alpha^n F(\alpha) = 0$.

Then a function $T : X \to X$ is said to be F-contraction if there exists a function $F \in \mathbf{F}$ such that for all $x, y, a \in X$,

$$\tau \in \mathbb{R}_+ \quad \Rightarrow \tau + F(d(Tx, Ty, a)) \le F(d(x, y, a)).$$

3. Preliminaries

Throughout the paper we denote the following:
(i) We write X as a 2-metric space.
(ii) $\Phi = \{\phi : \phi : \mathbb{R}_+ \to \mathbb{R}_+$ is Lebesgue integrable , summable on each compact subset of $\mathbb{R}_+$ satisfying the conditions:
(a) $\int_0^\epsilon \phi(t)dt > 0$ for each ϵ and
(b) $\int_0^{a+b} \phi(t)dt \le \int_0^a \phi(t)dt + \int_0^b \phi(t)dt\}$.
(iii) $F : F \in \mathbf{F}$.

Lemma 1 ([8]). *Let $\phi \in \Phi$ and s_n be a sequence of non-negative reals with $\lim_{n\to+\infty} s_n = s$. Then*

$$\lim_{n\to+\infty} \int_0^{s_n} \phi(t)dt = \int_0^a \phi(t)dt.$$

Lemma 2 ([8]). *Let $\phi \in \Phi$ and s_n be a sequence of non-negative reals. Then*

$$\lim_{n\to+\infty} \int_0^{s_n} \phi(t)dt = 0$$

if and only if $\lim_{n\to+\infty} s_n = 0$.

4. Main part

Samet et al. [15] introduced the concept of $\alpha - adimissible$ in metric space. We have generalized it in 2-metric space as follows:

α-2admissible: Let $T : X \to X$ be a self map on a 2-metric space (X, d) and $\alpha : X \times X \times X \to [0, +\infty)$ be a function. We say that T is an $\alpha - 2admissible$ mapping if $\forall x, y, a \in X,\ \alpha(x, y, a) \ge 1 \Rightarrow \alpha(Tx, Ty, a) \ge 1$.

Now we are to define F_α-contraction as follows:

F_α-contraction: Let (X, d) be a 2-metric space and T be a self map on X. Then T is called a F_α-contraction if for $\tau > 0$,

$$\tau + F(\alpha(x, y, a)d(Tx, Ty, a)) \leq F(d(x, y, a)),$$

where $F \in \mathbf{F}$.

Example 1. Let $X = [0, 2]$ and d be given by $d(x, y, a) = \min\{|x-y|, |y-a|, |a-x|\}$. Then clearly (X, d) is a 2-metric space. Let $\alpha : X \times X \times X \to [0, +\infty)$ be given by

$$\alpha(x, y, a) = \begin{cases} e^{x+y+a}, & \forall x, y, a \in [0, 1.5] \\ \frac{1}{4}, & \text{otherwise.} \end{cases}$$

Clearly $\alpha(x, y, a) \geq 1$, $\forall x, y, a \in X$. Let $T : X \to X$ be given by $Tx = kx$, where $\alpha(x, y, a)k < 1$.

Suppose $d(x, y, a) = |x - y|$. Then $|x - y| \leq |y - a|$
i.e., $y \leq x \leq a$ i.e., $y \leq x \leq a \leq \frac{a}{k}$
i.e., $|x - y| \leq |y - \frac{a}{k}|$ i.e., $k|x - y| \leq k|y - \frac{a}{k}|$.
Similarly $|x - y| \leq |a - x| \Rightarrow k|x - y| \leq k|x - \frac{a}{k}|$.

Now $d(Tx, Ty, a) = \min\{|Tx, Ty|, |Ty-a|, |a-Tx|\} = \min\{|kx-ky|, |ky-a|, |a - kx|\} = k|x - y|$.
Thus $F(\alpha(x, y, a)d(x, y, a)) = F(\alpha(x, y, a)k|x-y|) < F(|x-y|) = F(d(x, y, a))$.
Then there exist a $\tau > 0$ such that

$\tau + F(\alpha(x, y, a)d(x, y, a)) \leq F(d(x, y, a))$.

Thus T is a F_α-contraction mapping.

Lemma 3. *If T is a F_α-contraction, then T is also a F-contraction.*

Proof. If $\alpha(x, y, a) = 1$, there is nothing to proof. So we consider the case $\alpha \neq 0$.

Let us first suppose T is a F_α. Then for $\tau > 0$,

$\tau + F(\alpha(x, y, a)d(Tx, Ty, a)) \leq F(d(x, y, a))$.

Since,

$\tau + F(d(Tx, Ty, a)) \leq \tau + F(\alpha(x, y, a)d(Tx, Ty, a)) \leq F(d(x, y, a))$ i.e., $\tau + F(d(Tx, Ty, a)) \leq F(d(x, y, a))$ i.e., T is a f-contraction.

For the converse part, let T is F-contraction. Then for $\tau > 0$,

$\tau + F(d(Tx, Ty, a)) \leq F(d(x, y, a))$.

Since $F(d(Tx, Ty, a)) < F(\alpha(x, y, a)d(Tx, Ty, a))$, then we cannot find a $\tau > 0$ such that the relation $\tau+F(\alpha(x, y, a)d(Tx, Ty, a)) < \tau+F(d(Tx, Ty, a)) \leq F(d(x, y, a))$ hold.

Thus F-contraction does not imply F_α-contraction. Hence the lemma. ∎

Example 2. Let $X = [0, +\infty)$ and $d : X \times X \times X \to [0, +\infty)$ be given by $d(x, y, a) = \min\{|x-y|, |y-a|, |a-x|\}$. Then (X, d) is a 2-metric space. Let $\alpha : X \times X \times X \to [0, +\infty)$ be defined by

$$\alpha(x, y, a) = \begin{cases} 2\,, & \forall\ x, y, a \in [0, 5]; \\ 0\,, & \text{otherwise} \end{cases}$$

and $Fx = x$. Let T be defined by $Tx = \frac{x}{3} \forall x \in x$. Then for all $x, y, a \in [0, 5]$ where $x < y < a$, $d(x, y, a) = \min\{|x-y|, |y-x|, |a-x|\} = |x-y|$(say). Then

$$\begin{aligned} &|x-y| \leq |y-a| \text{ and } |x-y| \leq |a-x| \\ \text{implies, } &|x-y| \leq |y-3a| \ \text{ and } |x-y| \leq |x-3a| [\text{since } x < y < a < 3a] \\ \text{i.e., } &\frac{1}{3}|x-y| \leq \frac{1}{3}|y-3a| \ \text{ and } \frac{1}{3}|x-y| \leq \frac{1}{3}|x-3a|. \end{aligned}$$

Now,

$$\begin{aligned} d(Tx, Ty, a) &= d(\frac{x}{3}, \frac{y}{3}, a) = \min\{|\frac{x}{3} - \frac{y}{3}|, |\frac{y}{3} - a|, |a - \frac{x}{3}|\} \\ &= \min\{\frac{1}{3}|x-y|, \frac{1}{3}|y-3a|, \frac{1}{3}|x-3a|\} = \frac{1}{3}|x-y|. \end{aligned}$$

Therefore,

$$F(d(Tx, Ty, a)) = F(\frac{1}{3}|x-y|) = \frac{1}{3}|x-y|.$$

So,

$$F(\alpha(x, y, a)d(Tx, Ty, a)) = F(2(\frac{1}{3}|x-y|) = \frac{2}{3}|x-y|.$$

Thus

$$F(d(Tx, Ty, a)) \leq F(\alpha(x, y, a)d(Tx, Ty, a)).$$

Therefore there exists a $\tau = \frac{1}{5}|x-y| > 0$ such that,

$$\begin{aligned} \tau + F(d(Tx, Ty, a)) &\leq \tau + F(\alpha(x, y, a)d(Tx, Ty, a)) \\ &\leq F(|x-y|) = F(d(x, y, a)) \end{aligned}$$

i.e., F_α-contraction T is also a F-contraction.

Clearly, converse is not hold.

If $\alpha(x, y, a) = 1$, then for $\tau = \frac{1}{3}|x-y| > 0$

$$\begin{aligned} \tau + F(\alpha(x, y, a)d(Tx, Ty, a)) &= \tau + F(d(Tx, Ty, a)) = \tau + F(\frac{1}{3}|x-y|) \\ &= \frac{1}{3}|x-y| + \frac{1}{3}|x-y| = \frac{2}{5}|x-y| < |x-y| \\ &= d(x, y, a). \end{aligned}$$

Thus for $\alpha(x, y, a) = 1$, F_α-contraction T is also a F-contraction.

In the next part we have proved some common fixed point theorems.

Theorem 1. *Let (X,d) be a complete 2-metric space and $\{f_n\}_{n=1}^{\infty}$ be a sequence of self-maps satisfying the following relation*

$$\int_0^{d(f_ix,f_jy,a)} \phi(t)dt \leq \int_0^{M(x,y,a)} \phi(t)dt,$$

where $\phi \in \Phi$ and $M(x,y,a) = \alpha \max\{d(x,y,a), d(x,f_ix,a), d(y,f_jy,a)\}$ $+\beta \max\{d(x,f_ix,a), d(x,f_jy,a)\} + \gamma \frac{d(y,f_jy,a)}{1+d(y,f_ix,a)}$, $\alpha+\beta+\gamma<1$. Then $\{f_n\}_{n=1}^{\infty}$ have a unique common fixed point in X.

Proof. Let us construct a sequence $\{x_n\}$ for a fixed $i \in \mathbb{N}$ in X such that $x_{n+1} = f_ix_n, n \in \mathbb{N} \cup \{0\}$, with an initial approximation $x_0 \in X$.

If $x_n = f_ix_n$ i.e., $x_{n+1} = x_n$ then x_n is a common fixed point of $\{f_n\}_{n=1}^{\infty}$. So we assume that $x_{n+1} \neq x_n$.

At first we assume that $\lim\limits_{n \to +\infty} d(x_{n+1}, x_n, a) = 0$.

Since,

$$(1) \quad \int_0^{d(x_{n+1},x_n,a)} \phi(t)dt = \int_0^{d(f_ix_n,f_jx_{n-1},a)} \phi(t)dt \leq \int_0^{M(x_n,x_{n-1},a)} \phi(t)dt,$$

where,

$$\begin{aligned}
(2) \quad & M(x_n, x_{n-1}, a) \\
&= \alpha \max\{d(x_n,x_{n-1},a), d(x_n,f_ix_n,a), d(x_{n-1},f_jx_{n-1},a)\} \\
&\quad + \beta \max\{d(x_n,f_ix_n,a), d(x_n,f_jx_{n-1},a)\} \\
&\quad + \gamma \frac{d(x_{n-1},f_jx_{n-1},a)}{1+d(x_{n-1},f_ix_n,a)} \\
&= \alpha \max\{d(x_n,x_{n-1},a), d(x_n,x_{n+1},a), d(x_{n-1},x_n,a)\} \\
&\quad + \beta \max\{d(x_n,x_{n+1},a), d(x_n,x_n,a)\} \\
&\quad + \gamma \frac{d(x_{n-1},x_n,a)}{1+d(x_{n-1},x_{n+1},a)} \\
&\leq \alpha \max\{d(x_n,x_{n-1},a), d(x_n,x_{n+1},a)\} + \beta d(x_n,x_{n+1},a) \\
&\quad + \gamma d(x_{n-1},x_n,a).
\end{aligned}$$

If $d(x_n,x_{n-1},a) \leq d(x_n,x_{n+1},a)$, then from (2) we get

$$M(x_n,x_{n-1},a) = (\alpha+\beta+\gamma)d(x_n,x_{n+1},a).$$

So from (1) we have

$$\int_0^{d(x_{n+1},x_n,a)} \phi(t)dt \leq \int_0^{(\alpha+\beta+\gamma)d(x_n,x_{n+1},a)} \phi(t)dt$$

$$\text{implies,}\quad d(x_{n+1}, x_n, a) \leq (\alpha+\beta+\gamma) d(x_n, x_{n+1}, a)$$

$$\text{implies,}\quad 1 \leq \alpha+\beta+\gamma, \quad \text{a contradiction.}$$

Therefore $d(x_n, x_{n+1}, a) \leq d(x_n, x_{n-1}, a)$. Thus $\{d(x_{n+1}, x_n, a)\}$ is a monotone decreasing sequence of real numbers and bounded below.

Suppose $\lim\limits_{n\to+\infty} d(x_{n+1}, x_n, a) = r$.

Then

$$\begin{aligned}
\int_0^r \phi(t)dt &= \lim_{n\to+\infty} \int_0^{d(x_{n+1},x_n,a)} \phi(t)dt \\
&\leq \lim_{n\to+\infty} \int_0^{(\alpha+\beta+\gamma)d(x_n,x_{n+1},a)} \phi(t)dt \ [\text{using}(2)] \\
&\leq \lim_{n\to+\infty} \int_0^{(\alpha+\beta+\gamma)^2 d(x_n,x_{n+1},a)} \phi(t)dt \\
&\ \ \vdots \\
&\leq \lim_{n\to+\infty} \int_0^{(\alpha+\beta+\gamma)^n d(x_n,x_{n+1},a)} \phi(t)dt \leq 0
\end{aligned}$$

$$\text{implies, } r = 0.$$

Thus $\lim\limits_{n\to+\infty} d(x_{n+1}, x_n, a) = 0$. Next, let $n, m \in \mathbb{N}; n > m$.

Since

$$d(x_n, x_m, a) \leq d(x_n, x_m, x_{n-1}) + d(x_n, x_{n-1}, a) + d(x_{n-1}, x_m, a)$$

taking $\lim\limits_{n,m\to+\infty}$ we get from above

$$\begin{aligned}
\lim_{n,m\to+\infty} d(x_n, x_m, a) &\leq \lim_{n,m\to+\infty} d(x_n, x_m, x_{n-1}) + \lim_{n,m\to+\infty} d(x_n, x_{n-1}, a) \\
&\quad + \lim_{n,m\to+\infty} d(x_{n-1}, x_m, a) \\
&= \lim_{n,m\to+\infty} d(x_{n-1}, x_m, a) \\
&\ \ \vdots \\
&\leq \lim_{n,m\to+\infty} d(x_m, x_m, a) = 0.
\end{aligned}$$

Thus $\{x_n\}$ is a Cauchy sequence in X. Since X is a complete 2-metric space, there exists a $x \in X$ such that $\lim\limits_{n\to+\infty} d(x_n, x, a) = 0$. Next we show that x is a common fixed point of $\{f_n\}_{n=1}^{\infty}$.

Since,

$$\begin{aligned}
\lim_{n\to+\infty} d(f_i x, x, a) &\leq \lim_{n\to\infty} d(f_i, x, x_n) + \lim_{n\to+\infty} d(f_i x, x_n, a) \\
&\quad + \lim_{n\to+\infty} d(x_n, x, a) = \lim_{n\to+\infty} d(f_i x, x_n, a).
\end{aligned}$$

Thus

$$(3) \quad \int_0^{d(f_i x, x, a)} \phi(t)dt \leq \int_0^{d(f_i x, x_n, a)} \phi(t)dt = \int_0^{d(f_i x, f_j x_{n-1}, a)} \phi(t)dt \leq \int_0^{M(x, x_{n-1}, a)} \phi(t)dt,$$

where,

$$\begin{aligned} M(x, x_{n-1}, a) &= \alpha \max\{d(x, x_{n-1}, a), d(x, f_i x, a), d(x_{n-1}, f_j x_{n-1}, a)\} \\ &\quad + \beta \max\{d(x, f_i x, a), d(x, f_j x_{n-1}, a)\} \\ &\quad + \gamma \frac{d(x_{n-1}, f_j x_{n-1}, a)}{1 + d(x_{n-1}, f_i x, a)} \\ &\leq \alpha \max\{d(x, x_{n-1}, a), d(x, f_i x, a), d(x_{n-1}, x_n, a)\} \\ &\quad + \beta \max\{d(x, f_i x, a), d(x, x_n, a)\} + \gamma d(x_{n-1}, x_n, a) \end{aligned}$$

Therefore

$$\begin{aligned} \lim_{n \to +\infty} M(x, x_n, a) &\leq \lim_{n \to +\infty} \alpha \max\{d(x, x_n, a), d(x, f_i x, a), d(x_n, x_{n+1}, a)\} \\ &\quad + \lim_{n \to +\infty} \beta \max\{d(x, f_i x, a), d(x, x_{n+1}, a)\} \\ &\quad + \lim_{n \to +\infty} \gamma d(x_n, x_{n+1}, a) \\ &= \alpha d(x, f_i x, a) + \beta d(f_i x, x, a) + \gamma .0. \end{aligned}$$

Therefore from (3) we get

$$\int_0^{d(f_i x, x, a)} \phi(t)dt \leq \int_0^{(\alpha+\beta) d(f_i x, x, a)} \phi(t)dt.$$

Thus

$$\begin{aligned} & d(f_i x, x, a) \leq (\alpha + \beta) d(f_i x, x, a) \\ & \text{implies,} \quad d(f_i x, x, a) = 0 \\ & \text{implies,} \quad f_i x = x. \end{aligned}$$

Thus x is a common fixed point of $\{f_n\}_{n=1}^{\infty}$. Let y be another common fixed point. Then

$$(4) \qquad \int_0^{d(x, y, a)} \phi(t)dt = \int_0^{d(f_i x, f_j y, a)} \phi(t)dt \leq \int_0^{M(x, y, a)} \phi(t)dt,$$

where,

$$\begin{aligned} M(x,y,a) &= \alpha\max\{d(x,y,a),d(x,f_ix,a),d(y,f_jy,a)\} \\ &\quad +\beta\max\{d(x,f_ix,a),d(x,f_jy,a)\}+\gamma\frac{d(y,f_jy,a)}{1+d(y,f_ix,a)} \\ &\leq \alpha\max\{d(x,y,a),d(x,x,a),d(y,y,a)\} \\ &\quad +\beta\max\{d(x,x,a),d(x,y,a)\}+\gamma d(y,y,a) \\ &= \alpha d(x,y,a)+\beta d(x,y,a)+\gamma.0. \end{aligned}$$

Therefore from (4) we get

$$\int_0^{d(x,y,a)}\phi(t)dt=\int_0^{(\alpha+\beta)d(x,y,a)}\phi(t)dt$$

$$\begin{aligned} &\text{implies,}\quad d(x,y,a)\leq(\alpha+\beta)d(x,y,a) \\ &\text{implies,}\quad d(x,y,a)=0 \\ &\text{implies,}\quad x=y. \end{aligned}$$

Thus x is a unique common fixed point of $\{f_n\}_{n=1}^{\infty}$.
Hence the theorem. ■

Corollary 1. *Let (X,d) be a complete 2-metric space, f_1 and f_2 be a two self-maps satisfying the following relation*

$$\int_0^{d(f_1x,f_2y,a)}\phi(t)dt\leq\int_0^{M(x,y,a)}\phi(t)dt,$$

where $\phi\in\Phi$ and $M(x,y,a)=\alpha\max\{d(x,y,a),d(x,f_1x,a),d(y,f_2y,a)\}$ $+\beta\max\{d(x,f_1x,a),d(x,f_2y,a)\}+\gamma\frac{d(y,f_2y,a)}{1+d(y,f_1x,a)}$, $\alpha+\beta+\gamma<1$. Then f_1 and f_2 have a unique common fixed point in X.

Proof. Putting $f_i=f_1$ and $f_j=f_2$ in the Theorem 1 we get the result. ■

Corollary 2. *Let (X,d) be a complete 2-metric space, f be a self-map satisfying the following relation*

$$\int_0^{d(fx,fy,a)}\phi(t)dt\leq\int_0^{M(x,y,a)}\phi(t)dt,$$

where $\phi\in\Phi$ and $M(x,y,a)=\alpha\max\{d(x,y,a),d(x,fx,a),d(y,fy,a)\}+$ $\beta\max\{d(x,fx,a),d(x,fy,a)\}+\gamma\frac{d(y,fy,a)}{1+d(y,fx,a)}$, $\alpha+\beta+\gamma<1$. Then f have a unique fixed point in X.

Proof. Putting $f_i = f$ in the Theorem 1 we get the result. ∎

Example 3. Let, $X = [0, 1)$ and d be defined by $d(x, y, a) = \min\{|x-y|, |y-a|, |a-x|\}$ where $x, y, a \in X$. Then clearly d is a 2-metric and so (X, d) is a 2-metric space.

Now let us consider the sequence of functions $\{f_n\}_{n=1}^{\infty}$ given by $f_i(x) = x^i$ and the sequence $\{x_n\}$ given by $x_{n+1} = f_i(x_n)$ for a fixed $i \in \mathbb{N}$ with the initial approximation $x_0 \in X$.
Thus for $1 \le i < j \in \mathbb{N}$, $f_i(x_n) = x_0^{i^{n+1}}$, $f_j(x_n) = x_0^{j^{n+1}}$.

Now $d(f_i x_n, f_j x_{n-1}, a) = d(x_0^{i^{n+1}}, x_0^{j^n}, a)$.
Again,

$$\begin{aligned}
&M(x_n, x_{n-1}, a)\\
&= \alpha \max\{d(x_n, x_{n-1}, a), d(x_n, f_i x_n, a), d(x_{n-1}, f_j x_{n-1}, a)\}\\
&\quad + \beta \max\{d(x_n, f_i x_n, a), d(x_n, f_j x_{n-1}, a)\} + \gamma \frac{d(x_{n-1}, f_j x_{n-1}, a)}{1 + d(x_{n-1}, f_i x_n, a)}\\
&\quad [\text{where } \alpha, \beta, \gamma \ge 0 \text{ and } \alpha + \beta + \gamma < 1]\\
&= \alpha \max\{d(f_i x_{n-1}, f_j x_{n-2}, a), d(f_i x_n, x_n, a), d(f_i x_{n-2}, f_j x_{n-1}, a)\}\\
&\quad + \beta \max\{d(f_i x_n, x_n, a), d(f_i x_{n-1}, f_j x_{n-1}, a)\} + \gamma \frac{d(f_i x_{n-2}, f_j x_{n-1}, a)}{1 + d(f_i x_n, x_{n-1}, a)}\\
&= \alpha \max\{d(f_i x_{n-1}, f_j x_{n-2}, a), d(f_i x_n, f_j x_{n-1}, a), d(f_i x_{n-2}, f_j x_{n-1}, a)\}\\
&\quad + \beta \max\{d(f_i x_n, f_j x_{n-1}, a), d(f_i x_{n-1}, f_j x_{n-1}, a)\}\\
&\quad + \gamma \frac{d(f_i x_{n-2}, f_j x_{n-1}, a)}{1 + d(f_i x_n, f_j x_{n-2}, a)}\\
&\le \alpha \max\{d(x_0^{i^n}, x_0^{j^{n-1}}, a), d(x_0^{i^{n+1}}, x_0^{j^n}, a), d(x_0^{i^{n-1}}, x_0^{j^n}, a)\}\\
&\quad + \beta \max\{d(x_0^{i^{n+1}}, x_0^{j^n}, a), d(x_0^{i^n}, x_0^{j^n}, a)\} + \gamma \frac{d(x_0^{i^{n-1}}, x_0^{j^n}, a)}{1 + d(x_0^{i^{n+1}}, x_0^{j^{n-1}}, a)}\\
&\le \alpha d(x_0^{i^{n-1}}, x_0^{j^n}, a) + \beta d(x_0^{i^{n+1}}, x_0^{j^n}, a) + \gamma d(x_0^{i^{n-1}}, x_0^{j^n}, a)\\
&\le (\alpha + \beta + \gamma) d(x_0^{i^{n-1}}, x_0^{j^n}, a)\\
&\le d(x_0^{i^{n-1}}, x_0^{j^n}, a).
\end{aligned}$$

Thus $d(f_i x_n, f_j x_{n-1}, a) \le d(x_0^{i^{n+1}}, x_0^{j^n}, a) \le M(x_n, x_{n-1}, a)$.
Therefore,

$$\int_0^{d(f_i x_n, f_j x_{n-1}, a)} \phi(t)dt \le \int_0^{M(x_n, x_{n-1}, a)} \phi(t)dt.$$

So by Theorem 1, X has a unique fixed point $x_0 = 0$.

Theorem 2. *Let (X, d) be a complete 2-metric space and $\{f_n\}_{n=1}^{\infty}$ be a sequence of self-maps satisfying the relation,*

$$\int_0^{d(f_ix,f_jy,a)} \phi(t)dt \leq \int_0^{\psi(x,y,a)} \phi(t)dt,$$

where $\phi \in \Phi$ and $\psi(x, y, a) = \alpha d(x, y, a) + \beta \max\{d(x, f_ix, a), d(x, f_jy, a)\}+ \gamma \min\{d(y, f_jy, a), d(y, f_ix, a)\}; \alpha + \beta + \gamma < 1$. Then $\{f_n\}_{n=1}^{\infty}$ have a unique common fixed point in X.

Proof. Let $\{x_n\}$ be a sequence such that for a fixed $i \in \mathbb{N}$, $x_{n+1} = f_ix_n$ for $n \in \mathbb{N} \cup \{0\}$ where $x_0 \in X$ is an initial approximation. If $x_{n+1} = x_n$ i.e., $f_ix_n = x_n$, then x_n is a common fixed point of $\{f_n\}_{n=1}^{\infty}$ and this completes the theorem. So we assume that $x_{n+1} \neq x_n$.

At first we will show that $\lim_{n\to+\infty} d(x_{n+1}, x_n, a) = 0$.
Since

$$(5) \quad \int_0^{d(x_{n+1},x_n,a)} \phi(t)dt = \int_0^{d(f_ix_n,f_jx_{n-1},a)} \phi(t)dt \leq \int_0^{\psi(x_n,x_{n-1},a)} \phi(t)dt,$$

where,

$$\begin{aligned} \psi(x_n, x_{n-1}, a) &= \alpha d(x_n, x_{n-1}, a) + \beta \max\{d(x_n, f_ix_n, a), d(x_n, f_jx_{n-1}, a)\} \\ &\quad + \gamma \min\{d(x_{n-1}, f_jx_{n-1}, a), d(x_{n-1}, f_ix_n, a)\} \\ &= \alpha d(x_n, x_{n-1}, a) + \beta \max\{d(x_n, x_{n+1}, a), d(x_n, x_n, a)\} \\ &\quad + \gamma \min\{d(x_{n-1}, x_n, a), d(x_{n-1}, x_{n+1}, a)\} \\ &= \alpha d(x_n, x_{n-1}, a) + \beta d(x_n, x_{n+1}, a) + \gamma d(x_{n-1}, x_n, a) \end{aligned}$$

If $d(x_{n-1}, x_n, a) \leq d(x_n, x_{n+1}, a)$, then $\psi(x_n, x_{n-1}, a) \leq (\alpha + \beta + \gamma) \times d(x_{n+1}, x_n, a)$. From (5) we get

$$\int_0^{d(x_{n+1},x_n,a)} \phi(t)dt \leq \int_0^{(\alpha+\beta+\gamma)d(x_{n+1},x_n,a)} \phi(t)dt$$

implies, $d(x_{n+1}, x_n, a) \leq (\alpha + \beta + \gamma)d(x_{n+1}, x_n, a)$
implies, $1 \leq \alpha + \beta + \gamma$, a contradiction.

Therefore $d(x_{n+1}, x_n, a) \leq d(x_n, x_{n-1}, a)$. Thus $\{d(x_{n+1}, x_n, a)\}$ is a sequence of real numbers monotone decreasing and bounded below.

Suppose $\lim\limits_{n\to+\infty} d(x_{n+1}, x_n, a) = r$.
Now

$$\begin{aligned} (6) \int_0^r \phi(t)dt &= \lim_{n\to+\infty} \int_0^{d(x_{n+1},x_n,a)} \phi(t)dt = \lim_{n\to+\infty} \int_0^{d(f_ix_n,f_jx_{n-1},a)} \phi(t)dt \\ &\leq \lim_{n\to+\infty} \int_0^{\psi(x_n,x_{n-1},a)} \phi(t)dt, \end{aligned}$$

where

$$\begin{aligned}\psi(x_n, x_{n-1}, a) &= \alpha d(x_n, x_{n-1}, a) + \beta \max\{d(x_n, f_i x_n, a), d(x_n, f_j x_{n-1}, a)\} \\ &\quad + \gamma \min\{d(x_{n-1}, f_j x_{n-1}, a), d(x_{n-1}, f_i x_n, a)\} \\ &= \alpha d(x_n, x_{n-1}, a) + \beta \max\{d(x_n, x_{n+1}, a), d(x_n, x_n, a)\} \\ &\quad + \gamma \min\{d(x_{n-1}, x_n, a), d(x_{n-1}, x_{n+1}, a)\} \\ &= \alpha d(x_n, x_{n-1}, a) + \beta d(x_n, x_{n+1}, a) + \gamma d(x_{n-1}, x_n, a) \\ &\leq (\alpha + \beta + \gamma) d(x_n, x_{n-1}, a).\end{aligned}$$

Therefore from (6) we have

$$\begin{aligned}\int_0^r \phi(t)dt &= \lim_{n\to+\infty} \int_0^{(\alpha+\beta+\gamma)d(x_n,x_{n-1},a)} \phi(t)dt \\ &\leq \lim_{n\to+\infty} \int_0^{(\alpha+\beta+\gamma)^2 d(x_n,x_{n-1},a)} \phi(t)dt \\ &\quad\vdots \\ &\leq \lim_{n\to+\infty} \int_0^{(\alpha+\beta+\gamma)^n d(x_1,x_0,a)} \phi(t)dt = 0 \\ &\text{implies,} \quad r = 0.\end{aligned}$$

Thus $\lim\limits_{n\to+\infty} d(x_{n+1}, x_n, a) = 0$.

Next, let $n, m \in \mathbb{N}$; $n > m$.
Since $d(x_n, x_m, a) \leq d(x_n, x_m, X_{n-1}) + d(x_n, x_{n-1}, a) + d(x_{n-1}, x_m, a)$.
Taking limit as $n \to +\infty$ we get,

$$\begin{aligned}\lim_{n\to+\infty} d(x_n, x_m, a) &\leq \lim_{n\to+\infty} d(x_n, x_m, X_{n-1}) + \lim_{n\to+\infty} d(x_n, x_{n-1}, a) \\ &\quad + \lim_{n\to+\infty} d(x_{n-1}, x_m, a) \\ &= \lim_{n\to+\infty} d(x_{n-1}, x_m, a) \\ &\quad\vdots \\ &= \lim_{n\to+\infty} d(x_m, x_m, a) = 0.\end{aligned}$$

Thus $\{x_n\}$ is a Cauchy sequence in X. Since X is complete space, there exists an $x \in X$ such that

$$\lim_{n\to+\infty} x_n = x \text{ i.e., } \lim_{n\to+\infty} d(x_n, x, a) = 0.$$

Again,

$$\begin{aligned}\lim_{n\to+\infty} d(f_i x, x, a) &\leq \lim_{n\to+\infty} [d(f_i x, x, x_n) + d(f_i x, x_n, a) + d(x_n, x, a)] \\ &= \lim_{n\to+\infty} d(f_i x, x_n, a).\end{aligned}$$

Therefore

$$\lim_{n\to+\infty}\int_0^{d(f_ix,x,a)}\phi(t)dt \leq \lim_{n\to+\infty}\int_0^{d(f_ix,x_n,a)}\phi(t)dt = \lim_{n\to+\infty}\int_0^{d(f_ix,f_jx_{n-1},a)}\phi(t)dt \leq \lim_{n\to+\infty}\int_0^{\psi(x,x_{n-1},a)}\phi(t)dt, \tag{7}$$

where,

$$\begin{aligned}\psi(x,x_{n-1},a) &= \alpha d(x,x_{n-1},a)+\beta\max\{d(x,f_ix,a),d(x,f_jx_{n-1},a)\}\\&\quad+\gamma\min\{d(x_{n-1},f_jx_{n-1},a),d(x_{n-1},f_ix,a)\}\\&= \alpha d(x,x_{n-1},a)+\beta\max\{d(x,f_ix,a),d(x,x_n,a)\}\\&\quad+\gamma\min\{d(x_{n-1},x_n,a),d(x_{n-1},f_ix,a)\}.\end{aligned}$$

Therefore,

$\lim_{n\to+\infty}\psi(x,x_{n-1},a)=\alpha.0+\beta d(x,f_ix,a)+\gamma.0=\beta d(x,f_ix,a)$.

From (7) we get

$$\lim_{n\to+\infty}\int_0^{d(f_ix,x,a)}\phi(t)dt \leq \lim_{n\to+\infty}\int_0^{\beta d(x,f_ix,a)}\phi(t)dt$$

implies, $d(f_ix,x,a)\leq \beta d(f_ix,x,a)$

implies, $d(f_ix,x,a)=0$ i.e., $f_ix=x$.

Thus x is a common fixed point of $\{f_n\}_{n=1}^{\infty}$.

Let us suppose that y be another common fixed point.

Since

$$\int_0^{d(x,y,a)}\phi(t)dt=\int_0^{d(f_ix,f_jy,a)}\phi(t)dt\leq\int_0^{\psi(x,y,a)}\phi(t)dt, \tag{8}$$

where

$$\begin{aligned}\psi(x,y,a) &= \alpha d(x,y,a)+\beta\max\{d(x,f_ix,a),d(x,f_jy,a)\}\\&\quad+\gamma\min\{d(y,f_jy,a),d(y,f_ix,a)\}\\&= \alpha d(x,y,a)+\beta\max\{d(x,x,a),d(x,y,a)\}\\&\quad+\gamma\min\{d(y,y,a),d(y,x,a)\}\\&= \alpha d(x,y,a)+\beta d(x,y,a)+\gamma.0\\&= (\alpha+\beta)d(x,y,a).\end{aligned}$$

Therefore from (8) we get,

$$\int_0^{d(x,y,a)}\phi(t)dt\leq\int_0^{(\alpha+\beta)d(x,y,a)}\phi(t)dt$$

implies, $d(x,y,a) \leq (\alpha+\beta)d(x,y,a)$
implies, $d(x,y,a) = 0$
implies, $x = y$.

This completes the theorem. ■

Corollary 3. *Let (X,d) be a complete 2-metric space and f_1 and f_2 be a two self-maps satisfying the relation,*

$$\int_0^{d(f_1x,f_2y,a)} \phi(t)dt \leq \int_0^{\psi(x,y,a)} \phi(t)dt,$$

where $\phi \in \Phi$ and $\psi(,x,y,a) = \alpha d(x,y,a) + \beta \max\{d(x,f_1x,a), d(x,f_2y,a)\} + \gamma \min\{d(y,f_2y,a), d(y,f_1x,a)\}; \alpha+\beta+\gamma < 1$. Then f_1 and f_2 have a unique common fixed point in X.

Proof. Putting $f_i = f_1$ and $f_j = f_2$ in the above Theorem 2 the corollary hold. ■

Corollary 4. *Let (X,d) be a complete 2-metric space and f be a self-map satisfying the relation,*

$$\int_0^{d(fx,fy,a)} \phi(t)dt \leq \int_0^{\psi(x,y,a)} \phi(t)dt,$$

where $\phi \in \Phi$ and $\psi(,x,y,a) = \alpha d(x,y,a) + \beta \max\{d(x,fx,a), d(x,fy,a)\} + \gamma \min\{d(y,fy,a), d(y,fx,a)\}; \alpha+\beta+\gamma < 1$. Then f have a unique fixed point in X.

Proof. Putting $f_i = f_1$ and $f_j = f_2$ in the above Theorem 2 the corollary hold. ■

Theorem 3. *Let (X,d) be a complete 2-metric space and $\{f_n\}_{n=1}^{\infty}$ be a sequence of self-maps such that each of f_n be F-contraction and satisfies the relation*

$$\int_0^{\tau+F(d(f_ix,f_jy,a))} \phi(t)dt \leq \int_0^{F(d(x,y,a))} \phi(t)dt$$

for $\tau > 0$. Then $\{f_n\}_{n=1}^{\infty}$ have a unique common fixed point in X.

Proof. Let $x_0$be an initial approximation. Let for fixed $i \in \mathbb{N}$ the sequence $\{x_n\}$ be such that $x_{n+1} = f_ix_n$ for all $n \in \mathbb{N}\cup\{0\}$.
If $x_n = f_ix_n$ i.e., $x_n = x_{n+1}$, then x_n is a common fixed point of $\{f_n\}_{n=1}^{\infty}$.

So we assume that $x_n \neq x_{n+1}$.
Since

$$\begin{aligned}
\int_0^{F(d(x_{n+1},x_n,a))} \phi(t)dt &\leq \int_0^{\tau+F(d(x_{n+1},x_n,a))} \phi(t)dt \\
&= \int_0^{F(d(f_i x_n, f_j x_{n-1},a))} \phi(t)dt \\
&\leq \int_0^{F(d(x_n,x_{n-1},a))} \phi(t)dt
\end{aligned}$$

implies, $F(d(x_{n+1}, x_n, a)) \leq F(d(x_n, x_{n-1}, a))$
implies, $d(x_{n+1}, x_n, a) \leq d(x_n, x_{n-1}, a)$.

Thus $\{d(x_{n+1}, x_n, a)\}$ is a monotone decreasing bounded below sequence of real numbers and hence convergent.
Since

$$\tau + F(d(x_{n+1}, x_n, a)) = \tau + F(d(f_i x_n, f_j x_{n-1}, a)) \leq F(d(x_n, x_{n-1}, a)),$$

we have

$$\begin{aligned}
\int_0^{F(d(x_{n+1},x_n,a))} \phi(t)dt &\leq \int_0^{F(d(x_n,x_{n-1},a))-\tau} \phi(t)dt \\
&\leq \int_0^{F(d(x_{n-1},x_{n-2},a))-2\tau} \phi(t)dt \\
&\vdots \\
&\leq \int_0^{F(d(x_1,x_0,a))-n\tau} \phi(t)dt.
\end{aligned}$$

Thus

$$F(d(x_{n+1}, x_n, a)) \leq F(d(x_1, x_0, a)) - n\tau.$$

Taking limit as $n \to +\infty$ we get from above
$\lim\limits_{n\to+\infty} F(d(x_{n+1}, x_n, a)) = -\infty$ which implies, $\lim\limits_{n\to+\infty} d(x_{n+1}, x_n, a) = 0$.
Since (X, d) is 2-metric space, we have for $n, m \in \mathbb{N}, n > m$,

$$\begin{aligned}
\lim_{n,m\to+\infty} \int_0^{F(d(x_n,x_m,a))} \phi(t)dt &= \lim_{n,m\to+\infty} \int_0^{F(d(f_i x_{n-1},f_j x_{m-1},a))} \phi(t)dt \\
&\leq \lim_{n,m\to+\infty} \int_0^{F(d(x_{n-1},x_{m-1},a))-\tau} \phi(t)dt \\
&\leq \lim_{n,m\to+\infty} \int_0^{F(d(x_{n-2},x_{m-2},a))-2\tau} \phi(t)dt \\
&\vdots \\
&\leq \lim_{n,m\to+\infty} \int_0^{F(d(x_{n-m-1},x_0,a))-(m+1)\tau} \phi(t)dt
\end{aligned}$$

implies,
$\lim_{n,m\to\infty} F(d(x_n, x_m, a)) \leq \lim_{n,m\to+\infty} F(d(x_{n-m-1}, x_0, a)) - (m+1)\tau = -\infty$
implies, $\lim_{n,m\to+\infty} d(x_n, x_m, a) = 0$.
Therefore $\{x_n\}$ is a Cauchy sequence. Since (X, d) is a complete, there exists an $x \in X$ such that $\lim_{n\to+\infty} d(x_n, x, a) = 0$.
Again,

$$\begin{aligned}
\lim_{n\to+\infty}\int_0^{F(d(f_i x, x_n, a))}\phi(t)dt &= \lim_{n\to+\infty}\int_0^{F(d(f_i x, f_j x_{n-1}, a))}\phi(t)dt \\
&\leq \lim_{n\to+\infty}\int_0^{F(d(x, x_{n-1}, a))-\tau}\phi(t)dt \\
&\leq \lim_{n\to\infty}\int_0^{F(d(x, x_{n-1}, a))}\phi(t)dt
\end{aligned}$$

$$\begin{aligned}
&\text{implies,} \quad \lim_{n\to+\infty} F(d(f_i x, x_n, a)) \leq \lim_{n\to+\infty} F(d(x, x_{n-1}, a)) \\
&\text{implies,} \quad \lim_{n\to+\infty} d(f_i x, x_n, a)! \leq \lim_{n\to+\infty} d(x, x_{n-1}, a) = 0 \\
&\text{implies,} \quad \lim_{n\to+\infty} d(f_i x, x_n, a) = 0 \ \text{ i.e., } f_i x = \lim_{n\to+\infty} x_n = x.
\end{aligned}$$

Thus x is common fixed point of $\{f_n\}_{n=1}^{\infty}$.
Suppose $y \neq x$ be another common fixed point.
Then

$$\begin{aligned}
\int_0^{F(d(x,y,a))}\phi(t)dt &\leq \int_0^{\tau+F(d(x,y,a))}\phi(t)dt \\
&= \int_0^{\tau+F(d(f_i x, f_j y, a))}\phi(t)dt \\
&\leq \int_0^{F(d(x,y,a))}\phi(t)dt,
\end{aligned}$$

a contradiction.
Therefore, $x = y$. Hence the theorem. ■

Corollary 5. *Let (X, d) be a complete 2-metric space and f_1 and f_2 be a two of self-maps such that each of f_1 and f_2 be F-contraction and satisfies the relation*

$$\int_0^{\tau+F(d(f_1 x, f_2 y, a))}\phi(t)dt \leq \int_0^{F(d(x,y,a))}\phi(t)dt$$

for $\tau > 0$. Then f_1 and f_2 have a unique common fixed point in X.

Proof. From Theorem 3 by putting $f_i = f_1$ and $f_j = f_2$, we get the result. ■

Corollary 6. *Let (X, d) be a complete 2-metric space and f be a of self-map such that f be F-contraction and satisfies the relation*

$$\int_0^{\tau+F(d(fx,fy,a))} \phi(t)dt \leq \int_0^{F(d(x,y,a))} \phi(t)dt$$

for $\tau > 0$. Then f have a unique fixed point in X.

Proof. From Theorem 3 by putting $f_i = f$, we get the result. ■

Theorem 4. *Let (X, d) be a complete 2-metric space and $\{f_n\}_{n=1}^{\infty}$ be a sequence of self-maps such that each of f_n be F_α-contraction and satisfies the relation*

$$\int_0^{\tau+F(\alpha(x,y,a)d(f_ix,f_jy,a))} \phi(t)dt \leq \int_0^{F(d(x,y,a))} \phi(t)dt$$

for $\tau > 0$. Then $\{f_n\}_{n=1}^{\infty}$ have a unique common fixed point in X.

Proof. With an initial approximation $x_0 \in X$, let $\{x_n\}$ be a sequence such that $x_{n+1} = f_i x_n, n \in \mathbb{N} \cup \{0\}$ for a fixed $i \in \mathbb{N}$.

If $x_{n+1} = x_n$ i.e., $x_n = f_i x_n$, then x_n is a common fixed point of $\{f_n\}_{n=1}^{\infty}$. So we assume that $x_{n+1} \neq x_n$.
Now

$$\begin{aligned}
(9) \qquad & \int_0^{F(\alpha(x_n,x_{n-1},a)d(x_{n+1},x_n,a))} \phi(t)dt \\
&= \int_0^{\tau+F(\alpha(x_n,x_{n-1},a)d(f_ix_n,f_jx_{n-1},a))-\tau} \phi(t)dt \\
&\leq \int_0^{F(d(x_n,x_{n-1},a))-\tau} \phi(t)dt \\
&\leq \int_0^{F(\alpha(x_{n-1},x_{n-2},a)d(f_ix_{n-1},f_jx_{n-2},a))-\tau} \phi(t)dt \\
&\leq \int_0^{F(d(x_{n-1},x_{n-2},a))-2\tau} \phi(t)dt \\
&\vdots \\
&\leq \int_0^{F(d(x_1,x_0,a))-n\tau} \phi(t)dt.
\end{aligned}$$

Therefore

$$F(\alpha(x_n, x_{n-1}, a)d(x_{n+1}, x_n, a)) \leq F(d(x_1, x_0, a)) - n\tau.$$

Since by definition

$$\lim_{n\to+\infty} F(\alpha(x_n, x_{n-1}, a)d(x_{n+1}, x_n, a)) = -\infty$$

$$\text{implies,} \quad \lim_{n\to+\infty} \alpha(x_n, x_{n-1}, a)d(x_{n+1}, x_n, a) = 0$$

and we have from (9)

$$\lim_{n\to+\infty} \alpha(x_n, x_{n-1}, a)d(x_{n+1}, x_n, a) = 0$$

$$\text{i.e.,} \quad \lim_{n\to+\infty} d(x_{n+1}, x_n, a) = 0.$$

Again for $n > m \in \mathbb{N}$ we get,

$$\begin{aligned}
\int_0^{F(\alpha(x_n,x_m,a)d(x_{n+1},x_{m+1},a))} \phi(t)dt
&= \int_0^{\tau+F(\alpha(x_n,x_m,a)d(f_ix_n,f_jx_m,a))-\tau} \phi(t)dt \\
&\le \int_0^{F(d(x_n,x_m,a))-\tau} \phi(t)dt \\
&\le \int_0^{F(\alpha(x_{n-1},x_{m-1},a)d(f_ix_{n-1},f_jx_{m-1},a))-\tau} \phi(t)dt \\
&\le \int_0^{F(d(x_{n-1},x_{m-1},a))-2\tau} \phi(t)dt \\
&\vdots \\
&\le \int_0^{F(d(x_1,x_0,a))-n\tau} \phi(t)dt.
\end{aligned}$$

Therefore $F(\alpha(x_n, x_m, a)d(x_{n+1}, x_{m+1}, a)) \le F(d(x_1, x_0, a)) - n\tau$.
Taking limit as $n \to +\infty$ and by definition of F, we get

$$\lim_{n,m\to+\infty} \alpha(x_n, x_m, a)d(x_{n+1}, x_{m+1}, a) = 0$$

$$\text{i.e.,} \quad \lim_{n,m\to+\infty} d(x_{n+1}, x_{m+1}, a) = 0.$$

Thus $\{x_n\}$ is a Cauchy sequence in X. Since (X, d) is a complete 2-metric space, there exists an $x \in X$ such that $\lim_{n\to+\infty} d(x_n, x, a) = 0$.

Next,we are to show x is a fixed point.
Since

$$\begin{aligned}
\lim_{n\to+\infty} d(f_ix, x, a) &\le \lim_{n\to+\infty} d(f_ix, x, x_n) + \lim_{n\to+\infty} d(f_ix, x_n, a) \\
&\quad + \lim_{n\to+\infty} d(x_n, x, a) = \lim_{n\to+\infty} d(f_ix, x_n, a).
\end{aligned}$$

Now

$$\begin{aligned} F(\alpha(f_i x, x_n, a) d(f_i x, x_{n+1}, a)) &= F(\alpha(f_i x, x_n, a) d(f_i x, f_j x_n, a)) \\ &\leq F(d(x, x_n, a)) - \tau \\ &\leq F(d(x, x_n, a)). \end{aligned}$$

Taking limit as $n \to \infty$ we get

$\lim_{n\to\infty} F(\alpha(f_i x, x_n, a) d(f_i x, x_{n+1}, a)) \leq \lim_{n\to\infty} F(d(x, x_n, a))$

implies, $\lim_{n\to\infty} \alpha(f_i x, x_n, a) d(f_i x, x_{n+1}, a) \leq \lim_{n\to+\infty} d(x, x_n, a) = 0$

i.e., $\lim_{n\to+\infty} d(f_i x, x_{n+1}, a) = 0$

implies, $f_i x = \lim_{n\to+\infty} x_{n+1} = x$.

Therefore x is a common fixed point of $\{f_n\}_{n=1}^{\infty}$.

To show the uniqueness, let y be another common fixed point.
Since

$$\begin{aligned} \int_0^{F(\alpha(x,y,a)d(x,y,a))} \phi(t)dt &= \int_0^{\tau + F(\alpha(x,y,a)d(f_i x, f_j y, a)) - \tau} \phi(t)dt \\ &\leq \int_0^{F(d(x,y,a)) - \tau} \phi(t)dt \\ &\leq \int_0^{F(d(x,y,a))} \phi(t)dt. \end{aligned}$$

Therefore $F(\alpha(x, y, a) d(x, y, a)) \leq F(d(x, y, a))$

implies, $\alpha(x, y, a) d(x, y, a) \leq d(x, y, a)$

implies, $d(x, y, a) = 0$

implies, $x = y$.

Hence the result. ■

Corollary 7. *Let (X, d) be a complete 2-metric space and f_1 and f_2 be a two self-maps such that each of f_1 and f_2 be F_α-contraction and satisfies the relation*

$$\int_0^{\tau + F(\alpha(x,y,a)d(f_1 x, f_2 y, a))} \phi(t)dt \leq \int_0^{F(d(x,y,a))} \phi(t)dt$$

for $\tau > 0$. Then f_1 and f_2 have a unique common fixed point in X.

Proof. To get the result we have to put $f_i = f_1$ and $f_j = f_2$ in the Theorem 4. ■

Corollary 8. *Let (X, d) be a complete 2-metric space and f be a self-map such that f be F_α-contraction and satisfies the relation*

$$\int_0^{\tau+F(\alpha(x,y,a)d(fx,fy,a))} \phi(t)dt \leq \int_0^{F(d(x,y,a))} \phi(t)dt$$

for $\tau > 0$. Then f_1 and f_2 have a unique fixed point in X.

Proof. To get the result we have to put $f_i = f$ in the Theorem 4. ■

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